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A Prey-predator Model with Infection in Both Prey and Predator

S. P. Bera[a], A. Maiti[b], G. P. Samanta[c]

[a] *Department of Mathematics, U. J. Sri Siksha Niketan, Howrah-711405, INDIA*
[b] *Department of Mathematics, Darjeeling Government College, Darjeeling-734101, INDIA*
[c] *Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah-711103, INDIA*

Abstract. This paper aims to study the dynamical behaviours of a prey-predator system where both prey and predator populations are affected by diseases. A system of four differential equation has been proposed and analyzed. Stability of the equilibrium points of the model has been investigated. Computer simulations are carried out to illustrate our analytical findings. The biological implications of analytical and numerical findings are discussed critically.

1. Introduction

Ecological populations suffer from various diseases. These diseases often play significant roles in regulating the population sizes. Mathematical study of such populations has attracted attentions of both ecologists and mathematicians from several years past. As a result numerous mathematical models have been developed, and these models have become essential tools in analyzing the interaction of different populations, particularly the interaction between prey and predator.

Ever since the pioneering work of Lotka [19] and Volterra [24], a sufficient number of prey-predator models have been introduced and studied extensively in literature to describe the complex relationship between interacting Species of real ecosystems (for a detailed history, see [7, 9]). In last few decades, there has been a growing interest in the study of the effect of diseases in prey-predator systems. The influence of epidemics on predation was first studied by Anderson and May [1, 2]. They examined a modification of Lotka-Voltera prey-predator model with higher predation and no reproduction on infected prey. They established that invading disease tends to destabilize the prey-predator communities. Hadeler and Freedman [6] considered the prey-predator model in which predation is more likely on infected prey. They considered in their model that predators become infected only from infected prey by predation. Haque and Venturino [15] discussed models of diseases spreading in symbiotic communities. Numerous prey-predator models with infection in the prey population have been cultured by various researchers (interested readers might see [3–5, 10, 13, 16, 17, 20, 21, 25], and references therein).

The study of prey-predator dynamics with an infected predator has a great importance in controlling the predator population. But this area has been neglected for a long time in theoretical ecology. Recently, a few researchers have cultured some prey-predator models with infection in the predator [8, 11, 12, 14, 22].

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Email addresses: spbera@hotmail.com (S. P. Bera), alakesh_maity@hotmail.com (A. Maiti), g_p_samanta@yahoo.co.uk, gpsamanta@math.iiests.ac.in (G. P. Samanta)

There should be no denying that in real environment, it would be more relevant to consider prey-predator systems with infection in both the populations. Unfortunately, a little attention has been laid down in this area. One of the main reasons for this is the lack of mathematical machinery to handle the increasing number of differential equations involved with such modelling. Very recently, the papers [18] and [23] have tried to focus on this area. In this paper, we have considered a simple prey-predator model with disease in both prey and predator populations.

The rest of the paper is structured as follows. In section 2 we present the mathematical model with basic considerations. Boundedness and positivity of the solutions of the model are established in section 3. Section 4 deals with all the possible equilibrium points of the model and their feasibility conditions. Stability of the model at various equilibrium points is discussed in section 5. Computer simulations are carried out to illustrate our analytical findings numerically in section 6. Section 7 contains the general discussion and biological significance of our analytical findings.

2. The Mathematical Model

The model we introduce consists of two populations: the prey population and the predator population. Both the populations have two sub classes: susceptible and infected. At time T, let $S(T)$ denotes the density of the susceptible prey, and $I(T)$ denotes the density of the infected prey. The susceptible and infected predator densities are denoted by $X(T)$ and $Y(T)$, respectively. Now we discuss the basic assumptions that we have made in formulating the model.

1. In the absence of predator population and with no disease, the prey population grows logistically with intrinsic growth rate r and environmental carrying capacity $K(K > 0)$.

In presence of the disease in prey, the infected prey population contributes to the susceptible prey population growth towards the carrying capacity $K(K > 0)$.

2. Only the susceptible prey can reproduce.

3. The disease spreads among the susceptible prey when it comes in contact with the infected one. The prey, once became infected, never recovers. It will either die or will be removed by predation. The infected prey population have a disease induced death rate in excess.

4. We assume that the infected predators are unfit to be able to catch a healthy prey, and as such a healthy prey is caught by a healthy predator only. But an infected prey, being weak and more vulnerable, is available for predation by both susceptible and infected predators .

5. The disease spread over predator population by direct contact with an infected predator. An infected predator never becomes recovered or immune–it remains infected or dies out.

6. We further assume that the predator population have a natural death, whereas the infected population have a disease induce excess death rate also.

The above considerations motivate us to form the following set of four nonlinear ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dT} &= rS\left(1 - \frac{S+I}{K}\right) - a_1SI - b_1SX,\\
\frac{dI}{dT} &= a_1SI - d_1I - f_1IX - m_1IY,\\
\frac{dX}{dT} &= c_1SX + g_1IX - e_1XY - \delta_1X,\\
\frac{dY}{dT} &= e_1XY - (\delta_1 + \alpha_1)Y + n_1IY,
\end{aligned} \tag{2.1}$$

with

$$S(0) = S_0 > 0,\ I(0) = I_0 \geq 0,\ X(0) = X_0 \geq 0,\ Y(0) = Y_0 \geq 0.$$

Here the parameter a_1 is the infection rate for prey population, b_1 is the predation rate of susceptible prey by healthy predators. The infected prey population has a disease induced death d_1. f_1 and m_1 are respectively the predation rate of infected prey by susceptible and infected predators. The parameters c_1 and g_1 are the conversion rates for healthy and infected preys to healthy predator. δ_1 is the natural death rate for the predator population. The infected predator population has a disease induced death rate α_1. We have taken e_1 as the infection rate for predator population and n_1 as the conversion rate for infected prey to infected predator. We make an obvious assumption that all the parameters are positive.

The model we have just specified has thirteen parameters, which makes analysis difficult. To reduce the number of parameters and to determine which combinations of parameters control the behavior of the system, we non-dimensionalize the system (2.1) with the following scaling

$$s = \frac{S}{K}, \;\; i = \frac{I}{K}, \;\; x = \frac{X}{K}, \;\; y = \frac{Y}{K}, \;\; \text{and} \;\; t = rT.$$

Then the system (2.1) takes the form (after some simplification)

$$\begin{aligned} \frac{ds}{dt} &= s(1 - s - i) - asi - bsx, \\ \frac{di}{dt} &= asi - di - fxi - miy, \\ \frac{dx}{dt} &= csx + gix - exy - \delta x, \\ \frac{dy}{dt} &= exy - \alpha y + niy, \end{aligned} \tag{2.2}$$

where

$$a = \frac{a_1K}{r}, \;\; b = \frac{b_1K}{r}, \;\; c = \frac{c_1K}{r}, \;\; d = \frac{d_1}{r}, \;\; e = \frac{e_1K}{r},$$
$$f = \frac{f_1K}{r}, \;\; g = \frac{g_1K}{r}, \;\; m = \frac{m_1K}{r}, \;\; n = \frac{n_1K}{r}, \;\; \alpha = \frac{\delta_1 + \alpha_1}{r}, \;\; \delta = \frac{\delta_1}{r}.$$

3. Boundedness and Positivity

The following theorem ensures the boundedness of the system (2.2).

Theorem 3.1. *All solutions of the system* (2.2) *that start in* $\mathbb{R}^4_+$ *are uniformly bounded.*

Proof. Let $(s(t), i(t), x(t), y(t))$ be any solution of the system (2.2).
Since

$$\frac{ds}{dt} \leq s(1 - s),$$

we have

$$\limsup_{t\to\infty} s(t) \leq 1.$$

Let

$$W = s + i + x + y.$$

Therefore

$$\begin{aligned} \frac{dW}{dt} &\leq s - di - \delta x - \alpha y \\ &\leq 2s - RW, \;\; \text{where} \;\; R = \min\{1, d, \delta, \alpha\}. \end{aligned}$$

Hence

$$\frac{dW}{dt} + RW \leq 2s \leq 2.$$

Applying a theorem on differential inequalities, we obtain

$$0 \leq W(s,i,x,y) \leq \frac{2}{R} + \frac{W(s(0),i(0),x(0),y(0))}{e^{Rt}},$$

and for $t \to \infty$,

$$0 \leq W \leq \frac{2}{R}.$$

Thus, all the solutions of system (2.2) enter into the region

$$B = \left\{(s,i,x,y) : 0 \leq W \leq \frac{2}{R} + \epsilon, \text{ for any } \epsilon > 0\right\}.$$

Hence the theorem. □

Theorem 3.2. *All solutions of the system* (2.2) *that start in* $\mathbb{R}^4_+$ *remain positive forever.*

The proof is simple and so it is omitted.

4. Equilibrium Points and Feasibility Conditions

System (2.2) may have the following equilibrium points:

(A) The trivial equilibrium point $E_0(0,0,0,0)$
This equilibrium always exists.

(B) The axial equilibrium point $E_1(1,0,0,0)$
This disease and predator free equilibrium also exists unconditionally.

(C) The disease-free equilibrium point $E_2(\underline{s},0,\underline{x},0)$
This equilibrium exists when $c > \delta$. When this condition is satisfied, $\underline{s} = \frac{\delta}{c}$ and $\underline{x} = \frac{c-\delta}{bc}$. In terms of original parameters of the system, this condition $c > \delta$ indicates that $c_1 > \delta_1/K$. That is, if the predator is a high capacity consumer, then the disease-free equilibrium exists.

(D) The predator-free equilibrium point $E_3(\hat{s},\hat{i},0,0)$
This equilibrium exists if $a > d$. When this condition is satisfied, $\hat{s} = \frac{d}{a}$, $\hat{i} = \frac{a-d}{a(1+a)}$. In terms of the original parameters of the system, the condition $a > d$ implies $a_1K > d_1$. This indicates that predator-free equilibrium may exists if the disease induced death rate is low enough (less than the product of carrying capacity and the infection rate).

(E) The infected-predator-free equilibrium point $E_4(s',i',x',0)$
This exists when s', i', x' given by

$$\begin{aligned} s' &= \frac{g(bd+f) - \delta(f+af)}{g(ab+f) - c(f+af)}, \\ i' &= \frac{\delta(ab+f) - c(bd+f)}{g(ab+f) - c(f+af)}, \\ x' &= \frac{ag(bd+f) - gd(ab+f) + (f+af)(cd-a\delta)}{gf(ab+f) - cf(f+af)}, \end{aligned}$$

be all positive.

(F) The infected-prey-free equilibrium point $E_5(\bar{s}, 0, \bar{x}, \bar{y})$
This equilibrium exists if

$$e > b\alpha \quad \text{and} \quad ce > bc\alpha + e\delta,$$

where $\bar{s}, \bar{x}, \bar{y}$ are given by:

$$\begin{aligned} \bar{s} &= \frac{e - b\alpha}{e}, \\ \bar{x} &= \frac{\alpha}{e}, \\ \bar{y} &= \frac{ce - bc\alpha - e\delta}{e^2}. \end{aligned}$$

(G) The interior equilibrium point $E^*(s^*, i^*, x^*, y^*)$

Theorem 4.1. *The interior equilibrium point* $E^*(s^*, i^*, x^*, y^*)$ *of system* (2.2) *exists if*

$$(i) \quad |P_2| \geq \max\left\{|(1+a)P_1 + bP_3|, \left|\frac{(c + ac - g)P_1 + bcP_3}{c - \delta}\right|\right\},$$

$$(ii) \quad P_1, P_2 \text{ and } P_3 \text{ are of same sign,}$$

where P_1, P_2 *and* P_3 *are given by :*

$$\begin{aligned} P_1 &= e(ab\alpha + f\alpha + cm + de) - (bcm\alpha + me\delta + ae^2), \\ P_2 &= e(abn + fn + (1+a)cm) - (bcmn + a(1+a)e^2 + meg), \\ P_3 &= (aen + mn\delta + (1+a)cm\alpha) - (cmn + den + (1+a)ae\alpha + mg\alpha). \end{aligned}$$

When these conditions are satisfied, the values of s^*, i^*, x^*, y^* *are given by*

$$\begin{aligned} s^* &= \frac{P_2 - (1+a)P_1 - bP_3}{P_2}, \\ i^* &= \frac{P_1}{P_2}, \\ x^* &= \frac{P_3}{P_2}, \\ y^* &= \frac{(g - c - ac)P_1 + (c - \delta)P_2 - bcP_3}{eP_2}. \end{aligned}$$

5. Stability Analysis

The variational matrix $V(s, i, x, y)$ of system (2.2) at any point (s, i, x, y) is

$$V(s,i,x,y) = \begin{bmatrix} 1 - 2s - (1+a)i - bx & -(1+a)s & -bs & 0 \\ ai & as - d - fx - my & -fx & -mi \\ cx & gx & cs + gi - ey - \delta & -ex \\ 0 & ny & ey & ex + ni - \alpha \end{bmatrix}.$$

5.1. The Equilibrium Point $E_0(0, 0, 0, 0)$
At E_0, the variational matrix becomes

$$V(E_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \\ 0 & 0 & -\delta & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

The corresponding eigenvalues are $1, -d, -\delta, -\alpha$ and hence we have the following theorem.

Theorem 5.1. E_0 *is unstable.*

5.2. The Equilibrium Point $E_1(1,0,0,0)$

At E_1 the variational matrix $V(E_1)$ is given by

$$V(E_1) = \begin{bmatrix} -1 & -(1+a) & -b & 0 \\ 0 & a-d & 0 & 0 \\ 0 & 0 & c-\delta & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

The corresponding eigenvalues are $-1, a-d, c-\delta, -\alpha$. Hence we have the following theorem for local stability of E_1.

Theorem 5.2. *E_1 is locally asymptotically stable if $d > a$ and $\delta > c$.*

We have derived the following criterion for global stability of E_1.

Theorem 5.3. *If $(\delta/c) > (d/a) > 1$ and $n\delta a + \alpha a g > ncd$, then $E_1(1,0,0,0)$ is globally asymptotically stable.*

The proof is given in the appendix.

5.3. The Equilibrium Point $E_2(\underline{s}, 0, \underline{x}, 0)$

At E_2 the variational matrix $V(E_2)$ becomes

$$V(E_2) = \begin{bmatrix} -\frac{\delta}{c} & -\frac{(1+a)\delta}{c} & -\frac{b\delta}{c} & 0 \\ 0 & -\frac{(ab+f)\delta-(bd+f)c}{bc} & -\frac{f(c-\delta)}{bc} & 0 \\ -\frac{c-\delta}{b} & -\frac{g(c-\delta)}{bc} & 0 & -\frac{e(c-\delta)}{bc} \\ 0 & 0 & 0 & \frac{e(c-\delta)-bc\alpha}{bc} \end{bmatrix}.$$

If the corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ then

$$\lambda_1 = \frac{e(c-\delta) - bc\alpha}{bc},$$

and $\lambda_2, \lambda_3, \lambda_4$ are the roots of the cubic

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

where

$$\begin{aligned} A_1 &= \frac{(bd+f)c+b\delta-(ab+f)\delta}{bc}, \\ A_2 &= \frac{b^2c^2(bd+f)+fg(c-\delta)^2+b^2c\delta(c-\delta)-b\delta^2(ab+f)}{b^2c^2}, \\ A_3 &= \frac{fg\delta(c-\delta)^2-fc\delta(c-\delta)^2(1+a)-bc\delta(c-\delta)((ab+f)\delta-(bd+f)c)}{b^2c^3}. \end{aligned}$$

If $e\delta + bc\alpha > ec$, then λ_1 is negative. By Routh-Hurwitz's criterion, other eigenvalues have negative real parts if $A_1 > 0$, $A_3 > 0$ and $A_1A_2 - A_3 > 0$. Thus we have the following theorem.

Theorem 5.4. *E_2 is locally asymptotically stable if $e\delta + bc\alpha > ec$, $A_1 > 0$, $A_3 > 0$ and $A_1A_2 - A_3 > 0$.*

Remark. The existence of E_2 destabilizes E_1.

5.4. The Equilibrium Point $E_3(\hat{s}, \hat{i}, 0, 0)$

At E_3, the variational matrix $V(E_3)$ takes the form

$$V(E_3) = \begin{bmatrix} -\hat{s} & -(1+a)\hat{s} & -b\hat{s} & 0 \\ a\hat{i} & 0 & 0 & -m\hat{i} \\ 0 & 0 & c\hat{s} + g\hat{i} - \delta & 0 \\ 0 & 0 & 0 & n\hat{i} - \alpha \end{bmatrix}.$$

If the corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ then

$$\lambda_1 = \frac{n(a-d)}{a(1+a)} - \alpha,$$

$$\lambda_2 = \frac{g(a-d) + (cd - a\delta)(1+a)}{a(1+a)},$$

and

$$\lambda_{3,4} = \frac{-d \pm \sqrt{(1+4a^2)d^2 - 4a^3 d}}{2a}.$$

Theorem 5.5. *E_3 is locally asymptotically stable if $n(a-d) < a\alpha(1+a)$ and $g(a-d) + (cd - a\delta)(1+a) < 0$.*

5.5. The Equilibrium Point $E_4(s', i', x', 0)$

At E_4, the variational matrix $V(E_4)$ is given by

$$V(E_4) = \begin{bmatrix} -s' & -(1+a)s' & -bs' & 0 \\ ai' & 0 & -fx' & -mi' \\ cx' & gx' & 0 & -ex' \\ 0 & 0 & 0 & ex' + ni' - \alpha \end{bmatrix}.$$

If the corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ then

$$\lambda_1 = ex' + ni' - \alpha,$$

and $\lambda_2, \lambda_3, \lambda_4$ are the roots of the cubic

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0,$$

where

$$\begin{aligned} B_1 &= s', \\ B_2 &= bcs'x' + fgx'^2 s' + a(a+1)i's', \\ B_3 &= fgx'^2 s' + abgi's'x' - cf(1+a)x'^2 s'. \end{aligned}$$

Since $c < \delta$ for existence of $E_4(s', i', x', 0)$, we have $\lambda_1 < 0$. And hence by Routh Hurwitz's criterion, all the eigenvalues of $V(E_4)$ have negative real parts if

(*i*) $B_1 > 0$,
(*ii*) $B_3 > 0$, and
(*iii*) $B_1B_2 - B_3 > 0$.

Therefore we have the following theorem.

Theorem 5.6. *If $c < \delta$, $B_1 > 0$ $B_3 > 0$ and $B_1B_2 - B_3 > 0$, E_4 is locally asymptotically stable.*

5.6. The Equilibrium Point $E_5(\bar{s}, 0, \bar{x}, \bar{y})$

At E_5, the variational matrix $V(E_5)$ is given by

$$V(E_5) = \begin{bmatrix} -\bar{s} & -(1+a)\bar{s} & -b\bar{s} & 0 \\ 0 & a\bar{s} - d - f\bar{x} - m\bar{y} & -f\bar{x} & 0 \\ c\bar{x} & g\bar{x} & 0 & -e\bar{x} \\ 0 & n\bar{y} & e\bar{y} & 0 \end{bmatrix}.$$

The corresponding characteristic equation is given by

$$\lambda^4 + C_1\lambda^3 + C_2\lambda^2 + C_3\lambda + C_4 = 0,$$

where

$$\begin{aligned} C_1 &= \bar{s} - a\bar{s} + d + f\bar{x} - m\bar{y}, \\ C_2 &= fg\bar{x}^2 - \bar{s}(a\bar{s} - d - f\bar{x} - m\bar{y}), \\ C_3 &= fg\bar{s}\bar{x}^2 + e^2\bar{s}\bar{x}\bar{y}, \\ C_4 &= e\bar{s}\bar{x}\left\{e\bar{y}(d + f\bar{x} + m\bar{y} - a\bar{s}) - mf\bar{x}\bar{y}\right\}. \end{aligned}$$

By Routh Hurwitz's criterion, all the eigenvalues of $V(E_5)$ have negative real parts if

$$\begin{aligned} &(i) \quad C_1 > 0, \\ &(i) \quad C_3 > 0, \\ &(ii) \quad C_4 > 0, \text{ and} \\ &(iii) \quad C_1C_2C_3 > {C_3}^2 + {C_1}^2C_4. \end{aligned}$$

Therefore we have the following theorem.

Theorem 5.7. *If* $C_1 > 0,\ C_3 > 0,\ C_4 > 0$ *and* $C_1C_2C_3 > {C_3}^2 + {C_1}^2C_4$, E_5 *is locally asymptotically stable.*

Remark. The existence of E_5 destabilizes E_2.

5.7. The Equilibrium Point $E^*(s^*, i^*, x^*, y^*)$

At the interior equilibrium E^*, the variational matrix $V(E^*)$ is given by

$$V(E^*) = \begin{bmatrix} -s^* & -(1+a)s^* & -bs^* & 0 \\ ai^* & 0 & -fx^* & -mi^* \\ cx^* & gx^* & 0 & -ex^* \\ 0 & ny^* & ey^* & 0 \end{bmatrix}.$$

The corresponding characteristic equation is given by

$$\lambda^4 + D_1\lambda^3 + D_2\lambda^2 + D_3\lambda + D_4 = 0,$$

where

$$\begin{aligned} D_1 &= s^*, \\ D_2 &= e^2x^*y^* + fgx^{*2} + mni^*y^* + a(a+1)s^*i^* + bcs^*x^*, \\ D_3 &= e^2s^*x^*y^* + fgs^*x^{*2} + mns^*i^*y^* - enfx^{*2}y^* + megi^*x^*y^* + abgs^*i^*x^* - cf(a+1)s^*x^*, \\ D_4 &= -enfs^*x^{*2}y^* + \left[emg + a(a+1)e - aben + bcmn - cme(a+1)\right]s^*i^*x^*y^*. \end{aligned}$$

By Routh Hurwitz's criterion, all the eigenvalues of $V(E^*)$ have negative real parts if

(i) $D_1 > 0$,
(i) $D_3 > 0$,
(ii) $D_4 > 0$, and
(iii) $D_1D_2D_3 > {D_3}^2 + {D_1}^2D_4$.

Therefore we have the following theorem.

Theorem 5.8. *If $D_1 > 0$, $D_3 > 0$, $D_4 > 0$ and $D_1D_2D_3 > {D_3}^2 + {D_1}^2D_4$, E^* is locally asymptotically stable.*

We have derived the following criterion for global stability of E^*.

Theorem 5.9. *If E^* exists with $(g/f) = (n/m) = c(1 + a)/(ab)$, then E^* is globally asymptotically stable.*

The proof is given in the appendix.

6. Numerical Simulation

Numerical simulation are equally important beside the analytical findings to verify them. In this section, we present computer simulation of different solutions of the system (2.2) using MATLAB.

First we take the parameters of the system as $a = 0.4$, $b = 1$, $c = 0.1$, $d = 0.5$, $e = 0.2$, $f = 0.2$, $g = 0.15$, $m = 1.5$, $n = 1.4$, $\alpha = 0.5$, $\delta = 0.2$. Then the conditions of Theorem 5.2 are satisfied and consequently $E_1(1, 0, 0, 0)$ is locally asymptotically stable (see Fig. 1).

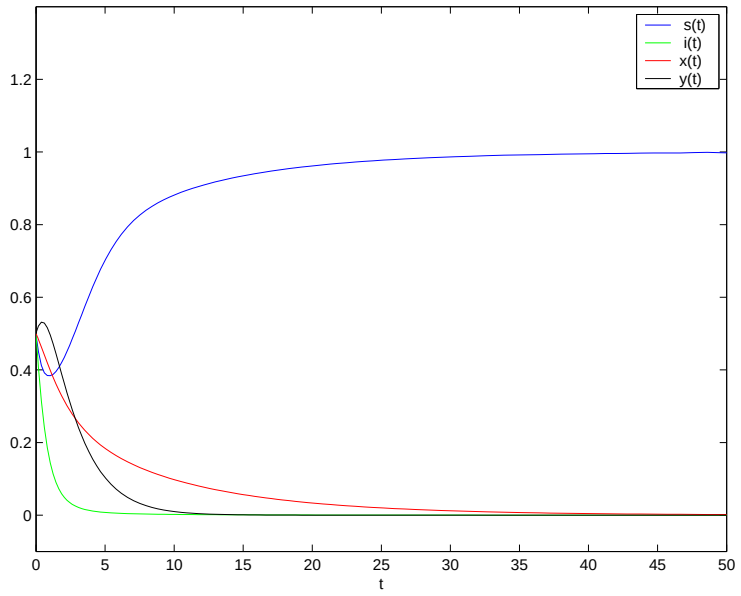

Fig.1. *For $a = 0.4$, $b = 1$, $c = 0.1$, $d = 0.5$, $e = 0.2$, $f = 0.2$, $g = 0.15$, $m = 1.5$, $n = 1.4$, $\alpha = 0.5$, $\delta = 0.2$ and $(s(0), i(0), x(0), y(0)) = (0.5, 0.5, 0.5, 0.5)$, $E_1(1, 0, 0, 0)$ is locally asymptotically stable.*

If we take the parameters of the system as $a = 0.4$, $b = 1$, $c = 0.5$, $d = 0.5$, $e = 0.2$, $f = 0.2$, $g = 0.15$, $m = 1.5$, $n = 1.4$, $\alpha = 0.5$, $\delta = 0.4$, then the conditions of Theorem 5.4 are satisfied and consequently $E_2(0.8, 0, 0.2, 0)$ is locally asymptotically stable. The corresponding behaviour of s, i, x, y with t is depicted in Fig. 2.

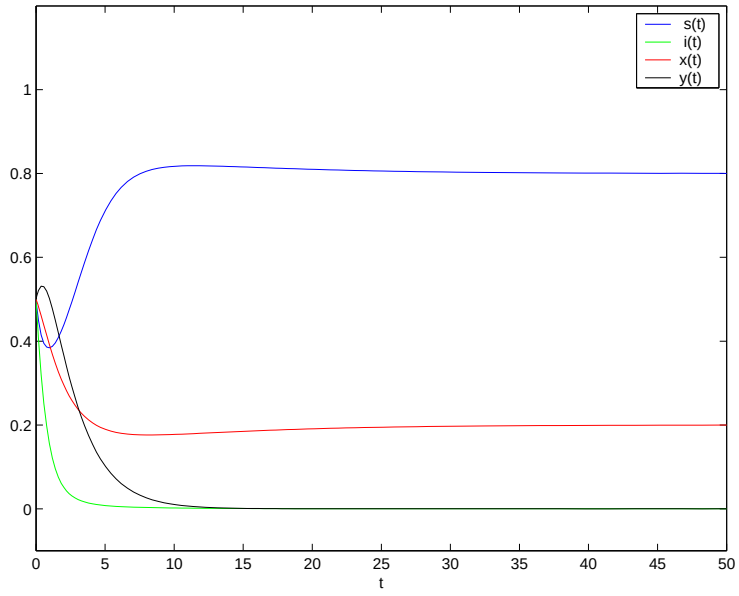

Fig.2. *For* $a = 0.4$, $b = 1$, $c = 0.5$, $d = 0.5$, $e = 0.2$, $f = 0.2$, $g = 0.15$, $m = 1.5$, $n = 1.4$, $\alpha = 0.5$, $\delta = 0.4$ *and* $(s(0), i(0), x(0), y(0)) = (0.5, 0.5, 0.5, 0.5)$, $E_2(0.8, 0, 0.2, 0)$ *is locally asymptotically stable.*

Now we take the parameters of the system as $a = 0.4$, $b = 1$, $c = 0.1$, $d = 0.1$, $e = 0.2$, $f = 0.2$, $g = 0.15$, $m = 1.5$, $n = 1.4$, $\alpha = 0.8$, $\delta = 0.4$. Then the conditions of Theorem 5.5 are satisfied and consequently $E_3(0.25, 0.5357, 0, 0)$ is locally asymptotically stable. In this case, the behaviour of s, i, x, y with t is depicted in Fig. 3.

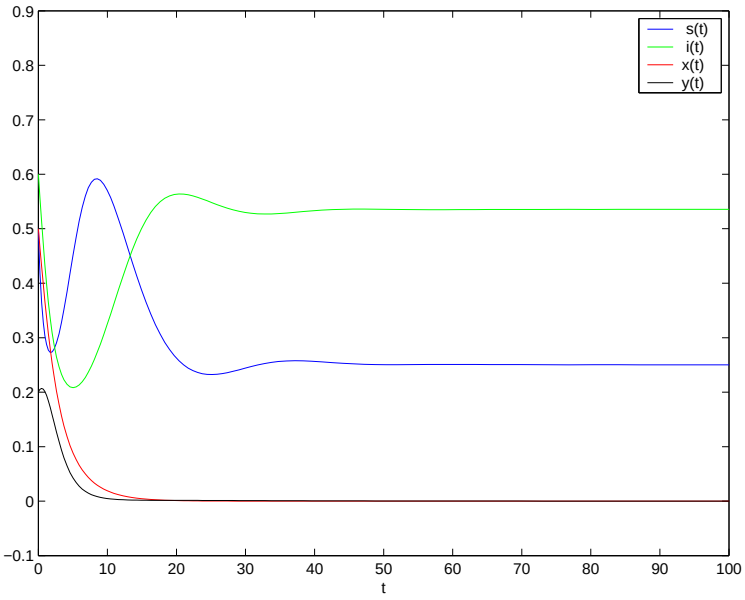

Fig.3. *If* $a = 0.4$, $b = 1$, $c = 0.1$, $d = 0.1$, $e = 0.2$, $f = 0.2$, $g = 0.15$, $m = 1.5$, $n = 1.4$, $\alpha = 0.8$, $\delta = 0.4$ *and* $(s(0), i(0), x(0), y(0)) = (0.5, 0.6, 0.5, 0.2)$ *then* $E_3(0.25, 0.5357, 0, 0)$ *is locally asymptotically stable.*

Let us take the parameters of the system as $a = 0.4$, $b = 1$, $c = 0.1$, $d = 0.1$, $e = 0.2$, $f = 0.6$, $g = 0.6$, $m = 1.5$, $n = 1.4$, $\alpha = 1$, $\delta = 0.2$. Then the conditions of Theorem 5.6 are satisfied and consequently $E_4(0.4884, 0.2519, 0.1589, 0)$ is locally asymptotically stable (see Fig. 4).

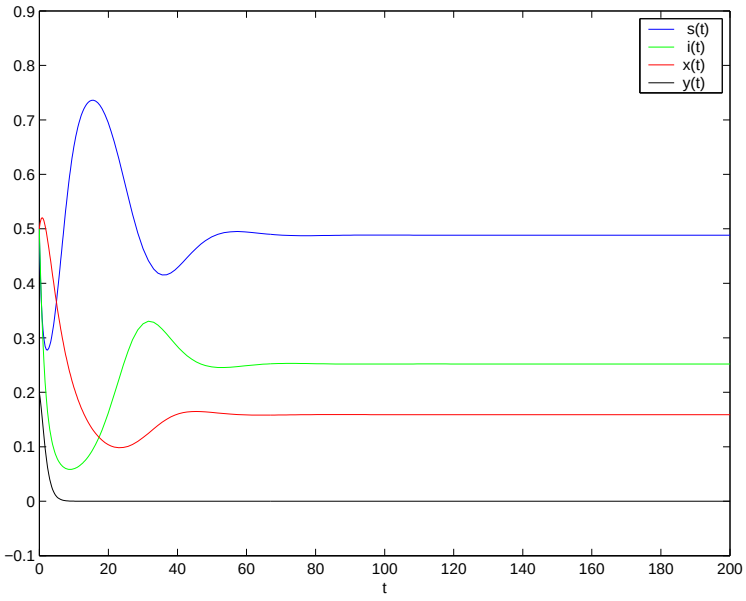

Fig.4. *Here* $a = 0.4$, $b = 1$, $c = 0.1$, $d = 0.1$, $e = 0.2$, $f = 0.6$, $g = 0.6$, $m = 1.5$, $n = 1.4$, $\alpha = 1$, $\delta = 0.2$ *and* $(s(0), i(0), x(0), y(0)) = (0.5, 0.5, 0.5, 0.2)$. *The figure shows that* $E_4(0.4884, 0.2519, 0.1589, 0)$ *is locally asymptotically stable.*

If we take the parameters of the system as $a = 0.4$, $b = 1$, $c = 1$, $d = 0.1$, $e = 1.2$, $f = 0.6$, $g = 0.4$, $m = 1.5$, $n = 1.4$, $\alpha = 0.3$, $\delta = 0.2$, then the conditions of Theorem 5.7 are satisfied and consequently $E_5(0.75, 0, 0.25, 0.4583)$ is locally asymptotically stable (see Fig. 5).

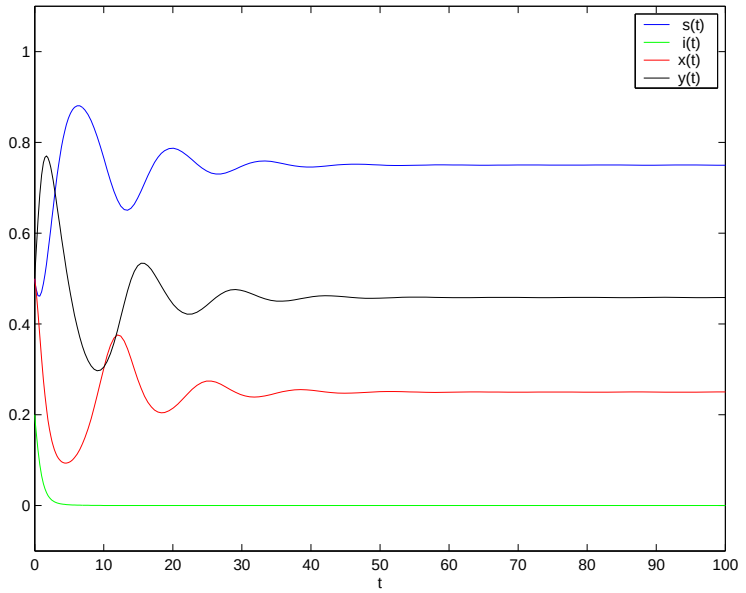

Fig.5. *Here* $a = 0.4$, $b = 1$, $c = 1$, $d = 0.1$, $e = 1.2$, $f = 0.6$, $g = 0.4$, $m = 1.5$, $n = 1.4$, $\alpha = 0.3$, $\delta = 0.2$ *and* $(s(0), i(0), x(0), y(0)) = (0.5, 0.2, 0.5, 0.5)$. *The figure shows that* $E_5(0.75, 0, 0.25, 0.4583)$ *is locally asymptotically stable.*

Finally, we take the parameters as $a = 1.9$, $b = 1.2$, $c = 0.8$, $d = 0.2$, $e = 0.2$, $f = 0.2$, $g = 0.15$, $m = 1.5$, $n = 1.4$, $\alpha = 0.3$, $\delta = 0.2$. Then conditions of Theorem 4.1 is satisfied, and hence $E^*(0.2568, 0.1921, 0.1550, 0.1713)$

exists. Also for such choices of parameters, the conditions of Theorem 5.8 are satisfied. Consequently, E^* is locally asymptotically stable. The stable behaviour of s, i, x, y with t is presented in Fig. 6.

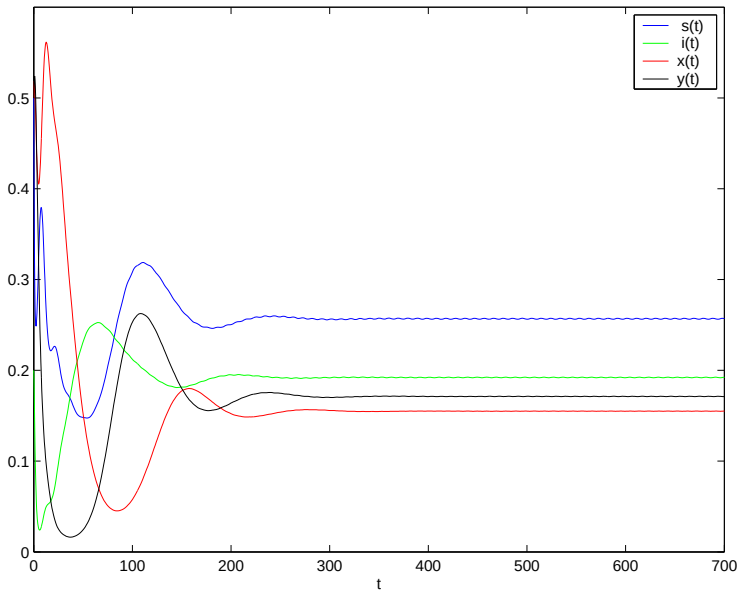

Fig.6. *Here* $a = 1.9$, $b = 1.2$, $c = 0.8$, $d = 0.2$, $e = 0.2$, $f = 0.2$, $g = 0.15$, $m = 1.5$, $n = 1.4$, $\alpha = 0.3$, $\delta = 0.2$ *and* $s(0) = 0.5$, $i(0) = 0.2$, $x(0) = 0.5$, $y(0) = 0.5$. *It shows that* E^* *is locally asymptotically stable and* s, i, x, y *approach their equilibrium values* $s^* = 0.2568$, $i^* = 0.1921$, $x^* = 0.1550$, $y^* = 0.1713$ *in finite time.*

7. Concluding Remarks

In the study of population dynamics, the role of ecological interactions such as competition, predation etc., are well recognized. On the other hand, parasite infection in the prey and predator populations is an issue which cannot be ignored, and also it has an important role in regulating population sizes. There are numerous field evidences where the prey and the predator are infected by parasites. It is observed that parasites affect the internal mechanism of the hosts. Parasites weaken the survival and fecundity of infected host. Sometimes parasites may also influence aspects of their hosts' sexual life cycle. For example, they may reduce the hosts' likelihood of finding mates or may increase or decrease the frequency with which a female produces ephippia and male offspring. Many parasites are able to redirect the metabolic pathways of the host to favor their own growth. Thus, reality dictates that, we should be concerned with prey-predator systems, where both the populations are disease affected.

In this paper, we have formulated a prey-predator model with disease in both the populations. In almost all the models with diseased prey, it is assumed that the predators eat only the infected preys (as they are weak and more vulnerable). But 'the susceptible preys are completely out of danger' is an oversimplification. In our model, we have made more realistic assumptions: (*i*) the susceptible predators are capable of catching both the susceptible and infected preys, and (*ii*) infected predators (being weak and having disturbed internal mechanism) can manage only infected preys (due to same reasons). The details of the construction of the model is presented in section 2. The results of section 3 indicates that the result is biologically well behaved. Stability of all the equilibrium points are cultured and validated by computer simulations.

The model may be used in many ways. For natural prey-predator systems, the equilibrium E_2 is very important. From the criterion for its existence, it is evident that if the predator is an high capacity consumer, then the existence of E_2 is guaranteed. The stability criterion of E_2 may be implemented for ecological balance in nature. In case when the diseased prey and predator can not be washed out, a rational use of

the stability criterion of E^* may be useful for ecological balance. In this case, the parameters of the system should be regulated in such a way that stability criterion of E^* is satisfied but i^* and y^* remain low enough.

If the model is used for pest management (by the methods of biological control, microbial control or IPM), the stability criterion of E_5 would be very useful. If $E_2(\underline{s}, 0, \underline{x}, 0)$ and $E_5(\bar{s}, 0, \bar{x}, \bar{y})$ both exist, it is interesting to notice that $\bar{s} - \underline{s} > 0$. Thus, for plant-pest interactions, infection in the pest enhances plant fitness. This result is in good agreement with many experimental evidences. Also this is interesting to notice that the existence of E_5 destabilizes E_2.

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Appendix

Proof of Theorem 5.3. We have

$$\limsup_{t\to\infty} s(t) \leq 1.$$

So, for $0 < \epsilon < (d-a)/a$, $\exists\ t_\epsilon > 0$ such that $s(t) < 1+\epsilon$, $\forall\ t > t_\epsilon$.
Then, for $t > t_\epsilon$, we have

$$\frac{di}{dt} < a(1+\epsilon)i - di < 0,$$

which implies that

$$\lim_{t\to\infty} i(t) = 0.$$

Therefore, given ξ satisfying $0 < \xi < (\delta a - cd)/(ag)$, $\exists\, t_\xi > 0$ such that $i(t) < \xi$, $\forall\, t > t_\xi$.
Now, for $t > T' = \max\{t_\epsilon, t_\xi\}$, we have

$$\frac{dx}{dt} < c\left(1 + \frac{d-a}{a}\right)x + g\xi x - \delta x.$$

$$\Longrightarrow \qquad \frac{dx}{dt} < \frac{1}{a}(cd - \delta a + ag\xi)x < 0.$$

This gives

$$\lim_{t\to\infty} x(t) = 0.$$

Consequently, given $\eta > 0$ satisfying $0 < \eta < (n\delta a + \alpha ag - ncd)/(ag)$ $\exists\, t_\eta > 0$ such that $x(t) < \eta$, $\forall\, t > t_\eta$.
Then, for $t > T'' = \max\{T', t_\eta\}$, we have

$$\frac{dy}{dt} < e\eta y - \alpha y - \frac{n(\delta a - cd)}{ag}y.$$

$$\Longrightarrow \qquad \frac{dy}{dt} < \frac{1}{ag}(ncd - n\delta a - \alpha ag + eag\eta)y < 0.$$

This implies that

$$\lim_{t\to\infty} y(t) = 0.$$

Finally, for $t > T''$, we have

$$\frac{ds}{dt} > s(1 - s - a\xi - b\eta),$$

which implies that

$$\liminf_{t\to\infty} s(t) \geq 1 - (a\xi + b\eta).$$

Since ξ, η are arbitrary, we have

$$\liminf_{t\to\infty} s(t) \geq 1.$$

We already have

$$\limsup_{t\to\infty} s(t) \leq 1.$$

Therefore,

$$\lim_{t\to\infty} s(t) = 1.$$

□

Proof of Theorem 5.9. Let us consider the following positive definite function about E^*:

$$\begin{aligned} V(s,i,x,y) &= \left(s - s^* - s^* \ln\frac{s}{s^*}\right) + L\left(i - i^* - i^* \ln\frac{i}{i^*}\right) \\ &\quad + M\left(x - x^* - x^* \ln\frac{x}{x^*}\right) + N\left(y - y^* - y^* \ln\frac{y}{y^*}\right), \end{aligned}$$

where L, M, N are positive constants to be specified later on.
Differentiating V with respect to t along the solution of (2.2), a little algebraic manipulation yields

$$\begin{aligned}\frac{dV}{dt} &= -(s-s^*)^2 + (s-s^*)(i-i^*)(aL-1-a) + (s-s^*)(x-x^*)(cM-b)\\ &\quad +(i-i^*)(x-x^*)(gM-fL) + (i-i^*)(y-y^*)(nN-mL)\\ &\quad +(x-x^*)(y-y^*)(eN-eM).\end{aligned}$$

We choose $L = (1+a)/a$, $M = N = b/c$. Then using the given condition, we see that dV/dt is negative definite. Consequently, V is a Lyapunov function and the theorem is established. □