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Weighted Asymptotic Behavior of Solutions to a Sobolev-Type Differential Equation with Stepanov Coefficients in Banach Spaces

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Abstract. In this paper, we investigate weighted asymptotic behavior of solutions to the Sobolev-type differential equation

$$\frac{d}{dt}[u(t) + f(t, u(t))] = A(t)u(t) + g(t, u(t)), \quad t \in \mathbb{R},$$

where $A(t) : D \subseteq \mathbb{X} \to \mathbb{X}$ for $t \in \mathbb{R}$ is a family of densely defined closed linear operator on a domain D, independent of t, and $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is a weighted pseudo almost automorphic function and $g : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is an S^p-weighted pseudo almost automorphic function and satisfying suitable conditions. Some sufficient conditions are established by properties of S^p-weighted pseudo almost automorphic functions combined with theories of asymptotically stable of operators.

1. Introduction

The concept of almost automorphy was first introduced by Bochner in [3] in relation to some aspects of differential geometry. Since then, this concept has undergone several interesting, natural and powerful generalizations, for more details about this topic we refer to [14, 26, 27] and references therein. The extensions of almost automorphy are very similar to those of almost periodicity. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata in [25]. Liang, Xiao and Zhang in [22, 32] presented the concept of pseudo almost automorhy. The concept of Stepanov-like almost automorphy was presented by Casarino in [5] and was further developed in [28] by N'Guérékata and Pankov. Blot et al. introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space in [4]. Xia and Fan presented the notation of Stepanov-like (or S^p-) weighted pseudo almost automorphic function in [30]. Zhang, Chang and N'Guérékata further investigated some new ergodic properties and composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [34, 35].

The above mentioned concepts have been applied to various abstract differental and partial functional differential equations by many scholars, see for instance [1, 2, 6–10, 13, 15–18, 20, 21, 23, 29, 31, 36] and

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references therein. Since Sobolev-type differential equations can find many applications in different disciplines such as wave propagations or dynamics of fliuds, Diagana in [14, Chapter 11] studied existence and uniqueness of pseudo almost automorphic solutions to the following Sobolev-type differential equation

$$\frac{d}{dt}[u(t)+f(t,u(t))] = A(t)u(t) + g(t,u(t)), \ \ t \in \mathbb{R}, \tag{1}$$

where $A(t) : D \subseteq \mathbb{X} \to \mathbb{X}$ for $t \in \mathbb{R}$ is a family of densely defined closed linear operator on a domain D, independent of t, and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a pseudo almost automorphic function and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is an S^p-pseudo almost automorphic function satisfying some Lipschitz type conditions.

Considering the Stepanov-like weighted pseudo almost automorphy is more general and has more rich information than pseudo almost automorphy, it is meaningful to consider the problem (1) when the forcing term g is Stepanov-like weighted pseudo almost automorphic. However, few results are available for weighted asymptotic behavior of solutions to the problem (1) under the condition that the forcing term g is Stepanov-like weighted pseudo almost automorphic. By the main theories developed in [14, 30, 34], the main aim of the present paper is to investigate weighted asymptotic behavior of solutions to the problem (1) with S^p-weighted pseudo almost automorphic coefficients. Some sufficient conditions are established via ergodicity and composition theorems of S^p-weighted pseudo almost automorphic functions combined with theories of asymptotically stable of operators.

The rest of this paper is organized as follows. In Section 2, we recall some preliminary results which will be used throughout this paper. In Section 3, we establish some sufficient conditions for weighted pseudo almost automorphic solutions to the problem (1).

2. Preliminaries

Throughout this paper, we assume that $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are two Banach spaces. We let $C(\mathbb{R},\mathbb{X})$ (respectively, $C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denote the collection of all continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the collection of all jointly continuous functions $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$). Furthermore, $BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) stands for the class of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the class of all jointly bounded continuous functions from $\mathbb{R}\times\mathbb{Y}$ into $\mathbb{X}$). Note that $BC(\mathbb{R},\mathbb{X})$ is a Banach space with the sup norm $\|\cdot\|_\infty$. Furthermore, we denote by $B(\mathbb{X})$ the space of bounded linear operators form $\mathbb{X}$ into $\mathbb{X}$ endowed with the operator topology.

First, we list some basic definitions, properties of some almost automorphic type functions in abstract spaces.

Definition 2.1. [27] A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\chi(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} \chi(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Definition 2.2. [27] A continuous function $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$ is said to be almost automorphic if $f(t,x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in \mathbb{K}$, where $\mathbb{K}$ is any bounded subset of $\mathbb{Y}$. The collection of all such functions will be denoted by $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

Definition 2.3. [33] A continuous function $f(t,s) : \mathbb{R}\times\mathbb{R} \to \mathbb{X}$ is called bi-almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\chi(t,s) := \lim_{n\to\infty} f(t+s_n, s+s_n)$$

is well defined for each $t, s \in \mathbb{R}$, and

$$\lim_{n\to\infty} \chi(t - s_n, s - s_n) = f(t, s)$$

for each $t, s \in \mathbb{R}$. The collection of all such functions will be denoted by $bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$.

Let $\mathbb{U}$ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and for each $\rho \in \mathbb{U}$, we set $m(r, \rho) := \int_{-r}^{r} \rho(t)dt$. Thus the space of weights $\mathbb{U}_\infty$ is defined by

$$\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \lim_{r\to\infty} m(r, \rho) = \infty\}.$$

For a given $\rho \in \mathbb{U}_\infty$, we define

$$PAA_0(\mathbb{X}, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t)\|\rho(t)dt = 0 \right\};$$

$$PAA_0(\mathbb{Y}, \mathbb{X}, \rho) := \Big\{ f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : f(\cdot, y) \text{ is bounded for each } y \in \mathbb{Y} \\ \text{and } \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t, y)\|\rho(t)dt = 0 \text{ uniformly in } y \in \mathbb{Y} \Big\}.$$

Definition 2.4. [4] Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ (respectively, $f \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) is called weighted pseudo almost automorphic if it can be expressed as $f = g + h$, where $g \in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) and $h \in PAA_0(\mathbb{X}, \rho)$ (respectively, $PAA_0(\mathbb{Y}, \mathbb{X}, \rho)$). We denote by $WPAA(\mathbb{R}, \mathbb{X})$ (respectively, $WPAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) the set of all such functions.

Lemma 2.5. [24, Theorem 2.15] Let $\rho \in \mathbb{U}_\infty$. If $PAA_0(\mathbb{X}, \rho)$ is translation invariant, then $(WPAA(\mathbb{R}, \mathbb{X}), \|\cdot\|_\infty)$ is a Banach space.

Remark 2.6. For the conditions to guarantee translation invariance, please refer to [19, (4.1)-(4.2), pp. 420].

Definition 2.7. [11, 14] The Bochner transform $f^b(t, s), t \in \mathbb{R}, s \in [0, 1]$, of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by

$$f^b(t, s) := f(t + s).$$

Definition 2.8. [11, 14] Let $p \in [1, \infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0, 1; \mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p d\tau \right)^{\frac{1}{p}}.$$

Definition 2.9. [11, 14] The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic (or S^p-almost automorphic) functions consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA(L^p(0, 1; \mathbb{X}))$. In other words, a function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; \mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ such that

$$\lim_{n\to\infty} \left(\int_0^1 \|f(t + s + s_n) - g(t + s)\|^p ds \right)^{\frac{1}{p}} = 0$$

and

$$\lim_{n\to\infty} \left(\int_0^1 \|g(t + s - s_n) - f(t + s)\|^p ds \right)^{\frac{1}{p}} = 0,$$

pointwise on $\mathbb{R}$.

Definition 2.10. [11, 14] A function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$, $(t, u) \to f(t, u)$ with $f(\cdot, u) \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$ if $t \to f(t, u)$ is S^p-almost automorphic for each $u \in \mathbb{Y}$. That means, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g(\cdot, u) \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ such that

$$\lim_{n\to\infty}\left(\int_0^1 \|f(t+s+s_n, u) - g(t+s, u)\|^p ds\right)^{\frac{1}{p}} = 0,$$

and

$$\lim_{n\to\infty}\left(\int_0^1 \|g(t+s-s_n, u) - f(t+s, u)\|^p ds\right)^{\frac{1}{p}} = 0,$$

pointwise on $\mathbb{R}$ and for each $u \in \mathbb{Y}$. We denote by $AS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ the set of all such functions.

Definition 2.11. [34] Let $\rho \in \mathbb{U}_\infty$. A function $f \in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g + h$, where $g \in AS^p(\mathbb{X})$ and $h^b \in PAA_0\left(L^p(0, 1; \mathbb{X}), \rho\right)$. In other words, a function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic relatively to the weight $\rho \in \mathbb{U}_\infty$, if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; \mathbb{X})$ is weighted pseudo almost automorphic in the sense that there exist two functions $g, h : \mathbb{R} \to \mathbb{X}$ such that $f = g + h$, where $g \in AS^p(\mathbb{X})$ and $h^b \in PAA_0\left(L^p(0, 1; \mathbb{X}), \rho\right)$. We denote by $WPAAS^p(\mathbb{R}, \mathbb{X})$ the set of all such functions.

Definition 2.12. [34] Let $\rho \in \mathbb{U}_\infty$. A function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$, $(t, u) \to f(t, u)$ with $f(\cdot, u) \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g + h$, where $g \in AS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $h^b \in PAA_0\left(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \rho\right)$. We denote by $WPAAS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ the set of all such functions.

Remark 2.13. [30, 34] It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{loc}(\mathbb{R}, \mathbb{X})$ is S^q-almost automorphic, then f is S^p almost automorphic. Also if $f \in AA(\mathbb{X})$, then f is S^p-almost automorphic for any $1 \le p < \infty$.

Lemma 2.14. [30, 35] Let $\rho \in \mathbb{U}_\infty$ be such that

$$\limsup_{t\to\infty} \frac{\rho(t+\iota)}{\rho(t)} < \infty \;\text{ and }\; \limsup_{r\to\infty} \frac{m(r+\iota, \rho)}{m(r, \rho)} < \infty, \tag{2}$$

for every $\iota \in \mathbb{R}$, then spaces $WPAAS^p(\mathbb{R}, \mathbb{X})$ and $PAA_0(L^p(0, 1; \mathbb{X}), \rho)$ are translation invariant.

Lemma 2.15. [34] Let $\rho \in \mathbb{U}_\infty$. Assume that $PAA_0(L^p(0, 1; \mathbb{X}), \rho)$ is translation invariant. Then the decomposition of an S^p-weighted pseudo almost automorphic function is unique.

Lemma 2.16. [30] $WPAA(\mathbb{R}, \mathbb{X}) \subseteq WPAAS^p(\mathbb{R}, \mathbb{X})$ and $WPAAS^q(\mathbb{R}, \mathbb{X}) \subseteq WPAAS^p(\mathbb{R}, \mathbb{X})$ for $1 \le p < q < +\infty$.

Lemma 2.17. [34] Let $\rho \in \mathbb{U}_\infty$ satisfy the condition (2), then the space $WPAAS^p(\mathbb{R}, \mathbb{X})$ equipped with the norm $\|\cdot\|_{S^p}$ is a Banach space.

Lemma 2.18. [34] Let $\rho \in \mathbb{U}_\infty$ and let $f = f_1 + f_2 \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ with $f_1 \in AS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$, $f_2^b \in PAA_0\left(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \rho\right)$. Assume that the following condition (i) and (ii) are satisfied:
(i) $f(t, x)$ is Lipschitzian in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{R}$; that is, there exists a constant $L_f > 0$ such that

$$\|f(t, x) - f(t, y)\| \le L_f \|x - y\|$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}$.
(ii) $f_1(t, x)$ is uniformly continuous in any bounded subset $K' \subseteq \mathbb{X}$ uniformly for $t \in R$.

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{R}, \mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0, 1; \mathbb{X}), \rho)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $\Lambda : \mathbb{R} \to \mathbb{X}$ defined by $\Lambda(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^p(\mathbb{R}, \mathbb{X})$.

Lemma 2.19. [34] Let $\rho \in \mathbb{U}_\infty$ and let $f = f_1 + f_2 \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ with $f_1 \in AS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$, $f_2^b \in PAA_0(\mathbb{Y}, L^p(0,1;\mathbb{X}), \rho)$. Assume that the following condition (i), (ii) and (iii) are satisfied:

(i) there exists a nonnegative function $L_f \in BS^p(\mathbb{R})$ with $p > 1$ such that for all $u, v \in \mathbb{R}$ and $t \in \mathbb{R}$.

$$\left(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds \right)^{\frac{1}{p}} \leq L_f(t)\|u - v\|;$$

(ii) $\rho \in L^q_{loc}(\mathbb{R})$ satisfies $\lim\limits_{T\to\infty} \sup \frac{T^{\frac{1}{p}} m_q(T,\rho)}{m(T,\rho)} < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $m_q(T,\rho) = \left(\int_{-T}^T \rho^q(t)dt \right)^{\frac{1}{q}}$;

(iii) $f_1(t,x)$ is uniformly continuous in any bounded subset $K \subseteq \mathbb{X}$.

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{R}, \mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}), \rho)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $\Lambda : \mathbb{R} \to \mathbb{X}$ defined by $\Lambda(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^p(\mathbb{R}, \mathbb{X})$.

Lemma 2.20. [30] Let $\rho \in \mathbb{U}_\infty$, $p > 1$ and let $f = \varphi + \chi \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ with $\varphi \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $\chi^b \in PAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}), \rho)$. Assume that the following conditions are satisfied:

(i) there exist nonnegative functions $\mathcal{L}_f(\cdot)$, $\mathcal{L}_\varphi(\cdot) \in AS^r(\mathbb{R}, \mathbb{R})$ with $r \geq \max\left\{p, \frac{p}{p-1}\right\}$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$

$$\left\| f(s,u) - f(s,v) \right\| \leq \mathcal{L}_f(t) \|u - v\|, \ \left\| \varphi(s,u) - \varphi(s,v) \right\| \leq \mathcal{L}_\varphi(t) \|u - v\|;$$

(ii) $u = u_1 + u_2 \in WPAAS^p(\mathbb{R}, \mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}), \rho)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact in $\mathbb{X}$.

Then there exists $\overline{q} \in [1, p)$ such that $\Lambda : \mathbb{R} \longrightarrow \mathbb{X}$ defined by $\Lambda(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^{\overline{q}}(\mathbb{R}, \mathbb{X})$.

3. Main Results

In this section, we investigate the weighted asymptotic behavior of solutions to the problem (1).

It is followed by [14, Chapter 11], we suppose that there exists a Banach space $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}})$ such that the embedding $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}}) \hookrightarrow (\mathbb{X}, \|\cdot\|)$ is continuous. Moreover, we assume that the following additional assumptions hold:

(H1) The system

$$u'(t) = A(t)u(t), \ t \geq s, \ u(s) = \varphi \in \mathbb{X}$$

has an associated evolution family of operators $\{U(t,s) : t \geq s$ with $t, s \in \mathbb{R}\}$. Further, we assume that the domains of the operators $A(t)$ are constant in t, that is, $D(A(t)) = D = \mathscr{Y}$ for all $t \in \mathbb{R}$ and that the evolution family $U(t,s)$ is asymptotically stable in the sense that there exist some constants $M, \delta > 0$ such that

$$\|U(t,s)\| \leq Me^{-\delta(t-s)}$$

for all $t, s \in \mathbb{R}$ with $t \geq s$.

(H2) The function $s \mapsto A(s)U(t,s)$ defined from $(-\infty, t)$ into $B(\mathbb{R} \times \mathscr{Y})$ is strongly measurable and there exist a measurable function $H : (0,\infty) \mapsto (0,\infty)$ with $H \in L^1(0,\infty)$ and a constant $\omega > 0$ such that

$$\|A(s)U(t,s)x\| \leq e^{-\omega(t-s)}H(t-s)\|x\|, \ t, s \in \mathbb{R}, \ t > s, \ x \in \mathscr{Y}.$$

(H3) The series

$$\sum_{k=1}^{\infty} \left(\int_{t-k}^{t-k+1} e^{-\omega q(t-s)} H^q(t-s) ds \right)^{\frac{1}{q}}$$

converges, $q > 1$, and let $\mathbb{K} = \left(\int_{-\infty}^t e^{-\omega q(t-s)} H^q(t-s) \right)^{\frac{1}{q}}$.

(H4) The function $\mathbb{R} \times \mathbb{R} \mapsto \mathbb{X}$, $(t,s) \mapsto U(t,s)x \in bAA(\mathbb{R} \times \mathbb{R}, \mathscr{Y})$ uniformly for $x \in \mathbb{X}$.

(H5) The function $\mathbb{R}\times\mathbb{R}\mapsto\mathbb{X}$, $(t,s)\mapsto A(s)U(t,s)x\in bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$ uniformly for $x\in\mathscr{Y}$.

(H6)The function $f\in WPAA(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and there exists a constant $L_f>0$ such that

$$\|f(t,u)-f(t,v)\|_{\mathscr{Y}}\le L_f\|u-v\|$$

for all $u,v\in\mathbb{X}$ and $t\in\mathbb{R}$.

(H7) The function $g\in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and there exists a constant $L_g>0$ such that

$$\|g(t,u)-g(t,v)\|\le L_g\|u-v\|$$

for all $u,v\in\mathbb{X}$ and $t\in\mathbb{R}$.

(H8) The function $g\in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and there exists a nonnegative function $\mathcal{L}_g(\cdot)\in BS^p(\mathbb{R})$, with $p>1$ such that

$$\|g(t,u)-g(t,v)\|\le\mathcal{L}_g(\cdot)\|u-v\|$$

for all $u,v\in\mathbb{X}$ and $t\in\mathbb{R}$.

(H9) $\rho\in L^q_{loc}(\mathbb{R})$ satisfies

$$\lim_{T\to\infty}\frac{T^{\frac{1}{p}}m_q(T,\rho)}{m(T,\rho)}<\infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

(H10)$g=g_1+g_2\in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, where $g_1\in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ are uniformly continuous in any bounded subset $M\subset\mathbb{X}$ in $t\in\mathbb{R}$ and $g_2^b\in PAA_0(\mathbb{X},L^p(0,1;\mathbb{X}),\rho)$.

(H11) The function $g=\varphi+\chi\in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})\,(p>1)$ with $\varphi\in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\chi^b\in PAA_0(\mathbb{X},L^p(0,1;\mathbb{X}),\rho)$, and there exist nonnegative functions $\mathfrak{L}_g(\cdot)$, $\mathcal{L}_\varphi(\cdot)\in AS^r(\mathbb{R},\mathbb{R})$ with $r\ge\max\left\{p,\frac{p}{p-1}\right\}$ such that for all $u,v\in\mathbb{X}$ and $t\in\mathbb{R}$

$$\left\|g(s,u)-g(s,v)\right\|\le\mathfrak{L}_g(t)\,\|u-v\|,\ \left\|\varphi(s,u)-\varphi(s,v)\right\|\le\mathcal{L}_\varphi(t)\,\|u-v\|.$$

Definition 3.1. [14, Definition 11.1] A continuous function $u:\mathbb{R}\mapsto\mathbb{X}$ is said to be a mild solution to Eq.(1.1) provided that the function $s\mapsto A(s)U(t,s)f(s,u(s))$ is integrable on (s,t),and

$$u(t)=-f(t,u(t))+U(t,s)\left(u(s)+f(s,u(s))\right)-\int_s^t A(s)U(t,s)f(s,u(s))ds+\int_s^t U(t,s)g(s,u(s))ds$$

for $t\ge s$ and for all $t,s\in\mathbb{R}$.

Under assumptions (H1)-(H2), it can be easily shown that the function u given by

$$u(t)=-f(t,u(t))+\int_{-\infty}^t U(t,s)g(s,u(s))ds-\int_{-\infty}^t A(s)U(t,s)f(s,u(s))ds$$

for each $t\in\mathbb{R}$ is a mild solution to Eq.(1).

Lemma 3.2. Suppose conditions (H1)-(H4) hold. If $g:\mathbb{R}\to\mathbb{X}$ is an Stepanov-like weighted pseudo almost automorphic function, and $F(\cdot)$ is given by

$$F(t)=\int_{-\infty}^t U(t,s)g(s)ds,t\in\mathbb{R},$$

then, $F(\cdot)\in WPAA(\mathbb{R}\times\mathbb{X})$.

Proof. Since $g \in WPAAS^p(\mathbb{R}\times\mathbb{X})$, we have $g = g_1 + g_2$ with $g_1 \in AS^p(\mathbb{X})$, $g_2^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$. Consider for each $k = 1, 2, \cdots$, the integrals

$$\begin{aligned} F_k(t) &= \int_{t-k}^{t-k+1} U(t,s)g(s)ds \\ &= \int_{t-k}^{t-k+1} U(t,s)g_1(s)ds + \int_{t-k}^{t-k+1} U(t,s)g_2(s)ds \\ &= X_k(t) + Y_k(t), \end{aligned}$$

where $X_k(t) = \int_{t-k}^{t-k+1} U(t,s)g_1(s)ds$, $Y_k(t) = \int_{t-k}^{t-k+1} U(t,s)g_2(s)ds$. In order to prove that each F_k is a weighted pseudo almost automorphic function, we only need to verify $X_k \in AA(\mathbb{X})$ and $Y_k \in PAA_0(\mathbb{X},\rho)$ for each $k = 1, 2, \cdots$.

Let us first show that $X_k \in AA(\mathbb{X})$. We have

$$\begin{aligned} \|X_k(t)\| &= \left\| \int_{t-k}^{t-k+1} U(t,s)g_1(s)ds \right\| \\ &\le \int_{t-k}^{t-k+1} Me^{-\delta(t-s)}\|g_1(s)\|ds \\ &\le Me^{-\delta(k-1)} \left(\int_{t-k}^{t-k+1} \|g_1(s)\|^p ds \right)^{\frac{1}{p}} \\ &\le Me^{-\delta(k-1)}\|g_1\|_{s^p}. \end{aligned}$$

Since $\sum_{k=1}^{\infty} e^{-\delta(k-1)} = \frac{1}{1-e^{-\delta}} < \infty$, we conclude that the series $\sum_{k=1}^{\infty} X_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$X(t) := \int_{-\infty}^{t} U(t,s)g_1(s)ds = \sum_{k=1}^{\infty} X_k(t).$$

Clearly, $X(t) \in C(\mathbb{R},\mathbb{X})$ and $\|X(t)\| \le \sum_{k=1}^{\infty} \|X_k(t)\| \le \sum_{k=1}^{\infty} Me^{-\delta(k-1)}\|g_1\|_{s^p}$.

Since $g_1 \in AS^p(\mathbb{R},\mathbb{X})$ and $U(t,s)x \in bAA(\mathbb{R}\times\mathbb{R},\mathscr{Y})$, then for every sequence of real numbers $\{s_{n'}\}_{n'\in\mathbb{N}}$ there exist a sequence $\{s_n\}_{n\in\mathbb{N}}$ and functions $g_1'(\cdot) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ and $U'(\cdot)$ such that the following equalities hold:

$$\lim_{n\to\infty} \left(\int_0^1 \|g_1(t+s+s_n) - g_1'(t+s)\|^p ds \right)^{\frac{1}{p}} = 0, \text{ for each } t \in \mathbb{R}, \tag{3}$$

$$\lim_{n\to\infty} \left(\int_0^1 \|g_1'(t+s-s_n) - g_1(t+s)\|^p ds \right)^{\frac{1}{p}} = 0, \text{ for each } t \in \mathbb{R}, \tag{4}$$

and

$$\lim_{n\to\infty} U(t+s_n, s+s_n)x = U'(t,s)x, \qquad t,s \in \mathbb{R}, \qquad x \in \mathbb{X}, \tag{5}$$

$$\lim_{n\to\infty} U'(t-s_n, s-s_n)x = U(t,s)x, \qquad t,s \in \mathbb{R}, \qquad x \in \mathbb{X}, \tag{6}$$

Let $X_k'(t) = \int_{t-k}^{t-k+1} U(t,s)g_1'(s)ds$. Then using the Hölder inequality, we have

$$\begin{aligned} \|X_k(t+s_n) - X_k'(t)\| &\le \left\| \int_{t-k}^{t-k+1} U(t+s_n, s+s_n)(g_1(s+s_n) - g_1'(s))ds \right\| \\ &+ \left\| \int_{t-k}^{t-k+1} (U(t+s_n, s+s_n) - U(t,s))g_1'(s))ds \right\| \\ &= I_k(t) + J_k(t), \end{aligned}$$

where

$$I_k(t) = \left\| \int_{t-k}^{t-k+1} U(t+s_n, s+s_n)(g_1(s+s_n) - g_1^{'}(s))ds \right\|$$

and

$$J_k(t) = \left\| \int_{t-k}^{t-k+1} (U(t+s_n, s+s_n) - U(t,s))g_1^{'}(s))ds \right\|.$$

Then using the Hölder inequality we get

$$\begin{aligned} I_k(t) &\le \int_{t-k}^{t-k+1} Me^{-\delta(t-s)} \|(g_1(s+s_n) - g_1^{'}(s)\|ds \\ &\le Me^{-\delta(k-1)} \left(\int_{t-k}^{t-k+1} \|(g_1(s+s_n) - g_1^{'}(s)\|)^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

Now using Eq.(3.1) it follows that $I_k(t) \to 0$ as $n \to \infty$ for each $t \in \mathbb{R}$. Similarly, using the Lebesgue Dominated Convergence Theorem and Eq.(3.3) it follows that $J_k(t) \to 0$ $n \to \infty$ for each $t \in \mathbb{R}$. Now,

$$\|X_k(t+s_n) - X_k^{'}(t)\| \to 0 \text{ as } n \to \infty.$$

Similarly, using Eqs.(3.2) and (3.4) it can be shown that

$$\|X_k^{'}(t-s_n) - X_k(t)\| \to 0 \text{ as } n \to \infty.$$

Thus, we conclude that each $X_k \in AA(\mathbb{X})$ and consequently their uniform limit $X(t) \in AA(\mathbb{X})$.

Next, we show that each $Y_k \in PAA_0(\mathbb{X}, \rho)$. For this, we note that

$$\begin{aligned} \|Y_k(t)\| &\le \int_{t-k}^{t-k+1} \|U(t,s)g_2(s)\|ds \\ &\le \int_{t-k}^{t-k+1} Me^{-\delta(t-s)} \|g_2(s)\|ds \\ &\le Me^{-\delta(k-1)} \left(\int_{t-k}^{t-k+1} \|g_2(s)\|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

Then, for $T > 0$, we see that

$$\frac{1}{m(T,\rho)} \int_{-T}^{T} \|Y_k(t)\|\rho(t)dt \le Me^{-\delta(k-1)} \frac{1}{m(T,\rho)} \int_{-T}^{T} \left(\int_{t-k}^{t-k+1} \|g_2(s)\|^p ds \right)^{\frac{1}{p}} \rho(t)dt.$$

Since $g_2^b \in PAA_0(L^p(0,1;\mathbb{X}), \rho)$, the above inequality leads to $Y_k \in PAA_0(\mathbb{X}, \rho)$ for each $k = 1, 2, \cdots$. By a similar way, we deduce that the uniform limit $Y(\cdot) = \sum_{k=1}^{\infty} Y_k(t) \in PAA_0(\mathbb{X}, \rho)$. Therefore, $F(t) := X(t) + Y(t)$ is weighted pseudo almost automorphic. The proof is complete. □

Lemma 3.3. Assume that conditions (H1)-(H3) and (H5) are satisfied. If $f : \mathbb{R} \to \mathbb{X}$ is an Stepanov-like weighted pseudo almost automorphic function, and $F(\cdot)$ is given by

$$F(t) = \int_{-\infty}^{t} A(s)U(t,s)f(s)ds, t \in \mathbb{R},$$

then, $F(\cdot) \in WPAA(\mathbb{R} \times \mathbb{X})$.

Proof. Since $f \in WPAAS^p(\mathbb{R}\times\mathbb{X})$, we have $f = f_1 + f_2$ with $f_1 \in AS^p(\mathbb{X})$, $f_2^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$. Consider for each $k = 1, 2, \cdots$, the integrals

$$\begin{aligned} F_k(t) &= \int_{t-k}^{t-k+1} A(s)U(t,s)f(s)ds \\ &= \int_{t-k}^{t-k+1} A(s)U(t,s)f_1(s)ds + \int_{t-k}^{t-k+1} A(s)U(t,s)f_2(s)ds \\ &= X_k(t) + Y_k(t), \end{aligned}$$

where $X_k(t) = \int_{t-k}^{t-k+1} A(s)U(t,s)f_1(s)ds$, $Y_k(t) = \int_{t-k}^{t-k+1} A(s)U(t,s)f_2(s)ds$. In order to prove that each F_k is a weighted pseudo almost automorphic function, we only need to verify $X_k \in AA(\mathbb{X})$ and $Y_k \in PAA_0(\mathbb{X},\rho)$ for each $k = 1, 2, \cdots$.

Let us first show that $X_k \in AA(\mathbb{X})$. We have

$$\begin{aligned} \|X_k(t)\| &= \left\| \int_{t-k}^{t-k+1} A(s)U(t,s)f_1(s)ds \right\| \\ &\le \int_{t-k}^{t-k+1} e^{-\omega(t-s)} H(t-s)\|f_1(s)\|ds \\ &\le \left(\int_{t-k}^{t-k+1} e^{-\omega q(t-s)} H^q(t-s)ds \right)^{\frac{1}{q}} \left(\int_{t-k}^{t-k+1} \|f_1(s)\|^p ds \right)^{\frac{1}{p}} \\ &\le \left(\int_{t-k}^{t-k+1} e^{-\omega q(t-s)} H^q(t-s)ds \right)^{\frac{1}{q}} \|f_1(s)\|_{S^p}. \end{aligned}$$

Using the fact the series given by

$$\sum_{k=1}^{\infty} \left(\int_{t-k}^{t-k+1} e^{-\omega q(t-s)} H^q(t-s)ds \right)^{\frac{1}{q}}$$

converges, we then deduce from the well-known Weierstrass theorem that the series$\sum_{k=1}^{\infty} X_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$X(t) := \int_{-\infty}^{t} A(s)U(t,s)f_1(s)ds = \sum_{k=1}^{\infty} X_k(t).$$

Clearly, $X(t) \in C(\mathbb{R},\mathbb{X})$ and

$$\|X(t)\| \le \sum_{k=1}^{\infty} \|X_k(t)\| \le \sum_{k=1}^{\infty} \left(\int_{t-k}^{t-k+1} e^{-\omega q(t-s)} H^q(t-s)ds \right)^{\frac{1}{q}} \|f_1(s)\|_{S^p}.$$

Since $f_1 \in AS^p(\mathbb{R},\mathbb{X})$ and $A(s)U(t,s)x \in bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$, then for every sequence of real numbers $\{s_{n'}\}_{n'\in\mathbb{N}}$ there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and functions $f_1'(\cdot) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ and $U'(\cdot)$ such that the following equalities hold:

$$\lim_{n\to\infty} \left(\int_0^1 \|f_1(t+s+s_n) - f_1'(t+s)\|^p ds \right)^{\frac{1}{p}} = 0, \text{ for each } t \in \mathbb{R}, \tag{7}$$

$$\lim_{n\to\infty} \left(\int_0^1 \|f_1'(t+s-s_n) - f_1(t+s)\|^p ds \right)^{\frac{1}{p}} = 0, \text{ for each } t \in \mathbb{R}, \tag{8}$$

and

$$\lim_{n\to\infty} A(s+s_n)U(t+s_n,s+s_n)x = U^{'}(t,s)x, \qquad t,s\in\mathbb{R}, \qquad x\in\mathbb{X}, \tag{9}$$

$$\lim_{n\to\infty} U^{'}(t-s_n,s-s_n)x = A(s)U(t,s)x, \qquad t,s\in\mathbb{R}, \qquad x\in\mathbb{X}, \tag{10}$$

Let $X_k^{'}(t)=\int_{t-k}^{t-k+1}A(s)U(t,s)f_1^{'}(s)ds$. Then using the Hölder inequality, we have

$$\begin{aligned}
\|X_k(t+s_n)-X_k^{'}(t)\| &\le \left\|\int_{t-k}^{t-k+1}A(s+s_n)U(t+s_n,s+s_n)(f_1(s+s_n)-f_1^{'}(s))ds\right\| \\
&+ \left\|\int_{t-k}^{t-k+1}(A(s+s_n)U(t+s_n,s+s_n)-A(s)U(t,s))f_1^{'}(s)ds\right\| \\
&= I_k(t)+J_k(t),
\end{aligned}$$

where

$$I_k(t)=\left\|\int_{t-k}^{t-k+1}A(s+s_n)U(t+s_n,s+s_n)(f_1(s+s_n)-f_1^{'}(s))ds\right\|$$

and

$$J_k(t)=\left\|\int_{t-k}^{t-k+1}(A(s+s_n)U(t+s_n,s+s_n)-A(s)U(t,s))f_1^{'}(s)ds\right\|.$$

Then using the Hölder inequality we get

$$\begin{aligned}
I_k(t) &\le \int_{t-k}^{t-k+1}e^{-\omega(t-s)}H(t-s)\|(f_1(s+s_n)-f_1^{'}(s)\|ds \\
&\le \left(\int_{t-k}^{t-k+1}e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\left(\int_{t-k}^{t-k+1}\|f_1(s+s_n)-f_1^{'}(s)\|^p ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Now using Eq.(3.5) it follows that $I_k(t)\to 0$ as $n\to\infty$ for each $t\in\mathbb{R}$. Similarly, using the Lebesgue Dominated Convergence Theorem and Eq.(3.7) it follows that $J_k(t)\to 0$ $n\to\infty$ for each $t\in\mathbb{R}$. Now,

$$\|X_k(t+s_n)-X_k^{'}(t)\|\to 0 \text{ as } n\to\infty.$$

Similarly, using Eqs.(3.6) and (3.8) it can be shown that

$$\|X_k^{'}(t-s_n)-X_k(t)\|\to 0 \text{ as } n\to\infty.$$

Thus, we conclude that each $X_k\in AA(\mathbb{X})$ and consequently their uniform limit $X(t)\in AA(\mathbb{X})$.

Next, we show that each $Y_k\in PAA_0(\mathbb{X},\rho)$. For this, we note that

$$\begin{aligned}
\|Y_k(t)\| &\le \int_{t-k}^{t-k+1}e^{-\omega(t-s)}H(t-s)\|f_2(s)\|ds \\
&\le \left(\int_{t-k}^{t-k+1}e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\left(\int_{t-k}^{t-k+1}\|f_2(s)\|^p ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Then, for $T>0$, we see that

$$\frac{1}{m(T,\rho)}\int_{-T}^{T}\|Y_k(t)\|\rho(t)dt\le\mathcal{K}\frac{1}{m(T,\rho)}\int_{-T}^{T}\left(\int_{t-k}^{t-k+1}\|f_2(s)\|^p ds\right)^{\frac{1}{p}}\rho(t)dt,$$

where $\mathcal{K}=\left(\int_{t-k}^{t-k+1}e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}$, since $f_2^b\in PAA_0(L^p(0,1;\mathbb{X}),\rho)$, the above inequality leads to $Y_k\in PAA_0(\mathbb{X},\rho)$ for each $k=1,2,\cdots$. By a similar way, we deduce that the uniform limit $Y(\cdot)=\sum_{k=1}^{\infty}Y_k(t)\in PAA_0(\mathbb{X},\rho)$. Therefore, $F(t):=X(t)+Y(t)$ is weighted pseudo almost automorphic. The proof is complete. □

We shall present and prove our main results.

Theorem 3.4. Let conditions (H1)-(H7) and (H10) be true, then the problem (1) has a unique weighted pseudo almost automorphic mild solution whenever $L_f + \frac{M}{\delta}L_g + L_f\mathbb{K} < 1$.

Proof. Consider the nonlinear operator Γ defined by

$$(\Gamma x)(t) = -f(t, x(t)) + \int_{-\infty}^{t} U(t,s)g(s,x(s))ds - \int_{-\infty}^{t} A(s)U(t,s)f(s,x(s))ds, t \in \mathbb{R}.$$

First, let us prove that $\Gamma(WPAA(\mathbb{R},\mathbb{X})) \subseteq WPAA(\mathbb{R},\mathbb{X})$. For each $x \in WPAA(\mathbb{R},\mathbb{X})$, by using [14, Theorem 6.18] and Lemma 2.18, one can easily see that $f(\cdot, x(\cdot)) \in WPAA(\mathbb{R},\mathscr{Y}) \subseteq WPAA(\mathbb{R},\mathbb{X})$, $g(\cdot, x(\cdot)) \in WPAAS^p(\mathbb{R},\mathbb{X})$. Hence, from the proof of Lemmas 2.16, 3.2 and 3.3 we know that $(\Gamma x)(\cdot) \in WPAA(\mathbb{R},\mathbb{X})$. That is, Γ maps $WPAA(\mathbb{R},\mathbb{X})$ into $WPAA(\mathbb{R},\mathbb{X})$.

Next, we prove that Γ is a strict contraction mapping on $WPAA(\mathbb{R},\mathbb{X})$. To this end, for each $t \in \mathbb{R}$, $x, y \in WPAAS^p(\mathbb{R},\mathbb{X})$, we have

$$\begin{aligned}\|\Gamma x - \Gamma y\| &\leq \|x-y\|_\infty\left(L_f + ML_g\int_{-\infty}^{t} e^{-\delta(t-s)}ds + L_f\int_{-\infty}^{t} e^{-\omega(t-s)}H(t-s)ds\right)\\ &\leq \left(L_f + \frac{M}{\delta}L_g + L_f\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\right)\|x-y\|_\infty.\end{aligned}$$

Hence

$$\|\Gamma x - \Gamma y\|_\infty \leq \left(L_f + \frac{M}{\delta}L_g + L_f\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\right)\|x-y\|_\infty,$$

which implies that Γ is a contraction on $WPAA(\mathbb{R},\mathbb{X})$. Therefore, by the Banach contraction principle, we draw a conclusion that there exists a unique fixed point $x(\cdot)$ for Γ in $WPAA(\mathbb{R},\mathbb{X})$ with $\Gamma x = x$. It is clear that x is the unique weighted pseudo almost automorphic mild solution of the problem (1). This completes the proof. □

Theorem 3.5. Suppose that conditions (H1)-(H6) and (H8)-(H10) hold, then the problem (1) admits a unique weighted pseudo almost automorphic mild solution whenever

$$\left(L_f + M\|\mathcal{L}_g\|_{S^p}\sqrt[q]{\frac{1}{\delta q}} + L_f\mathbb{K}\right) < 1.$$

Proof. Consider the nonlinear operator Γ is given by

$$(\Gamma x)(t) = -f(t, x(t)) + \int_{-\infty}^{t} U(t,s)g(s,x(s))ds - \int_{-\infty}^{t} A(s)U(t,s)f(s,x(s))ds, t \in \mathbb{R}.$$

For given $x \in WPAA(\mathbb{R},\mathbb{X})$, it follows from [14, Theorem 6.18] and Lemma 2.19 that the function $f(s,x(s)) \in WPAA(\mathbb{R},\mathscr{Y}) \subseteq WPAA(\mathbb{R},\mathbb{X}), g(s,x(s)) \in WPAAS^p(\mathbb{R},\mathbb{X})$. Moreover, from Lemmas 2.16, 3.2 and 3.3, we infer that $\Gamma x \in WPAA(\mathbb{R},\mathbb{X})$, that is, Γ maps $WPAA(\mathbb{R},\mathbb{X})$ into itself. Next, we prove that the operator Γ has a unique fixed point in $WPAA(\mathbb{R},\mathbb{X})$. Indeed, for each $t\in\mathbb{R}, x, y\in WPAA(\mathbb{R},\mathbb{X})$, we have

$$\begin{aligned}
\|\Gamma x(t)-\Gamma y(t)\| &\le L_f\|x-y\|_\infty + M\|x-y\|_\infty\left(\int_{-\infty}^{t} e^{-\delta q(t-s)}ds\right)^{\frac{1}{q}}\left(\int_{-\infty}^{t}\|\mathcal{L}_g(s)\|^p ds\right)^{\frac{1}{p}}\\
&\quad +\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}} L_f\|x-y\|_\infty\\
&\le L_f\|x-y\|_\infty + M\|x-y\|_\infty\|\mathcal{L}_g\|_{S^p}\sqrt[q]{\frac{1}{\delta q}}\\
&\quad +L_f\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\|x-y\|_\infty.
\end{aligned}$$

Hence

$$\|\Gamma x-\Gamma y\|_\infty \le \left(L_f + M\|\mathcal{L}_g\|_{S^p}\sqrt[q]{\frac{1}{\delta q}} + L_f\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\right)\|x-y\|_\infty.$$

Since $\left(L_f + M\|\mathcal{L}_g\|_{S^p}\sqrt[q]{\frac{1}{\delta q}} + L_f\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\right)<1$, Γ has a unique fixed point $x\in WPAA(\mathbb{R},\mathbb{X})$. The proof is finished. □

Theorem 3.6. Let assumptions (H1)-(H6) and (H11) hold, then the problem (1) admits a unique weighted pseudo almost automorphic mild solution whenever

$$\left(L_f + M\|\mathfrak{L}_g\|_{S^r}\sqrt[\bar{r}]{\frac{1}{\delta\bar{r}}} + L_f\mathbb{K}\right)<1,$$

where $\frac{1}{\bar{r}} = 1-\frac{1}{r}$.

Proof. Consider the nonlinear operator Γ is given by

$$(\Gamma x)(t) = -f(t,x(t)) + \int_{-\infty}^{t} U(t,s)g(s,x(s))ds - \int_{-\infty}^{t} A(s)U(t,s)f(s,x(s))ds,\ t\in\mathbb{R}.$$

For given $x\in WPAA(\mathbb{R},\mathbb{X})$, it follows from [14, Theorem 6.18] and Lemma 2.20 that the function $f(s,x(s))\in WPAA(\mathbb{R},\mathscr{Y})\subseteq WPAA(\mathbb{R},\mathbb{X}), g(s,x(s))\in WPAAS^{\bar{q}}(\mathbb{R},\mathbb{X})$. Moreover, from Lemmas 2.16, 3.2 and 3.3, we infer that $\Gamma x\in WPAA(\mathbb{R},\mathbb{X})$, that is, Γ maps $WPAA(\mathbb{R},\mathbb{X})$ into itself. Next, we prove that the operator Γ has a unique fixed point in $WPAA(\mathbb{R},\mathbb{X})$. Indeed, for each $t\in\mathbb{R}, x, y\in WPAA(\mathbb{R},\mathbb{X})$, we have

$$\begin{aligned}
\|\Gamma x(t)-\Gamma y(t)\| &\le L_f\|x-y\|_\infty + M\|x-y\|_\infty\left(\int_{-\infty}^{t} e^{-\delta\bar{r}(t-s)}ds\right)^{\frac{1}{\bar{r}}}\left(\int_{-\infty}^{t}\|\mathfrak{L}_g(s)\|^r ds\right)^{\frac{1}{r}}\\
&\quad +\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}} L_f\|x-y\|_\infty\\
&\le L_f\|x-y\|_\infty + M\|x-y\|_\infty\|\mathfrak{L}_g\|_{S^r}\sqrt[\bar{r}]{\frac{1}{\delta\bar{r}}}\\
&\quad +L_f\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\|x-y\|_\infty.
\end{aligned}$$

Hence

$$\|\Gamma x-\Gamma y\|_\infty \le \left(L_f + M\|\mathfrak{L}_g\|_{S^r}\sqrt[\bar{r}]{\frac{1}{\delta\bar{r}}} + L_f\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\right)\|x-y\|_\infty.$$

Since $\left(L_f + M\|\mathfrak{L}_g\|_{S^r}\sqrt[\bar{r}]{\frac{1}{\delta\bar{r}}} + L_f\left(\int_{-\infty}^{t} e^{-\omega q(t-s)}H^q(t-s)ds\right)^{\frac{1}{q}}\right) < 1$, Γ has a unique fixed point $x \in WPAA(\mathbb{R}, \mathbb{X})$. The proof is finished. □

Example 3.7. One can refer to [14, Chapter 11] for an interesting application involved in heat equation. Here we consider the following simple problem as

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + (-2+\sin t+\sin \pi t)u(t,x) + g(t,u(t,x)), \\ u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}, \quad x \in [0,\pi], \end{cases} \tag{11}$$

where $g \in L^2[0,\pi] \to L^2[0,\pi]$ given by

$$g(t,u(t,x)) = u(t,x)\sin\frac{1}{2+\cos t+\cos\pi t} + e^{-|t|}\sin u(t,x).$$

We take $X := L^2[0,\pi]$ equipped with its natural topology and define the operator A by
$D(A) := \{\varphi \in L^2[0,\pi] : u(0) = u(\pi) = 0, u'' \in L^2[0,\pi]\}$, and $A\varphi = \varphi''$, $\varphi \in D(A)$.

From the computation of [12, Theorem 4.5], we can see that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $L^2[0,\pi]$ with $\|T(t))\| \leq e^{-t}$ for $t \geq 0$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $\varphi_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$ for each $n = 1, 2\cdots$. Moreover, $T(t)\varphi = \Sigma_{n=1}^{\infty}e^{-n^2t}\langle\varphi,\varphi_n\rangle\varphi_n$, for each $\varphi \in L^2[0,\pi]$.

Define a family of linear operators $A(t)$ by

$$\begin{cases} D(A(t)) = D(A), \\ A(t)\varphi(\xi) := (A-2+\sin t+\sin\pi t)\varphi(\xi), \quad \xi \in [0,\pi], \quad \varphi \in D(A). \end{cases}$$

One sees that the system

$$\begin{cases} u'(t) = A(t)u(t), \quad t \geq s, \\ u(s) = \varphi \in L^2[0,\pi], \end{cases}$$

has an associated evolution family $(U(t,s))_{t\geq s}$ on $L^2[0,\pi]$, which can be explicitly given by:

$$U(t,s)\varphi(\xi) = T(t-s)e^{\int_s^t(-2+\sin\tau+\sin\pi\tau)d\tau}\varphi(\xi).$$

Hence

$$\|U(t,s)\| \leq e^{-(t-s)} \text{ for } t \geq s.$$

Thus the condition (H1) is true. Moreover, since $A(t+2\pi) = A(t)$ for all $t \in \mathbb{R}$, it follows that $U(t+2\pi, s+2\pi) = U(t,s)$ and $A(s+2\pi)U(t+2\pi, s+2\pi) = A(s)U(t,s)$ for all $t, s \in \mathbb{R}$ with $t \geq s$. Therefore $(t,s) \to U(t,s)w$ belongs to $bAA(\mathbb{R}\times\mathbb{R}, L^2[0,\pi])$ uniformly in $w \in L^2[0,\pi]$ and $(t,s) \to A(s)U(t,s)w$ belongs to $bAA(\mathbb{R}\times\mathbb{R}, D(A))$ uniformly in $w \in D(A)$. Hence conditions (H4)-(H5) hold. Note that the function $g \in WPAAS^p(\mathbb{R}, \mathbb{X})$ with weight $\rho(t) = 1+t^2$ for $t \in \mathbb{R}$, and

$$\|g(t,u) - g(t,v)\| \leq 2\|u-v\|.$$

According to Theorem 3.4, we obtain the following corollary.

Corollary 3.8. The Eq.(11) admits a unique weighted pseudo almost automorphic solution with weight $\rho(t) = 1+t^2$ provided that $\frac{M}{\delta} < \frac{1}{2}$.

References

[1] S. Abbas, Pseudo almost automorphic solutions of some nonlinear integro-differential equations, Comp. Math. Appl. 62 (2011) 2259–2272.

[2] S. Abbas, Y. Xia, Existence and attractivity of k-almost automorphic solutions of a model of cellular neural network with delay, Acta Mathematica Scientia 1 (2013) 290–302.

[3] S. Bochner, Continuous mappings of almost automorphic and almost periodic functions, Proc. Natl. Acad. Sci. USA 52 (1964) 907–910.
[4] J. Blot, G. M. Mophou, G. M. N'Guérékata, D. Pennequin, Weighted pseudo almost automorphic functions and applications to abstract differential equations, Nonlinear Anal. 71 (2009) 903–909.
[5] V. Casarino, Characterization of almost automorphic groups and semigroups, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. 5 (2000) 219–235.
[6] Y. K. Chang, Z. H. Zhao, J. J. Nieto, Pseudo almost automorphic and weighted pseudo almost automorphic mild solutions to semi-linear differential equations in Hilbert spaces, Rev. Mat. Complut. 24 (2011) 421–438.
[7] Y. K. Chang, R. Zhang, G. M. N'Guérékata, Weighted pseudo almost automorphic solutions to nonautonomous semilinear evolution equations with delay and S^p-weighted pseudo almost automorphic coefficients, Topol. Method Nonlinear Anal. 43 (2014) 69–88.
[8] A. Caicedo, C. Cuevas, G. Mophou, G. M. N'Guérékata, Asymptotic behavior of solutions of some semilinear functional differential and integro-differential equations with infinite delay in Banach spaces, J. Franklin Inst. 349 (2012) 1–24.
[9] C. Cuevas, M. Rabelo, H. Soto, Pseudo-almost automorphic solutions to a class of semilinear fractional differential equations, Communications Appl. Nonlinear Anal. 17 (2010) 31–47.
[10] C. Cuevas, A. Sepulveda, H. Soto, Almost periodic and pseudo-almost periodic solutions to fractional differential and integro-differential equations, Appl. Math. Comput. 218 (2011) 1735–1745.
[11] T. Diagana, G. M. N'Guérékata, Stepanov-like almost automorphic functions and applications to some semilinear equations, Appl. Anal. 86 (2007) 723–733.
[12] T. Diagana, E. Hernández, M. Rabello, Pseudo almost periodic solutions to some non-autonomous neutral functional differential equations with unbounded delay, Math. Comput. Modelling 45 (2007) 1241–1252.
[13] T. Diagana, H. Henríquez, E. Hernández, Almost automorphic mild solutions of some partial neutral functional differential equations and applications, Nonlinear Anal. 69 (2008) 1485–1493.
[14] T. Diagana, Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces, Springer, New York, 2013.
[15] H. S. Ding, J. Liang, G. M. N'Guérékata, T. J. Xiao, Existence of positive almost automorphic solutions to neutral nonlinear integral equations, Nonlinear Anal. 69 (2008) 1188–1199.
[16] H. S. Ding, W. Long, G. M. N'Guérékata, Almost automorphic solutions of nonautonomous evolution equations, Nonlinear Anal. 70 (2009) 4158–4164.
[17] H. S. Ding, J. Liang, T. J. Xiao, Almost automorphic solutions to nonautonomous semilinear evolution equations in Banach spaces, Nonlinear Anal. 73 (2010) 1426–1438.
[18] H. S. Ding, T. J. Xiao, J. Liang, Existence of positive almost automorphic solutions to nonlinear delay integral equations, Nonlinear Anal. 70 (2009) 2216–2231.
[19] H. S. Ding, J. Liang, T. J. Xiao, Weighted pseudo almost automorphic functions and WPAA solutions to semilinear evolution equations, J. Math. Anal. Appl. 409 (2014) 409–427.
[20] K. Ezzinbi, S. Fatajou, G. M. N'Guérékata, Pseudo almost automorphic solutions to some neutral partial functional differential equations in Banach space, Nonlinear Anal. 70 (2009) 1641–1647.
[21] H. Henríquez, C. Cuevas, Almost automorphy for abstract neutral differential equation via control theory, Annali di Matematica 192 (2013) 393–405.
[22] J. Liang, J. Zhang, T. J.Xiao, Composition of pseudo almost automorphic and asymptotically almost automorphic functions, J. Math. Anal. Appl. 340 (2008) 1493–1499.
[23] I. Mishra, D. Bahuguna, Weighted pseudo almost automorphic solution of an integro-differential equation with weighted Stepanov-like pseudo almost automorphic forcing term, Appl. Math. Comput. 219 (2013) 5345–5355.
[24] G. M. Mophou, Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations, Appl. Math. Comp. 217 (2011) 7579–7587.
[25] G. M. N'Guérékata, Sue les solutions presqu'automorphes d'équations différentielles abstraites, Annales des Sciences Mathématqiues du Québec 1 (1981) 69–79.
[26] G. M. N'Guérékata, Almost Automorphic and Almost Periodic Functions in Abstract Spaces, Kluwer Academic, New York, 2001.
[27] G. M. N'Guérékata, Topics in Almost Automorphy, Springer, New York, 2005.
[28] G. M. N'Guérékata, A. Pankov, Stepanov-like almost automorphic functions and monotone evolution, Nonlinear Anal. 68 (2008) 2658–2667.
[29] R. Sakthivel, P. Revathi, S. Marshal Anthoni, Existence of pseudo almost automorphic mild solutions to stochastic fractional differential equations, Nonlinear Anal. 75 (2012) 3339–3347.
[30] Z. N. Xia, M. Fan, Weighted Stepanov-like pseudo almost automorphy and applications, Nonlinear Anal. 75 (2012) 2378–2397.
[31] Z. N. Xia, Weighted pseudo almost automorphic solutions of hyperbolic semilinear integrodifferential equations, Nonlinear Anal. 95 (2014) 50–65.
[32] T. J. Xiao, J. Liang, J. Zhang, Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces, Semigroup Forum 76 (2008) 518–524.
[33] T. J. Xiao, X. X. Zhu, J. Liang, Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications, Nonlinear Anal. 70 (2009) 4079–4085.
[34] R. Zhang, Y. K. Chang, G. M. N'Guérékata, New composition theorems of Stepanov-like almost automorphic functions and applications to nonautonomous evolution equations, Nonlinear Anal. RWA 13 (2012) 2866–2879.
[35] R. Zhang, Y. K. Chang, G. M. N'Guérékata, Weighted pseudo almost automorphic solutions for non-autonomous neutral functional differential equations with infinite delay (in Chinese), Sci. Sin. Math. 43 (2013) 273–292, doi:10.1360/012013-9
[36] R. Zhang, Y. K. Chang, G. M. N'Guérékata, Existence of weighted pseudo almost automorphic mild solutions to semilinear integral equations with S^p-weighted pseudo almost automorphic coefficients, Discrete Contin. Dyn. Syst.-A 33 (2013) 5525–5537.