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Persistence and Extinction of a Stochastic Cooperative Model in a Polluted Environment with Pulse Toxicant Input

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Abstract. A cooperative model in a polluted environment with stochastic perturbations and impulsive toxicant input is proposed and studied. For each population, sufficient conditions for extinction, strong persistence in the mean and stochastic permanence are established. The threshold between strong persistence in the mean and extinction is obtained. Some simulation figures are worked out to illustrate the main results.

1. Introduction

In the world today, with the rapid development of industry and agriculture, lots of toxicants and contaminants enter into ecosystems. Organisms are often exposed to polluted environments and are affected by toxicants. This motivates scholars to investigate the effects of toxins on the species and to assess the risks taken by the population. Therefore, it is important to find a theoretical threshold value which determines extinction and persistence of a species or community.

Since Hallam and his coworkers [6–8] proposed toxicant-population systems in 1980s, a lot of deterministic mathematical models of single or multiple populations in polluted environments have been proposed, see e.g. [9]-[10]. It is important to point out that all the above papers have assumed that the exogenous input of toxicant is continuous. However, in many cases, toxicants are emitted in regular pulses. One example is the use of pesticides, another example is the pollution by heavy metals (see e.g. [4, 12]). Thus several population models in a polluted environment with pulse toxicant input have been proposed and studied, see e.g. [14]-[24]. Particularly, Liu, Chen and Zhang [14] proposed a single-species population model with impulsive toxicant input and obtained the survival threshold. Then Liu et al. [15] and Liu and Zhang [23] investigated a two-species Lotka-Volterra competition model with impulsive toxicant input. The authors obtained the persistence-extinction threshold. At the same time, Yang, Jin and Xue [35] studied a two-species Lotka-Volterra predator-prey system with impulsive toxicant input and obtained the persistence-extinction threshold.

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However, it is an usual phenomena in nature that one species enhances the growth rate of the other. As we know, one famous model for this type is the Lotka-Volterra cooperation system. Moreover, population models are inevitably affected by the stochastic noises, and in many cases, the noises should not be neglected, for example, when the population size is small or when the mean and variance of perturbations are large (May [28]). Thus many stochastic population models have been proposed (see e.g. [1]-[22]). For example, Liu and Wang [19] considered stochastic *single-species* population models in a polluted environment with impulsive toxicant input; Liu [22] studied stochastic *predator-prey* system with impulsive toxicant input. However, to the best of our knowledge, no results related to cooperation system (even in deterministic case) in polluted environments with impulsive toxicant input have been reported.

Motivated by these, in Section 2, we propose a stochastic cooperation system in polluted environments with impulsive toxicant input. Then in Section 3, we carry out the survival analysis for this model. Sufficient conditions for extinction, strong persistence in the mean and stochastic permanence are established. The threshold between strong persistence in the mean and extinction is obtained. In Section 4, we introduce some figures to support the results. We close the paper with conclusions in Section 5.

2. Model formulation

To begin with, we formulate the following deterministic system in polluted environments with impulsive toxicant input which is motivated by the systems in [14, 15, 23, 35]

$$\begin{cases} \left.\begin{aligned} &\dot{x}_1(t) = x_1(t)[r_{10} - r_{11}C_0(t) - a_{11}x_1(t) + a_{12}x_2(t)] \\ &\dot{x}_2(t) = x_2(t)[r_{20} - r_{21}C_0(t) + a_{21}x_1(t) - a_{22}x_2(t)] \\ &\dot{C}_0(t) = kC_e(t) - (g+m)C_0(t), \\ &\dot{C}_e(t) = -hC_e(t). \end{aligned}\right\} , t \neq n\tau,\ n \in Z^+. \\ \Delta x_i(t) = 0,\ \Delta C_0(t) = 0,\ \Delta C_e(t) = b,\ t = n\tau,\ n \in Z^+,\ i = 1, 2. \end{cases} \tag{1}$$

where all the parameters are positive constants, $\Delta x_i(t) = x_i(t^+) - x_i(t)$, $\Delta C_0(t) = C_0(t^+) - C_0(t)$, $\Delta C_e(t) = C_e(t^+) - C_e(t)$, $Z^+ = \{1, 2, ...\}$; $x_i(t)$ is the size of the ith population; r_{i0} stands for the growth rate of the ith population without toxicant; r_{i1} denotes the ith population response to the pollutant present in the organism; a_{ij} represents the action of species j upon the growth rate of species i (particularly, a_{ii} stands for the intraspecific competition coefficient of species i); $C_0(t)$ is the concentration of toxicant in the organism; $C_e(t)$ is the concentration of toxicant in the environment; $kC_e(t)$ stands for the organism's net uptake of toxicant from the environment; $gC_0(t)$ and $mC_0(t)$ represent the egestion and depuration rates of the toxicant in the organism, respectively; $hC_e(t)$ is the toxicant loss from the environment itself by volatilization and so on; τ is the period of the impulsive effect about the exogenous input of toxicant and b is the toxicant input amount at every time. In system (1), we have assumed that the capacity of the environment is so large that the change of toxicant in the environment that comes from the uptake and egestion by the organisms can be neglected ([8, 14, 15, 23, 35]), moreover, we have assumed that the individuals in the two species have the identical organismal toxicant concentration at time t ([13, 15, 23]).

Let us now take a further step by considering the stochastic fluctuations. Suppose that the population lives in an environment subjected to stochastic fluctuations which mainly affect the growth rate r_{i0} (see e.g. [1]-[22]). Thus r_{i0} can be written as an average rate plus an error term. Generally, by the famous central limit theorem, the error term follows a normal distribution; thus the error term can be approximated by a white noise $\alpha_i\dot{B}_i(t)$, where α_i^2 denotes the intensity of the noise, and $\dot{B}_i(t)$ is a Gaussian white noise process (i.e., $\{B_i(t),\ t \geq 0\}$ is a Brownian motion, $i = 1, 2$). Then

$$r_{i0} \to r_{i0} + \alpha_i\dot{B}_i(t).$$

Consequently we obtain the following stochastic system:

$$\begin{cases} \left.\begin{aligned} dx_1(t) &= x_1(t)[r_{10} - r_{11}C_0(t) - a_{11}x_1(t) + a_{12}x_2(t)]dt + \alpha_1 x_1(t)dB_1(t) \\ dx_2(t) &= x_2(t)[r_{20} - r_{21}C_0(t) + a_{21}x_1(t) - a_{22}x_2(t)]dt + \alpha_2 x_2(t)dB_2(t) \\ \frac{dC_0(t)}{dt} &= kC_e(t) - (g+m)C_0(t), \\ \frac{dC_e(t)}{dt} &= -hC_e(t). \end{aligned}\right\} , t \neq n\tau,\ n \in Z^+. \\ \Delta x_i(t) = 0,\ \Delta C_0(t) = 0,\ \Delta C_e(t) = b,\ t = n\tau,\ n \in Z^+,\ i = 1,2. \end{cases} \tag{2}$$

In order to establish our main result, we recall some classical concepts.

Definition 2.1. *(i) $x(t)$ is said to be extinctive if $\lim_{t\to+\infty} x(t) = 0$;*

(ii) $x(t)$ is said to be weakly persistent in the mean ([13]) if $\langle x\rangle^* > 0$, where $f^* = \limsup_{t\to+\infty} f(t)$, $\langle x\rangle = t^{-1}\int_0^t x(s)ds$;

(iii) $x(t)$ is said to be strongly persistent in the mean ([27]) if $\langle x\rangle_* > 0$, where $f_* = \liminf_{t\to+\infty} f(t)$;

(iv) Model (2) is said to be stochastically permanent if for any $\varepsilon \in (0,1)$, there exist positive constants $\beta = \beta(\varepsilon)$ and $\chi = \chi(\varepsilon)$ such that

$$\liminf_{t\to+\infty} \mathcal{P}\{x_1(t) \geq \beta\} \geq 1-\varepsilon, \quad \liminf_{t\to+\infty} \mathcal{P}\{x_2(t) \geq \beta\} \geq 1-\varepsilon; \tag{3}$$

$$\liminf_{t\to+\infty} \mathcal{P}\{x_1(t) \leq \chi\} \geq 1-\varepsilon, \quad \liminf_{t\to+\infty} \mathcal{P}\{x_2(t) \leq \chi\} \geq 1-\varepsilon. \tag{4}$$

3. Persistence and extinction

Throughout this paper, we suppose that $\{(B_1(t), B_2(t)),\ t \geq 0\}$ is a two-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Define:

$$R^2_+ = \{a | a_i > 0, a \in R^2,\ i = 1,2\};\ A = a_{11}a_{22} - a_{12}a_{21};\ b_i = r_{i0} - \alpha_i^2/2,\ i = 1,2;$$

$$B = r_{11}b_2 - r_{21}b_1;\ C_1 = a_{22}b_1 + a_{12}b_2;\ C_2 = a_{11}b_2 + a_{21}b_1;$$

$$D_1 = a_{22}r_{11} + a_{12}r_{21};\ D_2 = a_{11}r_{21} + a_{21}r_{11}.$$

To begin with, let us consider the following subsystem of (2):

$$\begin{cases} \left.\begin{aligned} \frac{dC_0(t)}{dt} &= kC_e(t) - (g+m)C_0(t) \\ \frac{dC_e(t)}{dt} &= -hC_e(t) \end{aligned}\right\} , t \neq n\tau,\ n \in Z^+. \\ \Delta C_0(t) = 0,\ \Delta C_e(t) = b,\ t = n\tau,\ n \in Z^+. \\ 0 \leq C_0(0) \leq 1,\ 0 \leq C_e(0) \leq 1. \end{cases} \tag{5}$$

Lemma 3.1. *([14]) System (5) has a unique positive τ-periodic solution $(\tilde{C}_0(t), \tilde{C}_e(t))^T$ and for every solution $(C_0(t), C_e(t))^T$ of (5), $C_0(t) \to \tilde{C}_0(t)$ and $C_e(t) \to \tilde{C}_e(t)$ as $t \to \infty$. Moreover, $C_0(t) > \tilde{C}_0(t)$ and $C_e(t) > \tilde{C}_e(t)$*

for all $t \geq 0$ *if* $C_0(0) > \tilde{C}_0(0)$ *and* $C_e(0) > \tilde{C}_e(0)$, *where*

$$\begin{cases} \tilde{C}_0(t) = \tilde{C}_0(0)e^{-(g+m)(t-n\tau)} + \dfrac{kb(e^{-(g+m)(t-n\tau)} - e^{-h(t-n\tau)})}{(h-g-m)(1-e^{-h\tau})}, \\ \tilde{C}_e(t) = \dfrac{be^{-h(t-n\tau)}}{1-e^{-h\tau}}, \\ \tilde{C}_0(0) = \dfrac{kb(e^{-(g+m)\tau} - e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})}, \\ \tilde{C}_e(0) = \dfrac{b}{1-e^{-h\tau}} \end{cases}$$

for $t \in (n\tau, (n+1)\tau]$ *and* $n \in Z^+$. *In addition,*

$$\lim_{t\to+\infty} t^{-1}\int_0^t \tilde{C}_0(s)ds = \frac{kb}{h(g+m)\tau} =: \delta. \tag{6}$$

Note that both $C_0(t)$ and $C_e(t)$ in (5) stand for concentrations, so we must have $0 \leq C_0(t) < 1$, $0 \leq C_e(t) < 1$ for all $t \geq 0$ to be realistic. In fact,

Lemma 3.2. *([14]) For model (5), if* $k \leq g+m$, $b \leq 1-e^{-h\tau}$, *then* $0 \leq C_0(t) \leq 1$ *and* $0 \leq C_e(t) \leq 1$ *for all* $t \geq 0$.

Consequently, from now on we always suppose $k \leq g+m$, $b \leq 1-e^{-h\tau}$.

Lemma 3.3. *([17]) Let* $x(t) \in C[\Omega \times [0,+\infty), R_+]$.
(I) If there are three constants $\lambda_0 > 0$, $T > 0$ *and* λ *such that*

$$\ln x(t) \leq \lambda t - \lambda_0 \int_0^t x(s)ds + \sum_{i=1}^{2} \beta_i B_i(t)$$

for $t \geq T$, *where* β_1 *and* β_2 *are constants, then: if* $\lambda \geq 0$, *then* $\langle x\rangle^* \leq \lambda/\lambda_0$ *almost surely (a.s.); if* $\lambda < 0$, *then* $\lim_{t\to+\infty} x(t) = 0$ *a.s.*
(II) If there are three positive constants λ_0, T *and* λ *such that*

$$\ln x(t) \geq \lambda t - \lambda_0 \int_0^t x(s)ds + \sum_{i=2}^{n} \beta_i B_i(t)$$

for $t \geq T$, *then* $\langle x\rangle_* \geq \lambda/\lambda_0$ *a.s.*

Now let us establish some conditions under which model (2) has a unique global positive solution.

Lemma 3.4. *Consider the first two equations of system (2), if* $A > 0$, *then for any given initial value* $x(0) = (x_1(0), x_2(0)) \in R^2_+$, *the two equations have a unique solution* $x(t) = (x_1(t), x_2(t))$ *on* $t \geq 0$ *and the solution will remain in* R^2_+ *with probability one (w.p.o.). Moreover,*

$$\{t^{-1}\ln x_1(t)\}^* \leq 0,\ \{t^{-1}\ln x_2(t)\}^* \leq 0,\ a.s. \tag{7}$$

From now on, we always suppose that $A > 0$. Now we are in the position to establish the threshold theorem.

Theorem 3.5. *Let*

$$\kappa_1 = \begin{cases} b_1/r_{11}, & B \leq 0; \\ C_1/D_1, & B \geq 0 \end{cases} ; \quad \kappa_2 = \begin{cases} C_2/D_2, & B \leq 0; \\ b_2/r_{21}, & B \geq 0. \end{cases}$$

(I) If $B \leq 0$ *(clearly,* $\kappa_1 \geq \kappa_2$ *in this case), then*

(i) If $\delta < \kappa_2$, then both x_1 and x_2 are strongly persistent in the mean w.p.o. and moreover,

$$\lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{C_1 - D_1\delta}{A},\quad \lim_{t\to+\infty}\langle x_2(t)\rangle = \frac{C_2 - D_2\delta}{A},\ w.p.o. \tag{8}$$

(ii) If $\kappa_2 < \delta < \kappa_1$, then x_1 is strongly persistent in the mean w.p.o. and

$$\lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{b_1 - r_{11}\delta}{a_{11}},\ w.p.o. \tag{9}$$

At the same time, x_2 is extinctive w.p.o.

(iii) If $\kappa_1 < \delta$, then both x_1 and x_2 are extinctive w.p.o.

(II) If $B > 0$ (clearly, $\kappa_1 < \kappa_2$ in this case), then

(iv) If $\delta < \kappa_1$, then both x_1 and x_2 are strongly persistent in the mean w.p.o. and

$$\lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{C_1 - D_1\delta}{A};\ \lim_{t\to+\infty}\langle x_2(t)\rangle = \frac{C_2 - D_2\delta}{A},\ w.p.o.$$

(v) If $\kappa_1 < \delta < \kappa_2$, then x_1 is extinctive w.p.o. and x_2 is strongly persistent in the mean w.p.o. and

$$\lim_{t\to+\infty}\langle x_2(t)\rangle = \frac{b_2 - r_{22}\delta}{a_{22}},\ w.p.o.$$

(vi) If $\kappa_2 < \delta$, then both x_1 and x_2 are extinctive w.p.o.

Remark 3.6. *Theorem 3.5 reveals some interesting and important biological results. Theorem 3.5 obtains the threshold between extinction and strongly persistence in the mean for each population:*

(a) Suppose that $B < 0$. From result (I) we can see that if $\delta > \kappa_1$, then both x_1 and x_2 are extinctive; If $\kappa_2 < \delta < \kappa_1$, then x_1 is strongly persistent in the mean and x_2 is extinctive; If $\delta < \kappa_2$, then both x_1 and x_2 are strongly persistent in the mean. That is to say, the persistence ability of x_1 is stronger than that of x_2. From the viewpoint of biology, this is reasonable. Note that $B < 0$, i.e. $r_{21}(r_{10} - \alpha_1^2/2) > r_{11}(r_{20} - \alpha_2^2/2)$, in other words the population x_1 has smaller environmental noise (i.e., α_1^2) and smaller dose-response parameter (i.e., r_{11}), then x_1 is more possible to be persistent.

(b) Suppose that $B = 0$, then $\kappa_1 = \kappa_2$. If $\delta > \kappa_1$, then both x_1 and x_2 are extinctive; If $\delta < \kappa_1$, then both x_1 and x_2 are strongly persistent in the mean. In other words, the persistence abilities of x_1 and x_2 are equal in this case.

(c) Suppose that $B > 0$. If $\delta > \kappa_2$, then both x_1 and x_2 are extinctive; If $\kappa_1 < \delta < \kappa_2$, then x_1 is extinctive and x_2 is strongly persistent in the mean; if $\delta < \kappa_1$, then both x_1 and x_2 are strongly persistent in the mean. That is to say the persistence ability of x_2 is stronger than that of x_1. The biological reason is similar to (a).

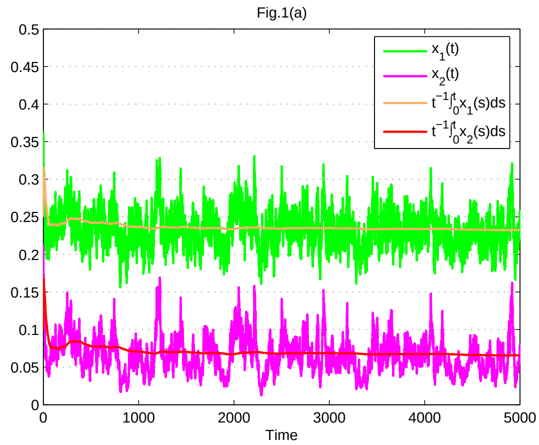
In the study of population system, it is well-known that permanence is one of the most desired properties. Now we shall show that if the white noises are sufficiently small, then system (2) is permanent.

Theorem 3.7. *If $b_i > r_{i1}\limsup_{t\to+\infty}\tilde{C}_0(t)$, $i = 1, 2$, then model (2) is stochastically permanent.*

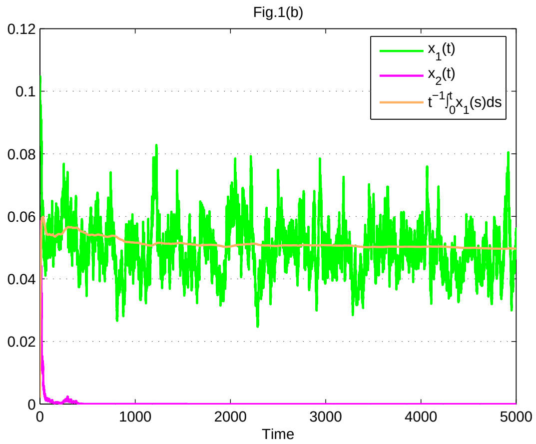
4. Numerical Simulations

Now let us use the famous Milstein method (see e.g. [11]) to support the analytical results. Here, we only give the case $B < 0$. When $B \geq 0$, the simulations can be obtained similarly.

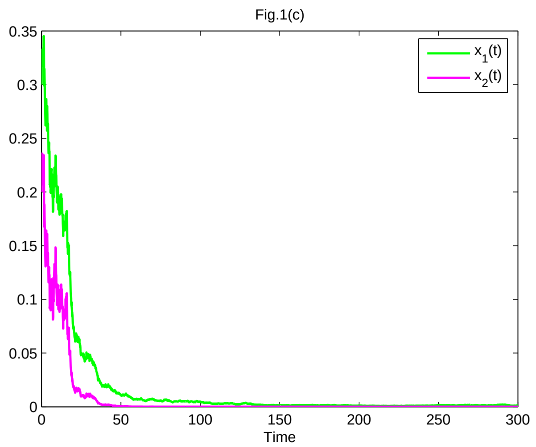
In Fig.1, we choose $r_{10} = 0.55$, $r_{20} = 0.45$, $r_{11} = r_{21} = 1$, $a_{11} = a_{22} = 1$, $a_{21} = a_{12} = 0.6$, $\alpha_2^2 = 0.8$, $k = g = m = 0.1, h = 0.5, b = 0.6, \tau = 6$. Then Lemma 3.2 holds and $A = a_{11}a_{22} - a_{12}a_{21} = 0.75 > 0$. At the same time, it follows from (6) that $\delta = 0.1$. The only difference between conditions of Fig.1(a), Fig.1(b) and Fig.1(c) is that the value of α_1^2 is different. In Fig.1(a), we choose $\alpha_1^2 = 0.5$. Clearly, $B = r_{11}b_2 - r_{21}b_1 = -0.25 < 0$,



(a)



(b)



(c)

Figure 1: Solutions of system (2) for $r_{10} = 0.55$, $r_{20} = 0.45$, $r_{11} = r_{21} = 1$, $a_{11} = a_{22} = 1$, $a_{21} = a_{12} = 0.5$, $\alpha_2^2 = 0.8$, $k = g = m = 0.1$, $h = 0.5$, $b = 0.6$, $\tau = 6$, $x_1(0) = 0.3$, $x_2(0) = 0.2$, $C_0(0) = C_e(0) = 0.1$, step size $\Delta t = 0.001$. (a) is with $\alpha_1^2 = 0.5$; (b) is with $\alpha_1^2 = 0.8$; (c) is with $\alpha_1^2 = 0.94$.

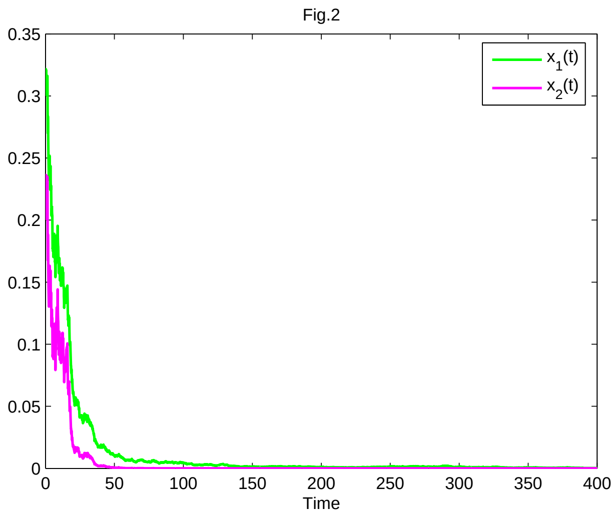


Figure 2: Solutions of system (2) for $r_{10} = 0.55$, $r_{20} = 0.45$, $r_{11} = r_{21} = 1$, $a_{11} = a_{22} = 1$, $a_{21} = a_{12} = 0.5$, $\alpha_1^2 = 0.5$, $\alpha_2^2 = 0.8$, $k = g = m = 0.1$, $h = 0.5$, $b = 0.6$, $\tau = 1.9$, $x_1(0) = 0.3$, $x_2(0) = 0.2$, $C_0(0) = C_e(0) = 0.1$, step size $\Delta t = 0.001$.

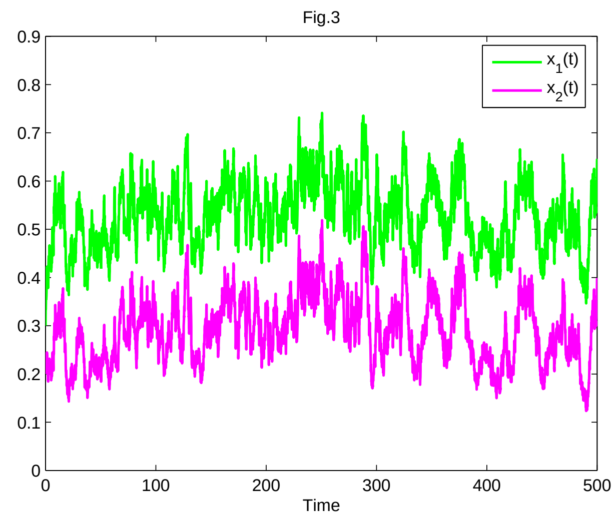


Figure 3: Solutions of system (2) for $r_{10} = 0.55$, $r_{20} = 0.45$, $\alpha_1^2 = 0.14$, $\alpha_2^2 = 0.64$, $r_{11} = r_{21} = 1$, $a_{11} = a_{22} = 1$, $a_{21} = a_{12} = 0.5$, $k = g = m = 0.1$, $h = 0.5$, $\tau = 6$, $b = 0.6$, $x_1(0) = 0.3$, $x_2(0) = 0.2$, $C_0(0) = C_e(0) = 0.1$, step size $\Delta t = 0.001$.

$\kappa_1 = b_1/r_{11} = 0.3$ and $\kappa_2 = C_2/D_2 = 0.133 > \delta$. By (i) in Theorem 3.5, we can obtain that both x_1 and x_2 are strongly persistent in the mean and

$$\lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{C_1 - D_1\delta}{A} = 0.233, \quad \lim_{t\to+\infty}\langle x_2(t)\rangle = \frac{C_2 - D_2\delta}{A} = 0.0667.$$

Fig.1(a) confirms these. In Fig.1(b), we choose $\alpha_1^2 = 0.8$. Note that $B = -0.1$, $\kappa_1 = 0.15$ and $\kappa_2 = 0.083$, then $\kappa_2 < \delta < \kappa_1$. In view of (ii) in Theorem 3.5, one can see that x_2 is extinctive and x_1 is strongly persistent in the mean and

$$\lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{b_1 - r_{11}\delta}{a_{11}} = 0.05.$$

See Fig.1(b). In Fig.1(c), we choose $\alpha_1^2 = 0.94$. Then $B = -0.03$ and $\kappa_1 = 0.08 < \delta$. Making use of (iii) in Theorem 3.5 gives that both x_1 and x_2 are extinctive. Fig.1(c) confirms these. By comparing Fig.1(a) with Fig.1(c), it is easy to obtain that with increasing α_1^2 value, x_1 is inclined to extinction. That is to say the stochastic noise of x_1 is unfavorable for the persistence of x_1. At the same time, by comparing Fig.1(a) with Fig.1(b), one can observe that with increasing α_1^2 value, x_2 is inclined to extinction. In other words the stochastic noise of x_1 is also unfavorable for the persistence of x_2.

In Fig.2, we choose $r_{10} = 0.55$, $r_{20} = 0.45$, $r_{11} = r_{21} = 1$, $a_{11} = a_{22} = 1$, $a_{21} = a_{12} = 0.5$, $\alpha_1^2 = 0.5$, $\alpha_2^2 = 0.8$, $k = g = m = 0.1$, $h = 0.5$, $b = 0.6$. The only difference between conditions of Fig.1(a) and Fig.2 is that the value of τ is different. In Fig.2, we choose $\tau = 1.9$. Then $\kappa_1 = 0.3 < \delta = 0.3158$. An application of (iii) in Theorem 3.5 leads to that both x_1 and x_2 are extinctive. Fig.2 confirms these. By comparing Fig.1(a) with Fig.2, one can see that the impulsive period τ plays a key role in determining the persistence and the extinction of x_1 and x_2.

In Fig.3, we choose $r_{10} = 0.55$, $r_{20} = 0.45$, $\alpha_1^2 = 0.14$, $\alpha_2^2 = 0.64$, $r_{11} = r_{21} = 1$, $a_{11} = a_{22} = 1$, $a_{21} = a_{12} = 0.5$, $k = g = m = 0.1$, $h = 0.5$, $\tau = 6$, $b = 0.6$. Then it follows from Theorem 3.7 that the model (2) is stochastically permanent. Fig.3 confirms this.

5. Conclusions and future directions

This paper has been devoted to a stochastic cooperative system in polluted environments with impulsive toxicant input. For each species, the threshold between strongly persistence in the mean and extinction has been established. Moreover, sufficient conditions for stochastic permanence have been obtained. These results have revealed that both the random perturbations and the impulsive period play key roles in determining the persistence and the extinction of the species.

Our results and numerical simulations reveal an important property of environmental noise: the stochastic noise of x_i is unfavorable for the persistence of both x_1 and x_2. From the viewpoint of biology, this is reasonable. Note that model (2) is a cooperative system, in which each member enhances the growth of others. Since the stochastic noise of x_i is unfavorable for the persistence of x_i, then x_j will obtain less supports. That is to say, stochastic noise of x_i is unfavorable for the persistence of x_j, $j \neq i$, $i, j = 1, 2$. Our results and numerical simulations also reveal that impulsive period τ play key roles in determining the persistence and the extinction of the species. Thus in order to conserve x_1 and x_2, we have the following approaches:

- To reduce the intensity of the white noises α_1^2 and α_2^2;
- To increase the impulsive period τ;
- To cut down the toxicant input amount at each time b.

Some interesting questions deserve further investigations. It is interesting to study n-species model. It is useful to point out that part methods developed in this paper are also applicable to n-species system. It is also interesting to consider others parameters, e.g., a_{ij}, are disturbed by stochastic noises.

Appendix

Proof. **of Lemma 3.4**: From Theorems 2.1 and 3.1 in Pang et al. [31], we only need to prove that there exist positive numbers p_1 and p_2 such that

$$\lambda^+_{max}(P\bar{A} + \bar{A}^T P) < 0,$$

where

$$\bar{A} = \begin{pmatrix} -a_{11} & a_{12} \\ a_{21} & -a_{22} \end{pmatrix}, \ P = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$$

and $\lambda^+_{max}(Q) = \sup_{x\in R^n_+,\ |x|=1} x^T Qx$ for a symmetric matrix Q. In fact, since $a_{11} > 0$, $a_{12} > 0$ and $a_{21} > 0$, then we can choose positive numbers p_1 and p_2 such that $\lambda_{max}(P\bar{A} + \bar{A}^T P) < 0$, where $\lambda_{max}(P\bar{A} + \bar{A}^T P)$ is the largest eigenvalue of $P\bar{A} + \bar{A}^T P$. On the other hand, for any symmetric matrix Q, it follows from the definition of λ^+_{max} that $\lambda^+_{max}(Q) \le \lambda_{max}(Q)$. Then the desired assertion follows. □

Proof. **of Theorem 3.5**: We only present the proof for (I), the proof of (II) is analogous. Note that $\kappa_1 = b_1/r_{11} \ge \kappa_2 = C_2/D_2$.

From Lemma 3.1, for $\forall\ \varepsilon > 0$, there exists a constant $T > 0$ such that

$$\tilde{C}_0(t) - \varepsilon \le C_0(t) \le \tilde{C}_0(t) + \varepsilon, \quad t > T. \tag{10}$$

Applying Itô's formula to Eq. (2) leads to

$$\frac{\ln(x_1(t)/x_1(0))}{t} = b_1 - r_{11}\langle C_0(t)\rangle - a_{11}\langle x_1(t)\rangle + a_{12}\langle x_2(t)\rangle + \frac{\alpha_1 B_1(t)}{t}; \tag{11}$$

$$\frac{\ln(x_2(t)/x_2(0))}{t} = b_2 - r_{21}\langle C_0(t)\rangle + a_{21}\langle x_1(t)\rangle - a_{22}\langle x_2(t)\rangle + \frac{\alpha_2 B_2(t)}{t}. \tag{12}$$

From (12)×a_{12}+(11)×a_{22}, we obtain

$$\begin{aligned} &a_{22}\frac{\ln(x_1(t)/x_1(0))}{t} + a_{12}\frac{\ln(x_2(t)/x_2(0))}{t} \\ &= C_1 - D_1\langle C_0(t)\rangle - A\langle x_1(t)\rangle \\ &+\frac{a_{22}\alpha_1 B_1(t) + a_{12}\alpha_2 B_2(t)}{t}, \end{aligned} \tag{13}$$

Similarly, from (12)×a_{11}+(11)×a_{21}, we have

$$\begin{aligned} &a_{11}\frac{\ln(x_2(t)/x_2(0))}{t} + a_{21}\frac{\ln(x_1(t)/x_1(0))}{t} \\ &= C_2 - D_2\langle C_0(t)\rangle - A\langle x_2(t)\rangle \\ &+\frac{a_{11}\alpha_2 B_2(t) + a_{21}\alpha_1 B_1(t)}{t}, \end{aligned} \tag{14}$$

Moreover, by the property of limit superior, it follows from (6), (10), (11) and (12) that

$$\frac{\ln(x_1(t)/x_1(0))}{t} \le b_1 + \varepsilon_1 - r_{11}\delta - a_{11}\langle x_1(t)\rangle + a_{12}\langle x_2(t)\rangle^* + \frac{\alpha_1 B_1(t)}{t};$$

$$\frac{\ln(x_2(t)/x_2(0))}{t} \le b_2 + \varepsilon_2 - r_{21}\delta + a_{21}\langle x_1(t)\rangle^* - a_{22}\langle x_2(t)\rangle + \frac{\alpha_2 B_2(t)}{t}.$$

Let

$$\lambda_1 = b_1 + \varepsilon_1 - r_{11}\delta + a_{12}\langle x_2\rangle^*;\ \lambda_2 = b_2 + \varepsilon_2 - r_{21}\delta + a_{21}\langle x_1\rangle^*.$$

Thus

$$\frac{\ln(x_1(t)/x_1(0))}{t} \le \lambda_1 - a_{11}\langle x_1(t)\rangle + \frac{\alpha_1 B_1(t)}{t}; \tag{15}$$

$$\frac{\ln(x_2(t)/x_2(0))}{t} \le \lambda_2 - a_{22}\langle x_2(t)\rangle + \frac{\alpha_2 B_2(t)}{t}. \tag{16}$$

(i) It follows from (7) that for arbitrarily given and sufficiently small $\varepsilon > 0$, there exists $T > 0$ such that for all $t \ge T$

$$D_1\langle C_0(t)\rangle \le D_1\delta + \varepsilon/2;\ a_{12}\frac{\ln(x_2(t)/x_2(0))}{t} \le a_{12}[\frac{\ln x_2(t)}{t}]^* + \varepsilon/2 \le \varepsilon/2.$$

Substituting the above inequalities into (13) yields

$$a_{22}\frac{\ln(x_1(t)/x_1(0))}{t} \ge C_1 - D_1\delta - \varepsilon - A\langle x_1(t)\rangle + \frac{a_{22}\alpha_1 B_1(t) + a_{12}\alpha_2 B_2(t)}{t}. \tag{17}$$

Since $C_1/D_1 \ge C_2/D_2 > \delta > 0$, then we can let ε be sufficiently small such that $C_1 - D_1\delta - \varepsilon > 0$. Then using (II) in Lemma 3.3 gives

$$\langle x_1\rangle_* \ge (C_1 - D_1\delta - \varepsilon)/A.$$

Then it follows from the arbitrariness of ε that

$$\langle x_1\rangle_* \ge (C_1 - D_1\delta)/A. \tag{18}$$

In other words, we have shown that x_1 is persistent in the mean, that is, $\langle x_1\rangle_* > 0$. Thus $\lambda_1 > 0$ (otherwise, inequalities (15) and Lemma 3.3 would lead to $\langle x_1\rangle^* = 0$). Similarly, making use of (14) yields

$$a_{11}\frac{\ln(x_2(t)/x_2(0))}{t} \ge C_2 - D_2\delta - \varepsilon - A\langle x_2(t)\rangle + \frac{a_{12}\alpha_1 B_1(t) + a_{11}\alpha_2 B_2(t)}{t}.$$

An application of (II) in Lemma 3.3, one can see that

$$\langle x_2\rangle_* \ge (C_2 - D_2\delta)/A > 0. \tag{19}$$

Thus $\lambda_2 > 0$. Then by (I) in Lemma 3.3, it follows from (15) and (16) that

$$\langle x_1\rangle^* \le \lambda_1/a_{11},\ \ \langle x_2\rangle^* \le \lambda_2/a_{22}.$$

That is to say

$$a_{11}\langle x_1\rangle^* - a_{12}\langle x_2\rangle^* \le b_1 - r_{11}\delta; \tag{20}$$

$$-a_{21}\langle x_1\rangle^* + a_{22}\langle x_2\rangle^* \le b_2 - r_{21}\delta.$$

Solving these two inequalities, we obtain

$$\langle x_1\rangle^* \le (C_1 - D_1\delta)/A,\ \langle x_2\rangle^* \le (C_2 - D_2\delta)/A.$$

Then the required assertion (8) follows from the above inequalities, (18) and (19).

(ii) Since $C_1/D_1 > \delta > 0$, then (18) holds. That is to say, the population x_1 is persistent in the mean: $\langle x_1\rangle_* > 0$. Thus $\lambda_1 > 0$. In other words, inequality (20) holds. If $\omega \in \{\langle x_2\rangle^* > 0\}$, then an application of Lemma 3.3 to inequality (16) results in

$$\langle x_2(\omega)\rangle^* \le \frac{\lambda_2}{a_{22}} = \frac{b_2 + \varepsilon_2 - r_{21}\delta + a_{21}\langle x_1(\omega)\rangle^*}{a_{22}}$$

Substituting (20) into the above inequality, we can see that

$$\begin{aligned}(a_{11}a_{22} - a_{12}a_{21})\langle x_2(\omega)\rangle^* &\le a_{11}b_2 + a_{21}b_1 - (a_{11}r_{21} + a_{21}r_{11})\delta + \varepsilon \\ &= C_2 - D_2\delta + \varepsilon,\end{aligned}$$

where $\varepsilon = a_{11}\varepsilon_2 + a_{21}\varepsilon_1$. Note that $A = a_{11}a_{22} - a_{12}a_{21} > 0$, then the left side of the above inequality is positive. Since ε is arbitrarily small, then $\delta \le C_2/D_2 = \kappa_2$, which is a contradiction with $\delta > \kappa_2$. Consequently, $\mathcal{P}\{\omega : \langle x_2\rangle^* > 0\} = 0$, that is to say, $\langle x_2\rangle^* = 0$ *a.s.*

Furthermore, substituting inequality (20) into inequality (16), one can derive that

$$\begin{aligned}\frac{\ln(x_2(t)/x_2(0))}{t} &\le b_2 + \varepsilon_2 - r_{21}\delta + \frac{a_{21}}{a_{11}}(b_1 + \varepsilon_1 - r_{11}\delta + a_{12}\langle x_2\rangle^*) \\ &-a_{22}\langle x_2(t)\rangle + \alpha_2 B_2(t)/t \\ &= [C_2 - D_2\delta + \varepsilon(t) + a_{11}\varepsilon_2 + a_{21}\varepsilon_1]/a_{11} + \alpha_2 B_2(t)/t,\end{aligned}$$

where $\varepsilon(t) = a_{12}a_{21}\langle x_2\rangle^* - a_{12}a_{21}\langle x_2(t)\rangle$. Since $\delta > C_2/D_2$, then we have $\langle x_2\rangle^* = 0$, which is to say, $\varepsilon(t) \to 0$. Thus applying Lemma 3.3 again leads to

$$\lim_{t\to+\infty} x_2(t) = 0.$$

In other words, we have shown that if $\delta > C_2/D_2$, then the population x_2 goes to extinction a.s.

Now let us prove (9). Since $\lim_{t\to+\infty} x_2(t) = 0$, then by (11), for sufficiently large t

$$\frac{\ln(x_1(t)/x_1(0))}{t} \le b_1 + \varepsilon - r_{11}\langle C_0(t)\rangle - a_{11}\langle x_1(t)\rangle + \frac{\alpha_1 B_1(t)}{t}; \tag{21}$$

$$\frac{\ln(x_1(t)/x_1(0))}{t} \ge b_1 - \varepsilon - r_{11}\langle C_0(t)\rangle - a_{11}\langle x_1(t)\rangle + \frac{\alpha_1 B_1(t)}{t}. \tag{22}$$

Making use of (21) and (I) in Lemma 3.3 one can see that

$$\langle x_1\rangle^* \le [b_1 + \varepsilon - r_{11}\delta]/a_{11}.$$

Similarly, using (22) and (II) in Lemma 3.3, we get

$$\langle x_1\rangle_* \ge [b_1 - \varepsilon - r_{11}\delta]/a_{11}.$$

Then the desired assertion (9) follows from the arbitrariness of ε.

(iii) To begin with, let us prove $\lim_{t\to+\infty} x_2(t) = 0$ *a.s.*.

Case (a): Suppose that $\langle x_1\rangle^* > 0$. Then $\lambda_1 > 0$. Consequently, similar to the proof of (ii), we can obtain $\lim_{t\to+\infty} x_2(t) = 0$.

Case (b): Suppose that $\langle x_1\rangle^* = 0$. Then it follows from (16) that

$$\frac{\ln(x_2(t)/x_2(0))}{t} \le b_2 + \varepsilon_2 - r_{21}\delta - a_{22}\langle x_2(t)\rangle + \frac{\alpha_2 B_2(t)}{t}$$

for sufficiently large t. Making use of $\delta > C_2/D_2 > b_2/r_{21}$ and Lemma 3.3, we obtain $\lim_{t\to+\infty} x_2(t) = 0$.

Now we are in the position to prove $\lim_{t\to+\infty} x_1(t) = 0$ *a.s.* In fact, since $\lim_{t\to+\infty} x_2(t) = 0$, then it follows from (15) that for sufficiently large t,

$$\frac{\ln(x_1(t)/x_1(0))}{t} \le b_1 + \varepsilon_1 - r_{11}\delta - a_{11}\langle x_1(t)\rangle + \frac{\alpha_1 B_2(t)}{t}.$$

Then the desired assertion follows from $\delta > b_1/r_{11}$ and Lemma 3.3. □

Proof. **of Theorem 3.7**: We shall divide the proof into two parts. To begin with, let us prove (3). Note that $b_i - r_{i1}\tilde{C}_0^* > 0,\ i = 1, 2$, we can choose a constant $\theta > 0$ such that

$$b_i - r_{i1}\tilde{C}_0^* > 0.5\theta\alpha_i^2,\ i = 1, 2$$

where $\tilde{C}_0^* = \limsup_{t\to+\infty} \tilde{C}_0(t)$. Define

$$V_1(x) = (1 + x_1^{-1})^\theta + (1 + x_2^{-1})^\theta.$$

Making use of Itô's formula gives

$$\begin{aligned} dV_1(x) &= \theta(1+x_1^{-1})^{\theta-2}\Big\{-\frac{1}{x_1^2}\Big(b_1 - r_{11}C_0(t) - 0.5\theta\alpha_1^2\Big) \\ &+\frac{1}{x_1}\Big(-r_{10} + r_{11}C_0(t) + a_{11} + \alpha_1^2\Big) + a_{11} - \frac{x_2}{x_1}\Big[a_{12} + \frac{a_{12}}{x_1}\Big]\Big\}dt \\ &+\theta(1+x_2^{-1})^{\theta-2}\Big\{-\frac{1}{x_2^2}\Big(b_2 - r_{21}C_0(t) - 0.5\theta\alpha_2^2\Big) \\ &+\frac{1}{x_2}\Big(-r_{20} + r_{21}C_0(t) + a_{22} + \alpha_2^2\Big) + a_{22} - \frac{x_1}{x_2}\Big[a_{21} + \frac{a_{21}}{x_2}\Big]\Big\}dt \\ &+\theta(1+x_1^{-1})^{\theta-1}\alpha_1 x_1^{-1}dB_1(t) + \theta(1+x_2^{-1})^{\theta-1}\alpha_2 x_2^{-1}dB_2(t) \\ &\le \theta(1+x_1^{-1})^{\theta-2}\Big\{-\frac{1}{x_1^2}\Big(b_1 - r_{11}C_0(t) - 0.5\theta\alpha_1^2\Big) + \frac{1}{x_1}\Big(r_{11} + a_{11} + \alpha_1^2\Big) + a_{11}\Big\}dt \\ &+\theta(1+x_2^{-1})^{\theta-2}\Big\{-\frac{1}{x_2^2}\Big(b_2 - r_{21}C_0(t) - 0.5\theta\alpha_2^2\Big) + \frac{1}{x_2}\Big(r_{21} + a_{22} + \alpha_2^2\Big) + a_{22}\Big\}dt \\ &+\theta(1+x_1^{-1})^{\theta-1}\alpha_1 x_1^{-1}dB_1(t) + \theta(1+x_2^{-1})^{\theta-1}\alpha_2 x_2^{-1}dB_2(t). \end{aligned}$$

Now, let κ be sufficiently small to satisfy

$$0 < \frac{\kappa}{\theta} < b_i - r_{i1}\tilde{C}_0^* - 0.5\theta\alpha_i^2,\ i = 1, 2.$$

Define

$$V_2(x(t)) = e^{\kappa t}V_1(x(t)) = e^{\kappa t}(1+x_1^{-1})^{\theta} + e^{\kappa t}(1+x_2^{-1})^{\theta}.$$

In view of Itô's formula, we obtain that for sufficiently large t,

$$\begin{aligned} dV_2(x(t)) &= \kappa e^{\kappa t}V_1(x)dt + e^{\kappa t}dV_1(x) \\ &\le \theta e^{\kappa t}(1+x_1^{-1})^{\theta-2}\Big\{\kappa(1+x_1^{-1})^2/\theta + \Big[-\frac{1}{x_1^2}\Big(b_1 - r_{11}C_0(t) - 0.5\theta\alpha_1^2\Big) \\ &+\frac{1}{x_1}\Big(r_{11} + a_{11} + \alpha_1^2\Big) + a_{11}\Big]\Big\}dt \\ &+\theta e^{\kappa t}(1+x_2^{-1})^{\theta-2}\Big\{\kappa(1+x_2^{-1})^2/\theta + \Big[-\frac{1}{x_2^2}\Big(b_2 - r_{21}C_0(t) - 0.5\theta\alpha_2^2\Big) \\ &+\frac{1}{x_2}\Big(r_{21} + a_{22} + \alpha_2^2\Big) + a_{22}\Big]\Big\}dt \\ &+\kappa e^{\kappa t}\theta(1+x_1^{-1})^{\theta-1}\alpha_1 x_1^{-1}dB_1(t) + \kappa e^{\kappa t}\theta(1+x_2^{-1})^{\theta-1}\alpha_2 x_2^{-1}dB_2(t) \\ &\le \theta e^{\kappa t}(1+x_1^{-1})^{\theta-2}\Big\{-\frac{1}{x_1^2}\Big(b_1 - r_{11}\tilde{C}_0^* - \varepsilon - 0.5\theta\alpha_1^2 - \kappa/\theta\Big) \\ &+\frac{1}{x_1}\Big(r_{11} + a_{11} + \alpha_1^2 + 2\kappa/\theta\Big) + a_{11} + \kappa/\theta\Big\}dt \\ &+\theta e^{\kappa t}(1+x_2^{-1})^{\theta-2}\Big\{-\frac{1}{x_2^2}\Big(b_2 - r_{21}\tilde{C}_0^* - \varepsilon - 0.5\theta\alpha_2^2 - \kappa/\theta\Big) \\ &+\frac{1}{x_2}\Big(r_{21} + a_{22} + \alpha_2^2 + 2\kappa/\theta\Big) + a_{22} + \kappa/\theta\Big\}dt \\ &+\kappa e^{\kappa t}\theta(1+x_1^{-1})^{\theta-1}\alpha_1 x_1^{-1}dB_1(t) + \kappa e^{\kappa t}\theta(1+x_2^{-1})^{\theta-1}\alpha_2 x_2^{-1}dB_2(t) \\ &=: e^{\kappa t}J(x)dt + \kappa e^{\kappa t}\theta(1+x_1^{-1})^{\theta-1}\alpha_1 x_1^{-1}dB_1(t) \\ &+\kappa e^{\kappa t}\theta(1+x_2^{-1})^{\theta-1}\alpha_2 x_2^{-1}dB_2(t). \end{aligned}$$

It then follows from the definition of κ that $J(x)$ is upper bounded in R_+^2, namely

$$K_1 := \sup_{x\in R_+^2} J(x) < +\infty.$$

Consequently,

$$\begin{aligned} dV_2(x(t)) &\leq K_1 e^{\kappa t}dt - \kappa e^{\kappa t}\theta(1+x_1^{-1})^{\theta-1}\alpha_1 x_1^{-1}dB_1(t) \\ &+\kappa e^{\kappa t}\theta(1+x_2^{-1})^{\theta-1}\alpha_2 x_2^{-1}dB_2(t) \end{aligned}$$

for sufficiently large t. That is to say,

$$\limsup_{t\to+\infty} E[x_1^{-\theta}(t)] \leq \limsup_{t\to+\infty} E\Big[(1+x_1^{-1}(t))^{\theta} + (1+x_2^{-1}(t))^{\theta}\Big] \leq \frac{K_1}{\kappa} = K;$$

$$\limsup_{t\to+\infty} E[x_2^{-\theta}(t)] \leq \limsup_{t\to+\infty} E\Big[(1+x_1^{-1}(t))^{\theta} + (1+x_2^{-1}(t))^{\theta}\Big] \leq K.$$

So for any fixed $\varepsilon > 0$, set $\beta = \varepsilon^{\frac{1}{\theta}}/K^{\frac{1}{\theta}}$. By Chebyshev's inequality (see e.g. Mao [26], P. 5), we can derive that

$$\mathcal{P}\{x_i(t) < \beta\} = \mathcal{P}\{x_i^{-\theta}(t) > \beta^{-\theta}\} \leq E[x_i^{-\theta}(t)]/\beta^{-\theta} = \beta^{\theta}E[x_i^{-\theta}(t)],\ i = 1,2.$$

Hence $\limsup_{t\to+\infty} \mathcal{P}\{x_i(t) < \beta\} \leq \beta^{\theta}K = \varepsilon$. Consequently $\liminf_{t\to+\infty} \mathcal{P}\{x_i(t) \geq \beta\} \geq 1-\varepsilon,\ i = 1,2.$

The proof of (4) is standard and hence is omitted (see e.g., [19]). □

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