Filomat 29:8 (2015), 1879–1892
DOI 10.2298/FIL1508879C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

General Univalence Criteria and Quasiconformal Extensions Starting from Loewner Chains Theory

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Abstract. The aim of this paper is to obtain general univalence conditions and quasiconformal extensions to C^n of holomorphic mappings defined on the Euclidian unit ball B. The asymptotical case of the quasiconformal extension results is also presented. We extend several results obtained by Hamada and Kohr(2011) in [15] to a more general case. In particular our results improve certain univalence criteria and quasiconformal extension results previously obtained by Pfaltzgraff [21], [22], Curt and Pascu [8], Curt [5], Hamada and Kohr [16], Curt and Kohr [6], [7] and Răducanu [24]. As applications we present general forms of the n-dimensional version of the well-known univalence criterion due to Lewandowski [19] and its quasiconformal extension.

1. Introduction

In 1972, Becker [2] showed that if $q \in [0, 1)$ and if f is a holomorphic function on the open unit disk which satisfies the inequality

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq q, \; z \in U \tag{1}$$

then f is univalent in U and can be extended to a quasiconformal homeomorphism of $\mathbb{R}^2$ onto itself.

A generalization of the univalence criterion given in (1) was obtained by Becker in [3]. He proved the following result.

Let $a(t)$ be an absolutely continuous function on $[0, \infty)$ with $a(0) = 1$, $\Re[a'(t)/a(t)] > 0$ a.e. on $[0, \infty)$ and $\lim\limits_{t\to\infty} a(t) = \infty$. Let f be a normalized holomorphic function on the open unit disk. Suppose that

$$\max_{|z|=e^{-t}} \left| 2\frac{a(t) - e^{-t}}{a'(t) + a(t)} \frac{zf''(z)}{f'(z)} - \frac{a'(t) - a(t)}{a'(t) + a(t)} \right| \leq q, \; \text{a.e. } t > 0, \; z \in U. \tag{2}$$

Then

2010 *Mathematics Subject Classification*. Primary 32H02; Secondary 30C45

Keywords. biholomorphic mapping, Loewner differential equation, subordination chain, univalence criterion, quasiregular mapping, quasiconformal mapping, quasiconformal extension

Received: 06 November 2013; Accepted: 12 April 2014

Communicated by H. M. Srivastava

Paula Curt has been supported by a Grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0899

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(i) If $q = 1$ the function f is univalent in U.

(ii) If $q < 1$ the function f extends to a quasiconformal homeomorphism of $\mathbb{R}^2$ onto itself.

During the time, various extensions of the univalence criterion due to Becker ([2], [3]) have been obtained (see [1], [25], [20], [26], [11], [27], [9]).

Recently, Hamada and Kohr [15] used the Loewner chains theory to obtain the n-dimensional version of the above result due to Becker. In the same paper the authors also considered the asymptotical case of their quasiconformal extension result.

In this paper we extend several results obtained in [15] to a more general case. In particular our results improve certain univalence criteria and quasiconformal extension results previously obtained by Pfaltzgraff [21], [22], Curt [5], Hamada and Kohr [16], Curt and Kohr [6], [7] and Răducanu [24]. As applications we present general forms of the n-dimensional version of the well-known univalence criterion due to Lewandowski [19] and its quasiconformal extension.

More results related to the problem of quasiconformal extensions for quasiregular holomorphic mappings on the unit ball of $\mathbb{C}^n$ can be found in [4], [14], [16], [22].

2. Preliminary Results

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \ldots, z_n)$ with the usual inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Let $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the open unit ball in $\mathbb{C}^n$ and $\overline{B}$ be the closed unit ball. In the case of one complex variable B will be denoted by U. Also denote by $\overline{\mathbb{R}^m} = \mathbb{R}^m \cup \{\infty\}$ the one point compactification of $\mathbb{R}^m$.

Let $\mathcal{L}(\mathbb{C}^n)$ be the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard norm

$$\|A\| = \sup\{\|Az\| : \|z\| = 1\}, \quad A \in \mathcal{L}(\mathbb{C}^n) \tag{3}$$

and let I be the identity in $\mathcal{L}(\mathbb{C}^n)$.

Denote by $\mathcal{H}(B)$ the class of holomorphic mappings f from B into $\mathbb{C}^n$.

If $f \in \mathcal{H}(B)$, let $Df(z)$ denotes the Fréchet derivative of f at $z \in B$ given by

$$Df(z) = \left(\frac{\partial f_j(z)}{\partial z_k}\right)_{1\le j,k\le n}. \tag{4}$$

A mapping $f \in \mathcal{H}(B)$ is said to be locally biholomorphic in B if $Df(z)$ is nonsingular at each point of B. Also, we say that $f \in \mathcal{H}(B)$ is normalized if $f(0) = 0$ and $Df(0) = I$.

The second Fréchet derivative of $f \in \mathcal{H}(B)$ at $z \in B$, denoted by $D^2 f(z)$ is a symmetric bilinear operator from $\mathbb{C}^n \times \mathbb{C}^n$ into $\mathbb{C}^n$. Clearly, $D^2 f(z)(z, .)$ is the linear operator obtained by restricting $D^2 f(z)$ to $\{z\} \times \mathbb{C}^n$ and has the matrix representation

$$D^2 f(z)(z, .) = \left(\sum_{m=1}^{n} \frac{\partial^2 f_k(z)}{\partial z_j \partial z_k} z_m\right)_{1\le j,k\le n}. \tag{5}$$

Definition 2.1. *Let $k \ge 1$. A mapping $f \in \mathcal{H}(B)$ is said to be k-quasiregular if*

$$\left\|Df(z)\right\|^n \le k|J_f(z)|, \; z \in B \tag{6}$$

where $J_f(z) = \det Df(z)$ is the complex jacobian determinant of f at $z \in B$.

If $f \in \mathcal{H}(B)$ is k-quasiregular for some $k \ge 1$, then it is called quasiregular.

It is known that quasiregular holomorphic mappings are locally biholomorphic.

Definition 2.2. *Let G and G' be two damains in $\mathbb{R}^m$. A homeomorphism $f : G \to G'$ is said to be k-quasiconformal ($k > 0$) if it is differentiable a.e., absolutely continuous on lines (ACL) and*

$$\left\|Df(x)\right\|^n \leq k|\det Df(x)| \text{ a.e., } x \in G \tag{7}$$

where $Df(x)$ is the real jacobian matrix of f.

Note that k-quasiregular biholomorphic mappings are k^2-quasiconformal.

A mapping $v \in \mathcal{H}(B)$ is called Schwarz mapping if $\|v(z)\| \leq \|z\|$ for all $z \in B$. If $f, g \in \mathcal{H}(B)$ we say that f is subordinate to g, written $f \prec g$, if there exists a Schwarz mapping v such that $f(z) = g(v(z)),\ z \in B$.

Definition 2.3. *A mapping $L : B \times [0, \infty) \to \mathbb{C}^n$ is called a subordination chain if the following two conditions are satisfied:*

(i) $L(0, t) = 0$ and $L(., t) \in \mathcal{H}(B)$ for all $t \geq 0$.

(ii) $L(z, s) \prec L(z, t)$ whenever $0 \leq s < t < \infty$.

If a subordination chain $L(z, t)$ is such that $L(., t)$ is biholomorphic on B for all $t > 0$ then, $L(z, t)$ is called a univalent subordination chain or a Loewner chain. In this case, there exists a biholomorphic Schwarz mapping $v = v(z, s, t)$ such that

$$L(z, s) = L(v(z, s, t), t),\ z \in B,\ 0 \leq s < t < \infty. \tag{8}$$

The Schwarz mapping $v = v(z, s, t)$ is called the transition mapping associated with $L(z, t)$.

A univalent subordination chain $L(z, t)$ is said to be normalized if $DL(0, t) = e^t I$.

The following two families of holomorphic mappings on B play an important role in our investigation:

$$\mathcal{N} = \{h \in \mathcal{H}(B) : h(0) = 0,\ \Re\,\langle h(z), z\rangle > 0,\ z \in B \setminus \{0\}\}$$

$$\mathcal{M} = \{h \in \mathcal{N},\ Dh(0) = I\}.$$

The next proposition was proved in [18], (see also [15]). Particular cases of it were obtained in [12], [13], [22] and [23].

Proposition 2.1. *([18]) Let $c_0(t) : [0, \infty) \to \mathbb{C}$ be a measurable function which is bounded a.e. on each interval $[0, T](T > 0)$ with $\Re c_0(t) > 0$ a.e on $[0, \infty)$ and*

$$\int_0^\infty \Re c_0(t)dt = \infty.$$

Also let $h = h(z, t) : B \times [0, \infty) \to \mathbb{C}^n$ be a mapping which satisfies the following conditions:

(i) $h(., t) \in \mathcal{N},\ Dh(0, t) = c_0(t)I$ for $t \geq 0$;

(ii) $h(z, .)$ is measurable on $[0, \infty)$ for $z \in B$.

Then, for each $s \geq 0$ and $z \in B$, the initial value problem

$$\frac{\partial v}{\partial t} = -h(v, t),\ \text{a.e. } t \geq s,\ v(z, s, s) = z \tag{9}$$

has a unique solution $v(z, s, t)$ such that $v(., s, t)$ is a univalent Schwarz mapping, $v(z, s, .)$ is locally absolutely continuous on $[0, \infty)$, locally uniform with respect to $z \in B$, $Dv(0, s, t) = \exp\left(-\int_s^t c_0(\tau)d\tau\right) I$ for $t \geq s \geq 0$ and

$$\exp\left(\int_0^t c_0(\tau)d\tau\right) \frac{\|v(z, s, t)\|}{(1 - \|v(z, s, t)\|)^2} \leq \exp\left(\int_0^s c_0(\tau)d\tau\right) \frac{\|z\|}{(1 - \|z\|)^2}. \tag{10}$$

In addition, the limit

$$\lim_{t\to\infty} e^{\int_0^t c_0(\tau)d\tau} v(z,s,t) = f(z,s) \tag{11}$$

exists locally uniformly on B *for each* $s \geq 0$. *Moreover,* $f(z,t)$ *is a univalent subordination chain such that*

$$\left\{ \exp\left(-\int_0^t c_0(\tau)d\tau \right) f(z,t) \right\}_{t\geq 0}$$

is a normal family,

$$Df(0,t) = \exp\left(\int_0^t c_0(\tau)d\tau \right) I$$

and $f(z,s) = f(v(z,s,t),t)$ *for* $z \in B,\ 0 \leq s \leq t < \infty$. *Also,* $f(z,.)$ *is locally absolutely continuous on* $[0,\infty)$, *locally uniformly with respect to* $z \in B$ *and satisfies*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t) \ a.e.\ t \geq 0,\ z \in B. \tag{12}$$

Definition 2.4. *([15])*

(i) A mapping $h(z,t)$ *which satisfies the assumptions (i) and (ii) of Proposition 2.1 is called generating vector field (cf. [10]).*

(ii) The univalent subordination chain $f(z,t)$ *given by (11) is called the canonical solution of the Loewner differential equation (12).*

(iii) Let $g : B \times [0,\infty) \to \mathbb{C}^n$ *be a mapping such that* $g(.,t) \in \mathcal{H}(B)$, $g(0,t) = 0$ *for* $t \geq 0$ *and* $g(z,.)$ *is locally absolutely continuous on* $[0,\infty)$, *locally uniformly with respect to* $z \in B$. *Assume that* $g(z,t)$ *satisfies the Loewner differential equation (12). Then, the mapping* $g(z,t)$ *is called standard solution of the Loewner differential equation (12).*

A sufficient condition for a standard solution $g(z,t)$ to be a canonical solution of the Loewner differential equation (12) is provided in the next theorem.

Theorem 2.1. *([15]) Let* $c_0(t) : [0,\infty) \to \mathbb{C}$ *be a measurable function which is bounded a.e. on each interval* $[0,T](T > 0)$, $\Re c_0(t) > 0$ *a.e.* $[0,\infty)$ *and*

$$\int_0^\infty \Re c_0(t)dt = \infty.$$

Let $h(z,t)$ *be a generating vector field such that* $Dh(0,t) = c_0(t)I,\ t \geq 0$. *Then, the Loewner differential equation (12) has a unique standard solution* $g(z,t)$ *such that*

$$Dg(0,t) = \exp\left(\int_0^t c_0(\tau)d\tau \right) I$$

and the family

$$\left\{ \exp\left(-\int_0^t c_0(\tau)d\tau \right) g(z,t) \right\}_{t\geq 0}$$

is a normal family.

In the following result is given a sufficient condition for a univalent subordination chain to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself. This result was obtained in [[15], Theorem2] (see also [7]).

Theorem 2.2. *([15]) Let $a : [0,\infty)) \to \mathbb{C}$ be an absolutely continuous function such that $a(0) = 1$, $\Re(a'(t)/a(t)) > 0$ a.e., $\lim_{t\to\infty} a(t) = \infty$ and $a'(t)/a(t)$ is bounded a.e. on each interval $[0,T](T > 0)$. Also, let $h = h(z,t) : B\times[0,\infty) \to \mathbb{C}^n$ be a mapping such that $Dh(0,t) = (a'(t)/a(t))I$ a.e. on $[0,\infty)$ and $h(z,.)$ is a measurable on $[0,\infty)$ for $z \in B$.*

Let $f(z,t) = a(t)z + \dots$ be a univalent subordination chain such that $f(z,.)$ is locally absolutely continuous on $[0,\infty)$, locally uniformly with respect to $z \in B$ and $f(z,t)$ satisfies the differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t) \text{ a.e. } t \in [0,\infty),\ z \in B.$$

Suppose that the following conditions hold:

(i) There exist some constants $M > 0$ and $\alpha \in [0,1)$ such that

$$\left\|Df(z,t)\right\| \le \frac{M|a(t)|}{(1-\|z\|)^{\alpha}},\ z \in B,\ t \in [0,\infty);$$

(ii) There exists a constant $c_1 > 0$ such that

$$\Re\,\langle h(z,t),z\rangle \ge c_1\,\|z\|^2\,,\ z \in B,\ \text{a.e. } t \in [0,\infty);$$

(iii) There exists a constant $c_2 > 0$ such that

$$\|h(z,t)\| \le c_2,\ z \in B,\ \text{a.e. } t \in [0,\infty);$$

(iv) There exists a constant $k > 0$ such that $f(.,t)$ is k-quasiregular for each $t \in [0,\infty)$.

Then, there exists a quasiconformal homeomorphism $\mathcal{F}$ of $\mathbb{R}^{2n}$ onto itself such that $\mathcal{F}|_B = f(.,0)$.

The asymptotical case of Theorem 2.2 was obtained in [[15], Theorem 3](see also [7]).

Theorem 2.3. *([15]) Let $c_0(t) : [0,T] \to \mathbb{C}$ be a measurable function such that it is bounded and $\Re c_0(t) \ge \epsilon > 0$ a.e. on $[0,T](T > 0)$. Also , let $h = h(z,t) : B\times[0,T] \to \mathbb{C}^n$ be a mapping such that $Dh(0,t) = c_0(t)I$ for $t \in [0,T]$ and $h(z,.)$ is measurable on $[0,T]$ for $z \in B$.*

Let $f(z,t) = a(t)z + \dots$ be a mapping such that $f(.,t) \in \mathcal{H}(B)$, $f(0,t) = 0$, $Df(0,t) = \exp\left(\int_0^t c_0(\tau)d\tau\right)I$ and $f(z,.)$ is locally absolutely continuous on $[0,T]$, locally uniformly with respect to $z \in B$. Suppose that $h(z,t)$ satisfies the differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t) \text{ a.e. } t \in [0,T],\ z \in B.$$

Assume that $f(.,0)$ is continuous and injective on $\overline{B}$. Also, assume that the following conditions hold:

(i) There exist some constants $M > 0$ and $\alpha \in [0,1)$ such that

$$\left\|Df(z,t)\right\| \le \frac{M|a(t)|}{(1-\|z\|)^{\alpha}},\ z \in B,\ t \in [0,T];$$

(ii) There exists a constant $c_1 > 0$ such that

$$\Re\,\langle h(z,t),z\rangle \ge c_1\,\|z\|^2\,,\ z \in B,\ \text{a.e. } t \in [0,T];$$

(iii) There exists a constant $c_2 > 0$ such that

$$\|h(z,t)\| \le c_2,\ z \in B,\ \text{a.e. } t \in [0,T];$$

(iv) There exists a constant $k > 0$ such that $f(.,t)$ is k-quasiregular for each $t \in [0,T]$.

Then, there exists a constant $\tau \in (0,T)$ such that $f(.,t)$ is continuous and injective on $\overline{B}$ for $t \in [0,\tau]$ and there exists a quasiconformal homeomorphism $\mathcal{F}$ of $\overline{\mathbb{R}^{2n}}$ onto itself such that $\mathcal{F}|_B = f(.,0)$.

3. Main Results

The main object of this section is to obtain a general sufficient condition for a normalized quasiregular mapping on B to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself. The asymptotical case will be also considered. These results generalize the results obtained in [15], [8], [7] and [24].

Definition 3.1. *Let $F = F(u,v) : B \times \mathbb{C}^n \to \mathbb{C}^n$ be a mapping of class C^1 with $F(0,0) = 0$ and let $\gamma > 0$. Also let $a = a(t)$ be an absolutely continuous complex valued function on $[0,\infty)$ such that $a(0) = 1$. We say that the conditions (P_a) are satisfied if the following assumptions hold:*

(i) $F(e^{-\gamma t}z, a(t)z) \in \mathcal{H}(B)$ *for all* $t \geq 0$;

(ii) For each $t \in [0,\infty)$ *there exists a complex number* $a_1(t) \geq 0$, *with* $a_1(0) = 1$ *such that*

$$e^{-\gamma t} D_u F(0,0) + a(t) D_v F(0,0) = a_1(t) I \tag{13}$$

where $D_u F(u,v)$ $(D_v F(u,v))$ *is the* $n \times n$ *matrix for which the* (i,j) *entry is given by*

$$\frac{\partial F_i}{\partial u_j}(u,v) \ \left(\frac{\partial F_i}{\partial v_j}(u,v)\right); \tag{14}$$

(iii) $D_v F(u,v)$ *is invertible for all* $(u,v) \in B \times \mathbb{C}^n$.

In the next theorem we give a sufficient condition for the univalence of the mapping $F(z,z)$, where $F(u,v)$ satisfies (P_a) conditions. Our result generalizes the result obtained in [15] for the function $F(u,v) = f(u) + G(u)(v-u)$, where $G(u)$ is a nonsingular $n \times n$ matrix, holomorphic as a function of $u \in B$ such that $G(0) = I$, and $\gamma = 1$. Moreover, when $\gamma = 1$ and $a(t) = e^t$ the next result is an improvement of Theorem 1 in [8].

Theorem 3.1. *Let $F = F(u,v) : B \times \mathbb{C}^n \to \mathbb{C}^n$ and let $a(t) : [0,\infty) \to \mathbb{C}$ be functions that satisfy (P_a) conditions. Suppose that $\alpha(t) = a'(t)/a(t)$ is bounded a.e. on each interval $[0,T]$ $(T > 0)$ and $\lim_{t\to\infty} |a(t)| = \infty$. Also, let $\gamma > 0$. If*

$$\left\| \frac{2\gamma}{\alpha(t)+\gamma} H(0,t) - \frac{\alpha(t)-\gamma}{\alpha(t)+\gamma} I \right\| < 1 \ a.e. \ t > 0 \tag{15}$$

and

$$\max_{\|z\|=1} \left\| \frac{2\gamma}{\alpha(t)+\gamma} H(z,t) - \frac{\alpha(t)-\gamma}{\alpha(t)+\gamma} I \right\| \leq 1 \ a.e. \ t > 0 \tag{16}$$

where

$$H(z,t) = \frac{e^{-\gamma t}}{a(t)} \left[D_v F(e^{-\gamma t}z, a(t)z) \right]^{-1} D_u F(e^{-\gamma t}z, a(t)z), \ z \in \overline{B}, \ t > 0 \tag{17}$$

then, $F(z,z)$ is a univalent mapping on B.

Proof. Consider the mapping $f : B \times [0,\infty) \to \mathbb{C}^n$ given by

$$f(z,t) = F(e^{-\gamma t}z, a(t)z). \tag{18}$$

We will prove that the mapping $f(z,t)$ is a standard solution of a Loewner differential equation. Since the conditions (P_a) are satisfied it follows that $f(.,t) \in \mathcal{H}(B)$, $f(0,t) = F(0,0) = 0$ and $a_1(t)I = Df(0,t)$, where $a_1(t) : [0,\infty) \to \mathbb{C}$, with $a_1(0) = 1$. Due to the fact that the mapping $F = F(u,v)$ is of class C^1 on $B \times \mathbb{C}^n$ we obtain that $f(z,.)$ is locally absolutely continuous on $[0,\infty)$, locally uniformly with respect to $z \in B$.

From the absolute continuity of the function $a(t)$ on $[0,\infty)$ it follows that $a(t)$ is a.e. differentiable and thus, there exists a set $N \subset [0,\infty)$ of measure zero such that $a(t)$ is differentiable on $[0,\infty) \setminus N$.

Making use of (18) we obtain that

$$Df(z,t) = e^{-\gamma t} D_u F(e^{-\gamma t} z, a(t)z) + a(t) D_v F(e^{-\gamma t} z, a(t)z)$$

$$= a(t) D_v F(e^{-\gamma t} z, a(t)z) \frac{\alpha(t)+\gamma}{2\gamma} [I - E(z,t)] \tag{19}$$

where for each $t \in (0,\infty) \setminus N$ and $z \in B$, $E(z,t)$ is a linear operator defined by

$$E(z,t) = \frac{\alpha(t)-\gamma}{\alpha(t)+\gamma} I - \frac{2\gamma}{\alpha(t)+\gamma} H(z,t). \tag{20}$$

In the sequence we will show that $I - E(z,t)$ is an invertible operator for each $t \in (0,\infty) \setminus N$. It is easy to see that $E(.,t) : \overline{B} \to \mathcal{L}(\mathbb{C}^n)$ is holomorphic for $t \in (0,\infty) \setminus N$. From (20) and (15) we deduce that

$$\|E(0,t)\| = \left\| \frac{2\gamma}{\alpha(t)+\gamma} H(0,t) - \frac{\alpha(t)-\gamma}{\alpha(t)+\gamma} I \right\| < 1, \; t \in (0,\infty) \setminus N. \tag{21}$$

Moreover, in view of the weak maximum modulus theorem (see [17]), we have

$$\|E(z,t)\| \le \max_{\|w\|=1} \left\| \frac{2\gamma}{\alpha(t)+\gamma} H(w,t) - \frac{\alpha(t)-\gamma}{\alpha(t)+\gamma} I \right\| \le 1, \; t \in (0,\infty) \setminus N, z \in B. \tag{22}$$

Hence, in virtue of (20) and (21), we obtain that $\|E(z,t)\| < 1$ for all $z \in B$ and $t \in (0,\infty) \setminus N$ and this proves that $I - E(z,t)$ is invertible.

By using elementary computations, we have

$$\frac{\partial f}{\partial t}(z,t) = \left[-\gamma e^{-\gamma t} D_u F(e^{-\gamma t} z, a(t)z) + a'(t) D_v F(e^{-\gamma t} z, a(t)z) \right](z)$$

$$= a(t) D_v F(e^{-\gamma t} z, a(t)z) \frac{\alpha(t)+\gamma}{2} \left[I + E(z,t) \right](z). \tag{23}$$

It follows that $f(z,t)$ satisfies the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t) h(z,t), \; t \in (0,\infty) \setminus N, \; z \in B,$$

where

$$h(z,t) = \begin{cases} \gamma \left[I - E(z,t) \right]^{-1} \left[I + E(z,t) \right](z), & t \in (0,\infty) \setminus N, \; z \in B \\ cz, & t \in N, \; z \in B \end{cases} \tag{24}$$

with c a positive constant.

The mapping $h(z,t)$ defined by (24) is measurable and holomorphic, with $h(0,t) = 0$ and

$$Dh(0,t) = \gamma \left[I - E(0,t) \right]^{-1} \left[I + E(0,t) \right] = \frac{a_1'(t)}{a_1(t)} I = c_0(t) I \tag{25}$$

for each $t \in (0,\infty) \setminus N$. Moreover,

$$\left\| \frac{h(z,t)}{\gamma} - z \right\| = \left\| E(z,t) \left(\frac{h(z,t)}{\gamma} + z \right) \right\| \le \|E(z,t)\| \left\| \frac{h(z,t)}{\gamma} + z \right\| \le \left\| \frac{h(z,t)}{\gamma} + z \right\|$$

and thus, $\Re \langle h(z,t), z \rangle \ge 0$. It follows that the mapping $h(z,t)$ is a generating vector field.

In the sequence, we will prove that the function $c_0(t) = a_1'(t)/a_1(t)$ satisfies the conditions of Theorem 2.1. First, from (25) we get that

$$E(0,t)\left(\gamma + \frac{a_1'(t)}{a_1(t)}\right) = \left(\frac{a_1'(t)}{a_1(t)} - \gamma\right)I$$

and thus,

$$\left|\gamma + \frac{a_1'(t)}{a_1(t)}\right| > \left|\frac{a_1'(t)}{a_1(t)} - \gamma\right|$$

which shows that $\Re\dfrac{a_1'(t)}{a_1(t)} > 0$ for $t \in (0,\infty) \setminus N$.

We also have that

$$\int_0^\infty \Re c_0(t)dt = \lim_{t\to\infty} \Re \int_0^t \frac{a_1'(s)}{a_1(s)}ds = \lim_{t\to\infty} \ln|a_1(t)|.$$

Since $|a_1(t)| \geq |a(t)| \left|\|D_vF(0,0)\| - \gamma\|D_uF(0,0)\|\right|$ and $\lim_{t\to\infty} |a(t)| = \infty$ it follows that $\lim_{t\to\infty} |a_1(t)| = \infty$ and therefore $\int_0^\infty \Re c_0(t)dt = \infty$.

It remains to prove that the function $c_0(t)$ is bounded a.e. on $[0,T]$, $T > 0$. Differentiating with respect to t the equality

$$a_1(t)I = e^{-\gamma t}D_uF(0,0) + a(t)D_vF(0,0)$$

we obtain

$$a_1'(t) = -\gamma e^{-\gamma t}D_uF(0,0) + a'(t)D_vF(0,0).$$

Dividing the last equality by $a_1(t)$, we have

$$\left|\frac{a_1'(t)}{a_1(t)}\right| \leq \frac{\gamma e^{-\gamma t}}{|a_1(t)|}\|D_uF(0,0)\| + \left|\frac{a'(t)}{a(t)}\right|\left|\frac{a(t)}{a_1(t)}\right|\|D_vF(0,0)\|.$$

Since the function $a_1(t)$ is continuous on $[0,T]$ and does not vanish on $[0,\infty)$ we obtain that $\dfrac{1}{|a_1(t)|}$ is bounded on $[0,T]$. From the hypothesis, we have that the mapping $F(u,v)$ is of class C^1 on $B \times [0,\infty)$ and $a'(t)/a(t)$ is bounded a. e. on $[0,T]$. Therefore, it follows that the function $c_0(t)$ is bounded a. e. on $[0,T]$.

In conclusion, the mapping $f(z,t)$ is a standard solution of the Loewner differential equation (12). Making use of Theorem 2.1, we obtain that the mapping $f(z,t)$ is also a canonical solution and therefore $f(z,t)$ is a univalent subordination chain. In particular $F(z,z) = f(z,0)$ is univalent. □

The next result provides a general quasiconformal extension of a normalized holomorphic mapping on B to $\mathbb{R}^{2n}$. Our result generalizes the result obtained in [15] for the function $F(u,v) = f(u) + G(u)(v-u)$, where $G(u)$ is a nonsingular $n \times n$ matrix, holomorphic as a function of $u \in B$ such that $G(0) = I$, and $\gamma = 1$. Moreover, when $\gamma = 1$ and $a(t) = e^t$ the next result is an improvement of Theorem 4.1 in [7].

Theorem 3.2. *Let $q \in (0,1)$. Let $F = F(u,v) : B\times\mathbb{C}^n \to \mathbb{C}^n$, $\gamma > 0$, and $a(t) : [0,\infty) \to \mathbb{C}$ be such that the conditions (P_a) given in Definition 3.1 are satisfied. Assume that*

(i) $\lim_{t\to\infty} |a(t)| = \infty$ *and* $\alpha(t) = \dfrac{a'(t)}{a(t)}$ *is bounded a. e. on each interval* $[0,T]$ $(T > 0)$*;*

(ii)

$$\max_{\|z\|=1}\left\|\frac{2\gamma}{\alpha(t)+\gamma}H(z,t) - \frac{\alpha(t)-\gamma}{\alpha(t)+\gamma}I\right\| \leq q < 1, \text{ a. e. } t > 0 \tag{26}$$

where $H(z,t)$ is defined by (17);

(iii) There exist some constants $M > 0$, $k \geq 1$ and $\alpha \in [0, 1)$ such that

$$\|D_v F(u, v)\| \leq \frac{M}{(1 - \|u\|)^\alpha}, \ u \in B, \ v \in \mathbb{C}^n \tag{27}$$

and

$$\|D_v F(u, v)\| \leq k \left|\det D_v F(u, v)\right|, \ u \in B, \ v \in \mathbb{C}^n. \tag{28}$$

Then, the mapping $f : B \to \mathbb{C}^n$ given by $f(z) = F(z, z)$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. It is easy to see that the inequality (26) imply the inequalities (15) and (16). As in the proof of Theorem 3.1 we obtain that the mapping $f(z, t) = F(e^{-\gamma t} z, a(t)z)$ is a univalent subordination chain which satisfies the Loewner differential equation (12), where $E(z, t)$ and $h(z, t)$ are defined by (19) and (23) respectively. Also, in the proof of Theorem 3.1 we showed that the function

$$a_1(t) = e^{-\gamma t} D_u F(0, 0) + a(t) D_v F(0, 0)$$

satisfies all the required conditions of Theorem 2.2.

Our result will follow from Theorem 2.2 if we prove that the conditions (i)-(iii) of the theorem are satisfied.

In view of (19), (27) and (28) we have

$$\begin{aligned} \left\|Df(z, t)\right\| &\leq |a_1(t)| \left\|[I - E(0, t)]^{-1}\right\| \left\|[D_v F(0, 0)]^{-1}\right\| \left\|D_v F(e^{-\gamma t} z, a(t)z)\right\| \|I - E(z, t)\| \\ &\leq \frac{1 + q}{1 - q} |a_1(t)| \frac{M}{(1 - \|e^{-\gamma t} z\|)^\alpha} \left\|[D_v F(0, 0)]^{-1}\right\| \leq \frac{M^* |a_1(t)|}{(1 - \|z\|)^\alpha}, \ \text{a. e. } t > 0, \ z \in B. \end{aligned}$$

Since $Df(z, t)$ is continuous with respect to the variable t, the last estimate holds true for all $t \geq 0$. Thus, the proof of (i) in Theorem 2.2 is completed.

The mapping $E(z, t)$ is holomorphic in the variable z with $E(0, t) = 0$ and $\|E(z, t)\| < 1$. Then, in view of Remark 2.3 from [6] it follows that the mapping $h(z, t)$ defined by (24) with $c = \dfrac{1 - q}{1 + q}$ satisfies the conditions (ii) and (iii) of Theorem 2.2.

In the sequence we will prove that the mapping $f(., t)$ is k^*-quasiregular, where k^* is a positive constant. In view of (19) and (28) we have

$$\begin{aligned} \left\|Df(z, t)\right\|^n &\leq |a(t)|^n \left|\frac{\alpha(t) + \gamma}{2\gamma}\right|^n \|I - E(z, t)\|^n \left\|D_v F(e^{-\gamma t} z, a(t)z)\right\|^n \\ &\leq |a(t)|^n \left|\frac{\alpha(t) + \gamma}{2\gamma}\right|^n \|I - E(z, t)\|^n k \left|\det D_v F(e^{-\gamma t} z, a(t)z)\right| \\ &= k |\det Df(z, t)| \frac{\|I - E(z, t)\|^n}{|\det[I - E(z, t)]|} \\ &\leq k \left(\frac{1 + q}{1 - q}\right)^{n-1} |\det Df(z, t)| = k^* |\det Df(z, t)|, \ \text{a. e. } t \geq 0, \ z \in B. \end{aligned}$$

The last inequality was obtained by using the following estimation from the linear operator theory.

If $A : \mathbb{C}^n \to \mathbb{C}^n$ is a linear operator such that $\|A\| \leq q < 1$ then,

$$\frac{\|A\|^n}{|\det A|} \leq \left(\frac{1 + q}{1 - q}\right)^{n-1}.$$

Since $Df(z,t)$ is continuous with respect to the variable t, the estimate (iv) of Theorem 2.2 holds true for all $t \geq 0$ with $k^* = k\left(\dfrac{1+q}{1-q}\right)^{n-1}$.

From the above arguments we can conclude that the mapping $f(z,t)$ satisfies all the assumptions of Theorem 2.2 and thus $f(z,0) = F(z,z)$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself, as desired. □

Making use of Theorem 2.3, we obtain the following asymptotical case of Theorem 3.2. This is a generalization of the result obtained in [15] for the function $F(u,v) = f(u) + G(u)(v-u)$, where $G(u)$ is a nonsingular $n \times n$ matrix, holomorphic as a function of $u \in B$ such that $G(0) = I$, and$\gamma = 1$. Moreover, when $\gamma = 1$ and $a(t) = e^t$ the next result is an improvement of Theorem 5.1 in [7].

Theorem 3.3. *Let $F = F(u,v) : B \times \mathbb{C}^n \to \mathbb{C}^n$, $\gamma > 0$ and $a(t) : [0,T] \to \mathbb{C}$ be such that the conditions (P_a) given in Definition 3.1 are satisfied. Assume that the mapping $F(z,z)$ is continuous and injective on $\overline{B}$. Suppose that*

(i) $\alpha(t) = \dfrac{a'(t)}{a(t)}$ *is bounded a. e. on* $[0,T]$;

(ii) $\max\limits_{\|z\|=1} \left\| \dfrac{2\gamma}{\alpha(t)+\gamma} H(z,t) - \dfrac{\alpha(t)-\gamma}{\alpha(t)+\gamma} I \right\| \leq q < 1,$ *a. e.* $0 < t \leq T,\ z \in B \setminus \{0\}$, *where* $H(z,t),\ t \in [0,T],\ z \in B$ *is defined by (17).*

Moreover, suppose that there exist some constants $M > 0,\ k > 0$ and $\alpha \in [0,1)$ such that the conditions (27) and (28) hold.

Then, the mapping $F(z,z)$ extends to a quasiconformal homeomorphism of $\overline{\mathbb{R}^{2n}}$ onto itself.

Proof. By the same reasoning as in the proof of Theorem 3.2 and by making use of Theorem 2.3, the result in our theorem follows. □

4. Applications

As aplications of Theorems 3.1, 3.2 and 3.3, we obtain the next results which are direct generalizations of Theorem 4 and Theorem 8 in [15].

Theorem 4.1. *Let $q \in (0,1]$ and $a : [0,\infty) \to \mathbb{C}$ be an absolutely continuous function such that $a(0) = 1$, $\Re[a'(t)/a(t)] > 0$ a.e. $t > 0$, $\lim\limits_{t\to\infty} |a(t)| = \infty$ and $a'(t)/a(t)$ is bounded a.e. on each interval $[0,T]$ $(T > 0)$. Let $G(z)$ be a nonsingular $n \times n$ matrix, holomorphic as a function of $z \in B$ normalized by $G(0) = I$ and let $\gamma > 0$. If $f : B \to \mathbb{C}^n$ is a normalized holomorphic mapping which satisfies:*

$$\max_{\|z\|=e^{-\gamma t}} \left\| \frac{2\gamma}{a'(t)+\gamma a(t)} \left\{ \|z\| \left[(G(z))^{-1} Df(z) - I \right] + (a(t) - \|z\|)(G(z))^{-1} D^2 G(z)(z,\cdot) \right\} \right.$$

$$\left. - \frac{a'(t)-\gamma a(t)}{a'(t)+\gamma a(t)} I \right\| \leq q, \text{ a.e. } t > 0,$$

then the followings are true:

(i) If $q = 1$, the mapping f is biholomorphic on B;

(ii) If $q < 1$ and if there exist some constants $k \geq 1$ and $\alpha \in (0,1)$ such that $\|G(z)\| = O((1-\|z\|)^{-\alpha})$ and $\|G(z)\|^n \leq k\,|\det G(z)|$, $z \in B$, hold then f is quasiregular on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. The results follow from Theorems 3.1 and 3.2 with

$$F(u,v) = f(u) + G(u)(v-u),\ u = e^{-\gamma t}z,\ v = a(t)z.$$

□

Theorem 4.2. *Let $q \in (0,1)$ and $a : [0,T] \to \mathbb{C}$ be an absolutely continuous function such that $a(0) = 1$, $\Re[a'(t)/a(t)] \geq \epsilon > 0$ and $a'(t)/a(t)$ is bounded a.e. on $[0,T]$ $(T > 0)$. Let $G(z)$ be a nonsingular $n \times n$ matrix, holomorphic as a function of $z \in B$ normalized by $G(0) = I$ and let $\gamma > 0$. If $f : \overline{B} \to \mathbb{C}^n$ is a normalized holomorphic mapping which is continuous and injective on $\overline{B}$ and satisfies:*

$$\max_{\|z\|=e^{-\gamma t}} \left\| \frac{2\gamma}{a'(t)+\gamma a(t)} \left\{ \|z\| \left[(G(z))^{-1} Df(z) - I \right] + (a(t) - \|z\|)(G(z))^{-1} D^2 G(z)(z,\cdot) \right\} \right.$$
$$\left. - \frac{a'(t)-\gamma a(t)}{a'(t)+\gamma a(t)} I \right\| \leq q,\ a.e.\ 0 < t \leq T,$$

then the following statement is true:

If there exist some constants $k \geq 1$ and $\alpha \in (0,1)$ such that $\|G(z)\| = O\left((1-\|z\|)^{-\alpha}\right)$ and $\|G(z)\|^n \leq k\,|detG(z)|$, $z \in B$, hold then f is quasiregular on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. The result follows from Theorem 3.3 with

$$F(u,v) = f(u) + G(u)(v-u),\ u = e^{-\gamma t}z,\ v = a(t)z.$$

□

Note that for $\gamma = 1$ Theorem 4.1 reduces to Theorem 4 in [15] and Theorem 4.2 reduces to Theorem 8 in [15].

In the sequence, by using Theorems 3.1, 3.2 and 3.3, we obtain the next results which are general n-dimensional versions of the well-known univalence criterion due to Lewandowski [19].

Theorem 4.3. *Let $q \in (0,1]$ and let $f : B \to \mathbb{C}^n$ be a normalized holomorphic mapping. Let $a : [0,\infty) \to \mathbb{C}$ be an absolutely continuous function such that $a(0) = 1$, $\Re[a'(t)/a(t)] > 0$ a.e. $t > 0$, $\lim_{t\to\infty} |a(t)| = \infty$ and $a'(t)/a(t)$ is bounded a.e. on each interval $[0,T]$ $(T > 0)$. Suppose that p is a complex valued holomorphic mapping on B with $p(0) = 1$ and Fréchet derivative denoted by $p'(z)$. Let $\gamma > 0$. If*

$$\max_{\|z\|=e^{-\gamma t}} \left\| \frac{2\gamma}{a'(t)+\gamma a(t)} \left\{ \|z\| \frac{1-p(z)}{1+p(z)} I + (a(t)-\|z\|) \left[\frac{p'(z)(z)}{1+p(z)} I + (Df(z))^{-1} D^2 f(z)(z,.) \right] \right\} \right.$$
$$\left. - \frac{a'(t)-\gamma a(t)}{a'(t)+\gamma a(t)} I \right\| \leq q,\ a.\ e.\ t > 0,\ z \in B \tag{29}$$

then, the followings are true:

(i) If $q = 1$, the mapping f is biholomorphic on B;

(ii) If $q < 1$ and if there exist some constants $M > 0$, $k \geq 1$ and $\alpha \in [0,1)$ such that

$$\left\| Df(z) \right\| \leq \frac{2}{|p(z)+1|} \frac{M}{(1-\|z\|)^{\alpha}},\ z \in B \tag{30}$$

and

$$\left\| Df(z) \right\|^n \leq \frac{2^{2n}}{|p(z)+1|^{2n}} k \left| \det\left(Df(z)\right) \right|,\ z \in B \tag{31}$$

then, the mapping f is quasiregular on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. The result follows from Theorems 3.1 and 3.2 with

$$F(u,v) = f(u) + \frac{p(u)+1}{2} Df(u)(v-u),\ u = e^{-\gamma t}z,\ v = a(t)z.$$

□

If we consider $\gamma = 1$ in Theorem 4.3, we obtain the following result.

Theorem 4.4. *Let $q \in (0,1]$ and let $f : B \to \mathbb{C}^n$ be a normalized holomorphic mapping. Let $a : [0,\infty) \to \mathbb{C}$ be an absolutely continuous function such that $a(0) = 1$, $\Re[a'(t)/a(t)] > 0$ a.e. $t > 0$, $\lim_{t\to\infty} |a(t)| = \infty$ and $a'(t)/a(t)$ is bounded a.e. on each interval $[0,T]$ $(T > 0)$. Also, let p be a complex valued holomorphic mapping on B with $p(0) = 1$ and Fréchet derivative denoted by $p'(z)$. Assume that:*

$$\max_{\|z\|=e^{-t}} \left\| \frac{2}{a'(t)+a(t)} \left\{ \|z\| \frac{1-p(z)}{1+p(z)} I + (a(t) - \|z\|) \left[\frac{p'(z)(z)}{1+p(z)} I + (Df(z))^{-1} D^2 f(z)(z,.) \right] \right\} - \frac{a'(t)-a(t)}{a'(t)+a(t)} I \right\| \le q$$

a. e. $t > 0$, $z \in B$. Then, the followings are true:

(i) If $q = 1$, the mapping f is biholomorphic on B;

(ii) If $q < 1$ and if there exist some constants $M > 0$, $k \ge 1$ and $\alpha \in [0,1)$ such that (30) and (31) hold then, f is quasiregular on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

If we take $a(t) = e^{-t}$ in Theorem 4.4, we obtain the next result which is the n-dimensional version of the well-known univalence criterion due to Lewandowski [19] and its quasiconformal extension.

Corollary 4.1. *Let $q \in (0,1]$ and let $f : B \to \mathbb{C}^n$ be a normalized holomorphic mapping. Let p be a complex valued holomorphic mapping on B with $p(0) = 1$. Assume that:*

$$\left\| \|z\|^2 \frac{1-p(z)}{1+p(z)} I + (1 - \|z\|^2) \left[\frac{p'(z)(z)}{1+p(z)} I + (Df(z))^{-1} D^2 f(z)(z,.) \right] \right\| \le q$$

a. e. $t > 0$, $z \in B$. Then, the followings are true:

(i) If $q = 1$, the mapping f is biholomorphic on B;

(ii) If $q < 1$ and if there exist some constants $M > 0$, $k \ge 1$ and $\alpha \in [0,1)$ such that (30) and (31) hold then, f is quasiregular on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Various generalizations of the Lewandowski's n-dimensional univalence criterion, given in Corollary 4.1, can be obtained by specializing the function $a(t)$ in Theorem 4.4.

If we consider $a(t) = e^{(\delta-1)t}$, $\delta \in \mathbb{C}$ and $\Re\delta > 1$ then, we have the next result.

Corollary 4.2. *Let $q \in (0,1]$ and let $\delta \in \mathbb{C}$ with $\Re\delta > 1$. Let $f : B \to \mathbb{C}^n$ be a normalized holomorphic mapping and let p be a complex valued holomorphic mapping on B with $p(0) = 1$. Assume that:*

$$\left\| \frac{2}{\delta} \left\{ \|z\|^\delta \frac{1-p(z)}{1+p(z)} I + (1 - \|z\|^\delta) \left[\frac{p'(z)(z)}{1+p(z)} I + (Df(z))^{-1} D^2 f(z)(z,.) \right] \right\} + \frac{2-\delta}{\delta} I \right\| \le q$$

a. e. $t > 0$, $z \in B$. Then, the followings are true:

(i) If $q = 1$, the mapping f is biholomorphic on B;

(ii) If $q < 1$ and if there exist some constants $M > 0,\ k \geq 1$ and $\alpha \in [0,1)$ such that (30) and (31) hold then, f is quasiregular on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Note that for $\delta = 2$, Corollary 4.2 reduces to Corollary 4.1.

Another particular case of Theorem 4.4 is obtained for

$$a(t) = \frac{e^t + ce^{-t}}{1+c},\ |c| \leq 1,\ c \neq -1.$$

Corollary 4.3. *Let $q \in (0,1]$ and let $c \in \mathbb{C}$ with $|c| \leq 1,\ c \neq -1$. Let $f : B \to \mathbb{C}^n$ be a normalized holomorphic mapping and let p be a complex valued holomorphic mapping on B with $p(0) = 1$. Assume that:*

$$\left\| \|z\|^2 \frac{1+2c-p(z)}{1+p(z)} I + (1-\|z\|^2)\left[\frac{p'(z)(z)}{1+p(z)} I + (Df(z))^{-1} D^2 f(z)(z,.)\right] \right\| \leq q$$

a. e. $t > 0,\ z \in B$. Then, the followings are true:

(i) If $q = 1$, the mapping f is biholomorphic on B;

(ii) If $q < 1$ and if there exist some constants $M > 0,\ k \geq 1$ and $\alpha \in [0,1)$ such that (30) and (31) hold then, f is quasiregular on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Note that for $c = 0$, Corollary 4.3 reduces to Corollary 4.1.

Another generalization of the Lewandowski's n-dimensional univalence criterion follows from Theorem 4.4 with

$$a(t) = \frac{e^t + be^{-t}}{b + e^{-2t}},\ \ |b-1| \leq 2,\ 1 \leq |2b+1|.$$

Corollary 4.4. *Let $q \in (0,1]$ and let $b \in \mathbb{C}$ with $|b-1| \leq 2,\ 1 \leq |2b+1|$. Let $f : B \to \mathbb{C}^n$ be a normalized holomorphic mapping and let p be a complex valued holomorphic mapping on B with $p(0) = 1$. Assume that:*

$$\left\| \|z\|^2 \frac{1-p(z)}{1+p(z)} I + \frac{1-\|z\|^4}{b+\|z\|^2}\left[\frac{p'(z)(z)}{1+p(z)} I + (Df(z))^{-1} D^2 f(z)(z,.)\right] + \frac{(b^2-1)\|z\|^2}{(b+\|z\|^2)^2} \right\| \leq q \frac{b + 2\|z\|^2 + b\|z\|^4}{(b+\|z\|^2)^2}$$

a. e. $t > 0,\ z \in B$. Then, the followings are true:

(i) If $q = 1$, the mapping f is biholomorphic on B;

(ii) If $q < 1$ and if there exist some constants $M > 0,\ k \geq 1$ and $\alpha \in [0,1)$ such that (30) and (31) hold then, f is quasiregular on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Note that for $b = 1$, Corollary 4.4 reduces to Corollary 4.1.

Finally we give the asymptotical case of Theorem 4.3.

Theorem 4.5. *Let $q \in (0,1)$ and let $f : \overline{B} \to \mathbb{C}^n$ be a normalized holomorphic mapping which is continuous and injective on $\overline{B}$. Let $a : [0,T] \to \mathbb{C}$ be an absolutely continuous function such that $a(0) = 1$, $\Re[a'(t)/a(t)] \leq \epsilon > 0$ a.e. on $[0,T]$ and $a'(t)/a(t)$ is bounded a.e. on $[0,T]$. Suppose that p is a complex valued holomorphic mapping on $\overline{B}$ with $p(0) = 1$. Let $\gamma > 0$. Assume that:*

$$\max_{\|z\|=e^{-\gamma t}} \left\| \frac{2\gamma}{a'(t)+\gamma a(t)} \left\{ \|z\| \frac{1-p(z)}{1+p(z)} I + (a(t) - \|z\|)\left[\frac{p'(z)(z)}{1+p(z)} I + (Df(z))^{-1} D^2 f(z)(z,.)\right]\right\} - \frac{a'(t)-\gamma a(t)}{a'(t)+\gamma a(t)} I \right\| \leq q < 1,\ a.\ e.\ 0 < t \leq T,\ z \in B.$$

If there exist some constants $M > 0,\ k \geq 1$ and $\alpha \in [0,1)$ such that the conditions(30) and (31) are satisfied then, the mapping f is quasiregular on $\overline{B}$ and extends to a quasiconformal homeomorphism of $\overline{\mathbb{R}^{2n}}$ onto itself.

Proof. The result follows from Theorem 3.3 with

$$F(u,v) = f(u) + \frac{p(u)+1}{2} Df(u)(v-u),\ u = e^{-\gamma t}z,\ v = a(t)z.$$

□

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