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A Note on the Analytic Representations of Convergent Sequences in S′

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Abstract. Using the generalised Cauchy representation for distributions of finite order and appropriate growth, we obtain some results concerning the boundary values of analytic representations of a sequence of distributions.

1. Introduction

The boundary values representation has been studied for a long time and from different points of view. One of the first results is that if $f \in L^1(\mathbb{R})$, then the function

$$\widehat{f}(z) = \frac{1}{2\pi i}\langle f(t), \frac{1}{t-z}\rangle,\ z = x + iy,\ x \notin \operatorname{supp} f, y \in \mathbb{R}$$

is the Cauchy representation of f i.e.

$$\lim_{y\to 0^+} \int_{-\infty}^{+\infty} [\widehat{f}(x+iy) - \widehat{f}(x-iy)]\varphi(x)dx = \int_{-\infty}^{+\infty} f(x)\varphi(x)dx \tag{1}$$

for every $\varphi \in D = D(\mathbb{R})$.

In [5], we gave boundary values characterization of distribution of the form

$$F(x) = \sum_{i=0}^{m} (P_i(x) f_i(x))^{(i)}, \tag{2}$$

where, for every $i \in \{0, 1, \ldots, m\}$, $f_i \in L^1(\mathbb{R})$ and P_i is a real analytic function on the real line $\mathbb{R}$ different from zero on $\mathbb{R}$ of suitable growth rate. There we generalized the equation (1) i.e. we proved that

$$\widehat{F}_P(z) = \frac{P(z)}{2\pi i}\langle F(t), \frac{1}{P(t)(t-z)}\rangle,\ z = x + iy \in \Omega\backslash \operatorname{supp} F \tag{3}$$

is the analytic function in $\Omega\backslash\operatorname{supp}F$ that satisfies

$$\widehat{F}_P(x+iy) - \widehat{F}_P(x-iy) \to F(x) \ \text{ in } \ D'(\mathbb{R}) \ \text{ as } \ y \to 0^+. \tag{4}$$

$(P(z) = 1 + \sum_{i=0}^{m} |P_i(z)|^2,\ z \in \Omega)$

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2. Main Results

Theorem 2.1. *Let $\{f_n(x)\}$ be a sequence of functions in $L^1(\mathbb{R})$ which converges to a function $f(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$, and, for every $n \in \mathbb{N}$, let $\hat{f}_n(z)$ be the analytic representation of $f_n(x)$. Suppose that $\{P_n(x)\}$ is a sequence of analytic functions on the real line that uniformly converges to a function $P(x)$ on each compact subset of the real line as $n \to \infty$. Then, as $n \to \infty$, the sequence $\{\hat{f}_n(z)\}$ uniformly converges on the sets of the form $\mathbb{R} \times [\delta, \infty)$ and $\mathbb{R} \times (-\infty, -\delta]$, $\delta > 0$ in the upper and lower half plane, respectively, the sequence of distributions $\{P_n f_n\}$ converges to the distribution Pf in D' and $P(z)\hat{f}(z)$ is the analytic representation of the distribution Pf.*

Proof. Let $\delta > 0$ be arbitrary chosen. For $z = x + iy \in \mathbb{R} \times [\delta, \infty)$, we have

$$
\begin{aligned}
|\hat{f}_n(z) - \hat{f}(z)| &= \left| \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f_n(t)}{t-z} dt - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{t-z} dt \right| \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|f_n(t) - f(t)|}{|t-z|} dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|f_n(t) - f(t)|}{\sqrt{(t-x)^2 + y^2}} dt \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|f_n(t) - f(t)|}{y} dt \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|f_n(t) - f(t)|}{\delta} dt \\
&= \frac{1}{2\pi\delta} \int_{-\infty}^{+\infty} |f_n(t) - f(t)| dt.
\end{aligned}
$$

Since $\{f_n\}$ converges to f in $L^1(\mathbb{R})$, the above estimates show that $\{\hat{f}_n(z)\}$ uniformly converges to $\hat{f}(z)$ on the set $\mathbb{R} \times [\delta, \infty)$ as $n \to \infty$.

Next, we will show that $\{P_n f_n\}$ converges to Pf in D'. Let $\varphi \in D$ be arbitrary chosen and let $\operatorname{supp}\varphi = [-a, a]$, $a > 0$. Then

$$
\begin{aligned}
&\langle P_n f_n, \varphi \rangle - \langle Pf, \varphi \rangle = \\
&= \int_{-\infty}^{+\infty} P_n(x) f_n(x) \varphi(x) dx - \int_{-\infty}^{+\infty} P(x) f(x) \varphi(x) dx = \\
&= \int_{-\infty}^{+\infty} [P_n(x) - P(x)] f_n(x) \varphi(x) dx + \int_{-a}^{a} [f_n(x) - f(x)] P(x) \varphi(x) dx.
\end{aligned}
$$

The convergence of the sequences $\{f_n\}$ and $\{P_n\}$, and the boundness of $P(x)$ on the sets $[-a, a]$ imply that $\{P_n f_n\}$ converges to Pf in D' as $n \to \infty$.

It remains to prove that the function $P(z)\hat{f}(z)$ is the analytic representation of the distribution Pf.

Let $\varphi \in D$ and $\varepsilon > 0$ be arbitrary chosen.

$$
\begin{aligned}
&\langle Pf, \varphi\rangle - \int_{-\infty}^{\infty} [P(x+iy)\hat{f}(x+iy) - P(x-iy)\hat{f}(x-iy)]\varphi(x)dx = \\
=&\{\langle Pf, \varphi\rangle - \langle P_n f_n, \varphi\rangle\} + \{\langle P_n f_n, \varphi\rangle - \\
-& \int_{-\infty}^{\infty} [P_n(x+iy)\hat{f}_n(x+iy) - P_n(x-iy)\hat{f}_n(x-iy)]\varphi(x)dx\} \\
+&\{\int_{-\infty}^{\infty} [P_n(x+iy)\hat{f}_n(x+iy) - P_n(x-iy)\hat{f}_n(x-iy)]\varphi(x)dx - \\
-& \int_{-\infty}^{\infty} [P(x+iy)\hat{f}(x+iy) - P(x-iy)\hat{f}(x-iy)]\varphi(x)dx\} \\
=&J_{1n} + J_{2n} + J_{3n}.
\end{aligned}
$$

We showed that $\lim_{n\to\infty}\langle P_n f_n, \varphi\rangle = \langle Pf, \varphi\rangle$, so there exists N_1 such that

$$|J_{1n}| < \frac{\varepsilon}{3}, \quad \text{for} \quad n > N_1. \tag{5}$$

By the Proposition 2.2. in [5], we have that there exists y_0, such that

$$|J_{2n}| < \frac{\varepsilon}{3}, \quad \text{for} \quad y < y_0. \tag{6}$$

$$
\begin{aligned}
J_{3n} =& \int_{-\infty}^{\infty} [P_n(x+iy)\hat{f}_n(x+iy) - P_n(x-iy)\hat{f}_n(x-iy)]\varphi(x)dx - \\
&- \int_{-\infty}^{\infty} [P(x+iy)\hat{f}(x+iy) - P(x-iy)\hat{f}(x-iy)]\varphi(x)dx = \\
&= \{\int_{-\infty}^{\infty} [P_n(x+iy) - P(x+iy)]\hat{f}_n(x+iy)\varphi(x)dx\} \\
&+ \{\int_{-\infty}^{\infty} [\hat{f}_n(x+iy) - \hat{f}(x+iy)]P(x+iy)\varphi(x)dx\} \\
&+ \{\int_{-\infty}^{\infty} [\hat{f}(x-iy) - \hat{f}_n(x-iy)]P(x-iy)\varphi(x)dx\} \\
&+ \{\int_{-\infty}^{\infty} [P(x-iy) - P_n(x-iy)]\hat{f}_n(x-iy)\varphi(x)dx\} \\
&= J_{3n}^1 + J_{3n}^2 + J_{3n}^3 + J_{3n}^4.
\end{aligned}
$$

For $M_1' = \int_{-\alpha}^{\alpha} |\hat{f}_n(x+iy)||\varphi(x)|dx$, $M_2' = \int_{-\alpha}^{\alpha} |P(x+iy)||\varphi(x)|dx$, $M_3' = \int_{-\alpha}^{\alpha} |P(x-iy)||\varphi(x)|dx$, $M_4' = \int_{-\alpha}^{\alpha} |\hat{f}_n(x-iy)||\varphi(x)|dx$ and using the uniform converges of the sequences $\{P_n\}$ and $\{\hat{f}_n\}$, we obtain that there exists N'

such that, for $k \in \{1,2,3,4\}$, $|J_{3n}^k| < \frac{\varepsilon}{12}$, for $n > N'$ which means that

$$|J_{3n}| < \frac{\varepsilon}{3}, \quad \text{for} \quad n > N'. \tag{7}$$

Together, (5), (6) and (7), complete the proof. □

Theorem 2.2. *Let $\{F_n\}$ be a sequence of distributions in S' of finite order and appropriate growth that converges to a distribution F in S' and, for every $n \in \mathbb{N}$, let $\hat{F}_{P,n}(z)$ be the analytic representation of F_n. Then the sequence $\{\hat{F}_{P,n}(z)\}$ uniformly converges to the function $\hat{F}_P(z)$ on each compact subset of $\mathbb{R} \times [\delta, \infty)$ and $\mathbb{R} \times (-\infty, -\delta]$, $\delta > 0$, in the upper and lower half plane respectively as $n \to \infty$, and $\hat{F}_P(z)$ is the analytic representation of the distribution F.*

Proof. For $n \in \mathbb{N}$, let $F_n \in S'$ be a distribution of finite order and appropriate growth, i.e. $F_n = \sum_{i=0}^{k} [P_i(t) f_{i,n}(t)]^{(i)}$, where $\{f_{i,n}\}_n$ is a sequence of functions of $L^1(\mathbb{R})$ such that $f_{i,n} \to f_i$ as $n \to \infty$, for every $i \in \{0,1,\dots,k\}, P_i(t) = (1+t^2)^{i/2}$, $P(t) = (1+t^2)^{k/2}$ and such that $\frac{P_i(t)F_{i,n}(t)}{P(t)} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

For z in compact subset K of $\mathbb{R} \times [\delta, \infty)$,

$$\begin{aligned}
&\hat{F}_{P,n}(z) - \hat{F}_P(z) = \frac{P(z)}{2\pi i}\langle F_n, \frac{1}{P(t)(t-z)}\rangle - \frac{P(z)}{2\pi i}\langle F, \frac{1}{P(t)(t-z)}\rangle \\
&= \frac{P(z)}{2\pi i}\langle F_n - F, \frac{1}{P(t)(t-z)}\rangle \\
&= \frac{P(z)}{2\pi i}\left\langle \sum_{i=0}^{k} [P_i(t) f_{i,n}(t) - P_i(t) f_i(t)]^{(i)}, \frac{1}{P(t)(t-z)} \right\rangle \\
&= \frac{P(z)}{2\pi i}\left\langle \frac{1}{P(t)} \sum_{i=0}^{k} (-1)^i [P_i(t) f_{i,n}(t) - P_i(t) f_i(t)], \left(\frac{1}{t-z}\right)^{(i)} \right\rangle \\
&= \frac{P(z)}{2\pi i}\left\langle \sum_{i=0}^{k} (-1)^i \ \frac{P_i(t) f_{i,n}(t) - P_i(t) f_i(t)}{P(t)}, \left(\frac{1}{t-z}\right)^{(i)} \right\rangle \\
&= \frac{P(z)}{2\pi i} \sum_{i=0}^{k} (-1)^i \int_{-\infty}^{+\infty} \frac{P_i(t)[f_{i,n}(t) - f_i(t)]}{P(t)} \left(\frac{1}{t-z}\right)^{(i)} dt
\end{aligned}$$

For arbitrary $\varepsilon > 0$, we get that

$$\begin{aligned}
&|\hat{F}_{P,n}(z) - \hat{F}_P(z)| \le \frac{1}{2\pi}|P(z)| \sum_{i=0}^{k} \int_{-\infty}^{+\infty} \frac{|P_i(t)||f_{i,n}(t) - f_i(t)|}{|P(t)|} \left| \left(\frac{1}{t-z}\right)^{(i)} \right| dt \\
&\le \frac{1}{2\pi}|P(z)| \sum_{i=0}^{k} \int_{-\infty}^{+\infty} \frac{|P(t)||f_{i,n}(t) - f_i(t)|}{|P(t)|} \ \frac{1}{\delta^i} \ dt \\
&= \frac{1}{2\pi}|P(z)| \sum_{i=0}^{k} \frac{1}{\delta^i} \int_{-\infty}^{+\infty} |f_{i,n}(t) - f_i(t)| \ dt \\
&\le \frac{1}{2\pi}|P(z)| \sum_{i=0}^{k} \frac{1}{\delta^i} \ \varepsilon \le \left(\frac{1}{2\pi} M \sum_{i=0}^{k} \frac{1}{\delta^i} \right) \varepsilon = M_1 \varepsilon,
\end{aligned}$$

where $M = \sup_{z \in K} |P(z)|$.

It remains to prove that $\hat{F}_P(z) = \frac{P(z)}{2\pi i}\langle F, \frac{1}{P(t)(t-z)}\rangle$ is analytic representation of the distribution $F \in S'$, i.e. that

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\left[\frac{P(z)}{2\pi i}\langle F, \frac{1}{P(t)(t-z)}\rangle - \frac{P(\bar{z})}{2\pi i}\langle F, \frac{1}{P(t)(t-\bar{z})}\rangle\right]\varphi(x)dx = \langle F, \varphi\rangle.$$

Let $\varepsilon > 0$ be arbitrary chosen.

$$\begin{aligned}
&\langle F, \varphi\rangle - \int_{-\infty}^{\infty}\left[\frac{P(z)}{2\pi i}\langle F, \frac{1}{P(t)(t-z)}\rangle - \frac{P(\bar{z})}{2\pi i}\langle F, \frac{1}{P(t)(t-\bar{z})}\rangle\right]\varphi(x)dx = \\
=&\{\langle F, \varphi\rangle - \langle F_n, \varphi\rangle\} \\
+&\{\langle F_n, \varphi\rangle - \int_{-\infty}^{\infty}\left[\frac{P(z)}{2\pi i}\langle F_n, \frac{1}{P(t)(t-z)}\rangle - \frac{P(\bar{z})}{2\pi i}\langle F_n, \frac{1}{P(t)(t-\bar{z})}\rangle\right]\varphi(x)dx\} \\
+&\{\int_{-\infty}^{\infty}\left[\frac{P(z)}{2\pi i}\langle F_n, \frac{1}{P(t)(t-z)}\rangle - \frac{P(\bar{z})}{2\pi i}\langle F_n, \frac{1}{P(t)(t-\bar{z})}\rangle\right]\varphi(x)dx- \\
-&\int_{-\infty}^{\infty}\left[\frac{P(z)}{2\pi i}\langle F, \frac{1}{P(t)(t-z)}\rangle - \frac{P(\bar{z})}{2\pi i}\langle F, \frac{1}{P(t)(t-\bar{z})}\rangle\right]\varphi(x)dx\} \\
=&J_1 + J_2 + J_3.
\end{aligned}$$

Since $\{F_n\}$ converges to F, we get that there exists N_1 such that

$$|J_1| < \frac{\varepsilon}{3}, \quad \text{for} \quad n > N_1. \tag{8}$$

From the condition in T.2.2. i.e. that $\hat{F}_{P,n}(z) = \frac{P(z)}{2\pi i}\langle F_n, \frac{1}{P(t)(t-z)}\rangle$ is the analytic representation of F_n, for every $n \in \mathbb{N}$, we conclude that there exists y_0 such that

$$|J_2| < \frac{\varepsilon}{3}, \quad \text{for} \quad y < y_0 \tag{9}$$

$$\begin{aligned}
J_3 = &\int_{-\infty}^{\infty}\frac{P(z)}{2\pi i}\langle F_n - F, \frac{1}{P(t)(t-z)}\rangle\varphi(x)dx+ \\
+ &\int_{-\infty}^{\infty}\frac{P(\bar{z})}{2\pi i}\langle F - F_n, \frac{1}{P(t)(t-\bar{z})}\rangle\varphi(x)dx.
\end{aligned}$$

Since $\{F_n\}$ converges to F, we get that there exists N_2, such that

$$|J_3| < \frac{\varepsilon}{3}, \quad \text{for} \quad n > N_3. \tag{10}$$

(8), (9) and (10) give that $\hat{F}_P(z)$ is the analytic representation of F.

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