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Alternative Proofs of Some Classical Tauberian Theorems for The Weighted Mean Method of Integrals

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Abstract. Let $0 \neq p(x)$ be a nondecreasing real valued differentiable function on $[0, \infty)$ such that $p(0) = 0$ and $p(x) \to \infty$ as $x \to \infty$. Given a real valued function $f(x)$ which is continuous on $[0, \infty)$ and

$$s(x) = \int_0^x f(t)dt.$$

We define the weighted mean of $s(x)$ as

$$\sigma_p(x) = \frac{1}{p(x)} \int_0^x p'(t)s(t)dt,$$

where $p'(t)$ is derivative of $p(t)$. It is known that if the limit $\lim_{x\to\infty} s(x) = s$ exists, then $\lim_{x\to\infty} \sigma_p(x) = s$ also exists. However, the converse is not always true. Adding some suitable conditions to existence of $\lim_{x\to\infty} \sigma_p(x)$ which are called Tauberian conditions may imply convergence of the integral $\int_0^\infty f(t)dt$.

In this work, we give some classical type Tauberian theorems to retrieve convergence of $s(x)$ out of weighted mean integrability of $s(x)$ with some Tauberian conditions.

1. Introduction

Let $0 \neq p(x)$ be a nondecreasing real valued differentiable function on $[0, \infty)$ such that $p(0) = 0$ and $p(x) \to \infty$ as $x \to \infty$. Given a real valued continuous function f on $[0, \infty)$ and $s(x) = \int_0^x f(t)dt$. The weighted mean of $s(x)$ is defined by

$$\sigma_p(x) = \frac{1}{p(x)} \int_0^x s(t)dp(t) = \frac{1}{p(x)} \int_0^x p'(t)s(t)dt.$$

The integral

$$\int_0^\infty f(t)dt$$

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is said to be integrable by weighted mean method determined by the function $p(x)$, in short; $(\overline{N}, p)$ integrable to a finite number s if

$$\lim_{x\to\infty} \sigma_p(x) = s. \tag{1}$$

If $p(x) = x$ in the definition, then the $(\overline{N}, p)$ integrability method reduces to Cesàro integrability method. If the integral

$$\int_0^\infty f(t)dt = s \tag{2}$$

exists, then limit (1) also exists. However, the converse is not always true. For example, $\lim_{x\to\infty} \int_0^x \cos t dt$ does not exist. Also, by a special case choosing $p(x) = x^2$, from

$$\begin{aligned}\sigma_p(x) = \frac{1}{p(x)}\int_0^x s(t)dp(t) &= \frac{1}{p(x)}\int_0^x \Big(\int_0^t f(u)du\Big)dp(t)\\ &= \frac{1}{p(x)}\int_0^x f(u)\Big(\int_u^x dp(t)\Big)du\\ &= \frac{1}{p(x)}\int_0^x (p(x)-p(u))f(u)du\\ &= \int_0^x (1-\frac{p(t)}{p(x)})f(t)dt\end{aligned}$$

it follows that

$$\lim_{x\to\infty} \sigma_p(x) = \lim_{x\to\infty} \int_0^x (1-\frac{t^2}{x^2})\cos t dt = 0.$$

Notice that (1) may imply (2) by adding some suitable conditions on $s(x)$. Such a condition is called a Tauberian condition and resulting theorem is said to be a Tauberian theorem.

The weighted De la Vallée Poussin means of $s(x)$ are defined by

$$\tau_p^>(x) = \frac{1}{p(\lambda x)-p(x)}\int_x^{\lambda x} p'(t)s(t)dt$$

for $\lambda > 1$, and

$$\tau_p^<(x) = \frac{1}{p(x)-p(\lambda x)}\int_{\lambda x}^{x} p'(t)s(t)dt$$

for $0 < \lambda < 1$.

The concept of slowly decreasing for a sequence of real numbers was introduced by Schmidt [9]. Similarly, we can define for a real function.

A function $s(x)$ is said to be slowly decreasing if

$$\lim_{\lambda\to 1^+} \liminf_{x\to\infty} \min_{x\le t\le \lambda x} (s(t)-s(x)) \ge 0, \tag{3}$$

for $\lambda > 1$. The condition (3) can be equivalently reformulated as follows:

$$\lim_{\lambda\to 1^-} \liminf_{x\to\infty} \min_{\lambda x\le t\le x} (s(x)-s(t)) \ge 0, \tag{4}$$

for $0 < \lambda < 1$.

If the functions $s(x)$ and $-s(x)$ are slowly decreasing, then $s(x)$ is slowly oscillating. An equivalent definition of slow oscillation is given as follows:

A real valued function $s(x)$ is slowly oscillating [1] if

$$\lim_{\lambda\to 1^+}\limsup_{x\to\infty}\max_{x\le t\le \lambda x}|s(t)-s(x)|=0, \tag{5}$$

for $\lambda > 1$.

In [1–4, 7], a number of authors presented some Tauberian theorems for Cesàro integrability method. Also, Çanak and Totur [8] obtained a Tauberian condition, known as the Landau's condition $\frac{p(x)}{p'(x)}f(x) = O(1)$ (see [6]), for weighted mean integrability order α, for some $\alpha > -1$.

In this paper, we establish that one-sided boundedness of the function $\frac{p(x)}{p'(x)}f(x)$ is a Tauberian condition for weighted mean integrability. Furthermore, we prove that slow decrease of $s(x)$ is a Tauber condition for weighted mean integrability.

2. Main Results

The results are some classical type Tauberian theorems for the weighted mean method of integrals.

Theorem 2.1. *Let*

$$\liminf_{x\to\infty}\frac{p(\lambda x)}{p(x)} > 1, \textit{ for } \lambda > 1, \tag{6}$$

and

$$\limsup_{x\to\infty}\frac{p(x)}{p(\lambda x)} > 1, \textit{ for } 0 < \lambda < 1. \tag{7}$$

If $\int_0^\infty f(t)dt$ is $(\overline{N}, p)$ integrable to s and

$$\frac{p(x)}{p'(x)}f(x) \ge -C,$$

for some $C \ge 0$ and enough large x, then the integral $\int_0^\infty f(t)dt$ converges to s.

Theorem 2.1 is a classical type Tauberian theorem known as the Hardy Littlewood's Tauberian theorem [5]. A special case of Theorem 2.1 can be obtained by choosing $p(x) = x$ as follows:

Corollary 2.2. *If $\int_0^\infty f(t)dt$ be Cesàro integrable to s. If $xf(x) \ge -C$ for some $C \ge 0$ and enough large x, then the integral $\int_0^\infty f(t)dt$ converges to s.*

Corollary 2.2 is given by Çanak and Totur [3].

The following theorem is a version of the generalized Littlewood theorem [9] for real functions.

Theorem 2.3. *Let the conditions (6) and (7) be satisfied. If $\int_0^\infty f(t)dt$ is $(\overline{N}, p)$ integrable to s and $s(x)$ is slowly decreasing, then the integral $\int_0^\infty f(t)dt$ converges to s.*

An obvious corollary of Theorem 2.3 is represented as follows:

Corollary 2.4. *Let the conditions (6) and (7) be satisfied. If $\int_0^\infty f(t)dt$ is $(\overline{N}, p)$ integrable to s and $s(x)$ is slowly oscillating, then the integral $\int_0^\infty f(t)dt$ converges to s.*

A special case of Theorem 2.3 can be obtained by choosing $p(x) = x$.

Corollary 2.5. *If $\int_0^\infty f(t)dt$ is Cesàro integrable to s and $s(x)$ is slowly oscillating, then the integral $\int_0^\infty f(t)dt$ converges to s.*

Corollary 2.5 is given by Çanak and Totur [3].

3. Proofs

We need the following lemma to be used in the proofs of main theorems.

Lemma 3.1. *(i) For* $\lambda > 1$,

$$s(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)}(\sigma_p(\lambda x) - \sigma_p(x)) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)(s(t) - s(x))dt$$

(ii) For $0 < \lambda < 1$,

$$s(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(x) - p(\lambda x)}(\sigma_p(x) - \sigma_p(\lambda x)) + \frac{1}{p(x) - p(\lambda x)} \int_{\lambda x}^{x} p'(t)(s(x) - s(t))dt$$

Proof. (i) From the definition of weighted de la Vallée Poussin means of $s(x)$, we have

$$s(x) = \tau_p^{>}(x) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)(s(t) - s(x))dt. \tag{8}$$

Substracting $\sigma_p(x)$ from the identity (8), we get

$$s(x) - \sigma_p(x) = \tau_p^{>}(x) - \sigma_p(x) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)(s(t) - s(x))dt. \tag{9}$$

Also $\tau_p^{>}(x)$ can be written as

$$\begin{aligned}
\tau_p^{>}(x) &= \frac{1}{p(\lambda x) - p(x)} \left(\int_0^{\lambda x} p'(t)s(t)dt - \int_0^{x} p'(t)s(t)dt \right) \\
&= \frac{1}{p(\lambda x) - p(x)}(\sigma_p(\lambda x)p(\lambda x) - \sigma_p(x)p(x)) \\
&= \frac{p(\lambda x)}{p(\lambda x) - p(x)}\sigma_p(\lambda x) - \frac{p(x)}{p(\lambda x) - p(x)}\sigma_p(x).
\end{aligned}$$

Therefore, we have

$$\tau_p^{>}(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)}\sigma_p(\lambda x) - \left(\frac{p(\lambda x)}{p(\lambda x) - p(x)} - 1 \right)\sigma_p(x).$$

Substracting $\sigma_p(x)$ from the last identity, we get

$$\tau_p^{>}(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)}\sigma_p(\lambda x) - \frac{p(\lambda x)}{p(\lambda x) - p(x)}\sigma_p(x)$$

Writing last identity in (9), we obtain

$$s(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)}(\sigma_p(\lambda x) - \sigma_p(x)) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)(s(t) - s(x))dt.$$

This completes the proof.

(ii) The proof of Lemma 3.1(ii) is similar to that of Lemma 3.1(i). □

Proof of Theorem 2.1
Suppose that $\frac{p(x)}{p'(x)}f(x) \geq -C$ for some $C \geq 0$. Then, we obtain $-s'(x) \leq C\frac{p'(x)}{p(x)}$ for all x. From Lemma 3.1 (i), we have

$$\begin{aligned} s(x)-\sigma_p(x) &= \frac{p(\lambda x)}{p(\lambda x)-p(x)}(\sigma_p(\lambda x)-\sigma_p(x)) - \frac{1}{p(\lambda x)-p(x)}\int_x^{\lambda x} p'(t)(s(t)-s(x))dt \\ &= \frac{p(\lambda x)}{p(\lambda x)-p(x)}(\sigma_p(\lambda x)-\sigma_p(x)) - \frac{1}{p(\lambda x)-p(x)}\int_x^{\lambda x}\left(\int_x^t s'(z)dz\right)p'(t)dt \\ &\leq \frac{p(\lambda x)}{p(\lambda x)-p(x)}(\sigma_p(\lambda x)-\sigma_p(x)) + \frac{1}{p(\lambda x)-p(x)}\int_x^{\lambda x}\left(\int_x^t C\frac{p'(z)}{p(z)}dz\right)p'(t)dt \\ &= \frac{p(\lambda x)}{p(\lambda x)-p(x)}(\sigma_p(\lambda x)-\sigma_p(x)) + \frac{C}{p(\lambda x)-p(x)}\int_x^{\lambda x}\log\frac{p(t)}{p(x)}p'(t)dt \\ &\leq \frac{p(\lambda x)}{p(\lambda x)-p(x)}(\sigma_p(\lambda x)-\sigma_p(x)) + C\log\frac{p(\lambda x)}{p(x)}, \end{aligned}$$

for $\lambda > 1$.

After taking lim sup of both sides as $x \to \infty$, we obtain

$$\begin{aligned} \limsup_{x\to\infty}\left(s(x)-\sigma_p(x)\right) &\leq \limsup_{x\to\infty}\left(\frac{p(\lambda x)}{p(\lambda x)-p(x)}\left(\sigma_p(\lambda x)-\sigma_p(x)\right) + C\log\frac{p(\lambda x)}{p(x)}\right) \\ &\leq \limsup_{x\to\infty}\frac{p(\lambda x)}{p(\lambda x)-p(x)}\limsup_{x\to\infty}\left(\sigma_p(\lambda x)-\sigma_p(x)\right) + \limsup_{x\to\infty}\left(C\log\frac{p(\lambda x)}{p(x)}\right). \end{aligned}$$

Since $s(x)$ is weighted mean integrable to s, we have $\sigma_p(x) \to s$ as $x \to \infty$. By the condition (6), we get

$$0 \leq \limsup_{x\to\infty}\frac{p(\lambda x)}{p(\lambda x)-p(x)} \leq 1 + (\liminf\frac{p(\lambda x)}{p(x)} - 1)^{-1} < \infty.$$

Therefore the first term on the right-hand side of the inequality above vanishes and we obtain

$$\limsup_{x\to\infty}\left(s(x)-\sigma_p(\lambda x)\right) \leq \limsup_{x\to\infty}\left(C\log\frac{p(\lambda x)}{p(x)}\right).$$

for some $C > 0$. After taking the limit of both sides as $\lambda \to 1^+$, we get

$$\limsup_{x\to\infty}\left(s(x)-\sigma_p(x)\right) \leq 0. \tag{10}$$

From Lemma 3.1 (ii) and the hypothesis $-s'(x) \leq \frac{Cp'(x)}{p(x)}$ for all x, we have

$$\begin{aligned} s(x)-\sigma_p(x) &= \frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x)) + \frac{1}{p(x)-p(\lambda x)}\int_{\lambda x}^{x} p'(t)(s(x)-s(t))dt \\ &= \frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x)) + \frac{1}{p(\lambda x)-p(x)}\int_x^{\lambda x}\left(\int_x^t s'(z)dz\right)p'(t)dt \\ &\geq \frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x)) - \frac{1}{p(\lambda x)-p(x)}\int_x^{\lambda x}\left(\int_x^t \frac{Cp'(z)}{p(z)}dz\right)p'(t)dt \\ &= \frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x)) - \frac{C}{p(\lambda x)-p(x)}\int_x^{\lambda x}\log\frac{p(t)}{p(x)}p'(t)dt \\ &\geq \frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x)) - C\log\frac{p(\lambda x)}{p(x)}. \end{aligned}$$

After taking lim inf of both sides as $x \to \infty$, we have

$$\begin{aligned}\liminf_{x\to\infty}\Big(s(x)-\sigma_p(x)\Big) &\geq \liminf_{x\to\infty}\left(\frac{p(\lambda x)}{p(x)-p(\lambda x)}\Big(\sigma_p(x)-\sigma_p(\lambda x)\Big)-C\log\frac{p(\lambda x)}{p(x)}\right)\\ &\geq \liminf_{x\to\infty}\frac{p(\lambda x)}{p(x)-p(\lambda x)}\liminf_{x\to\infty}\Big(\sigma_p(x)-\sigma_p(\lambda x)\Big)+\liminf_{x\to\infty}\left(-C\log\frac{p(\lambda x)}{p(x)}\right)\end{aligned}$$

By the condition (7), we have

$$0 \leq \liminf_{x\to\infty}\frac{p(\lambda x)}{p(x)-p(\lambda x)} = (\limsup\frac{p(x)}{p(\lambda x)}-1)^{-1} < \infty.$$

From $\sigma_p(x) \to s$ as $x \to \infty$, the first term on the right-hand side of the equality above vanishes and we obtain

$$\liminf_{x\to\infty}\Big(s(x)-\sigma_p(\lambda x)\Big) \geq \liminf_{x\to\infty}\left(-C\log\frac{p(\lambda x)}{p(x)}\right).$$

for some $C > 0$. After taking the limit of both sides as $\lambda \to 1^-$, we get

$$\liminf_{x\to\infty}\Big(s(x)-\sigma_p(x)\Big) \geq 0. \tag{11}$$

From (10) and (11), we obtain $\lim_{x\to\infty} s(x) = \lim_{x\to\infty}\sigma_p(x)$. □

Proof of Theorem 2.3

Let $s(x)$ be slowly decreasing. By Lemma 3.1 (i), we have

$$\begin{aligned} s(x)-\sigma_p(x) &= \frac{p(\lambda x)}{p(\lambda x)-p(x)}(\sigma_p(\lambda x)-\sigma_p(x)) - \frac{1}{p(\lambda x)-p(x)}\int_x^{\lambda x} p'(t)(s(t)-s(x))dt\\ &\leq \frac{p(\lambda x)}{p(\lambda x)-p(x)}(\sigma_p(\lambda x)-\sigma_p(x)) - \frac{1}{p(\lambda x)-p(x)}\int_x^{\lambda x} p'(t)\min_{x\leq t\leq \lambda x}(s(t)-s(x))dt\\ &\leq \frac{p(\lambda x)}{p(\lambda x)-p(x)}(\sigma_p(\lambda x)-\sigma_p(x)) - \min_{x\leq t\leq \lambda x}(s(t)-s(x))\end{aligned}$$

After taking lim sup of both sides as $x \to \infty$, we have

$$\begin{aligned}\limsup_{x\to\infty}\Big(s(x)-\sigma_p(x)\Big) &\leq \limsup_{x\to\infty}\left(\frac{p(\lambda x)}{p(\lambda x)-p(x)}(\sigma_p(\lambda x)-\sigma_p(x)) - \min_{x\leq t\leq \lambda x}(s(t)-s(x))\right)\\ &\leq \limsup_{x\to\infty}\frac{p(\lambda x)}{p(\lambda x)-p(x)}\limsup_{x\to\infty}(\sigma_p(\lambda x)-\sigma_p(x)) + \limsup_{x\to\infty}\Big(-\min_{x\leq t\leq \lambda x}(s(t)-s(x))\Big)\end{aligned}$$

Since $\sigma_p(x) \to s$ as $x \to \infty$, by the condition (6), the first term on the right-hand side of the equality above vanishes and we obtain

$$\limsup_{x\to\infty}\Big(s(x)-\sigma_p(x)\Big) \leq -\liminf_{x\to\infty}\min_{x\leq t\leq \lambda x}(s(t)-s(x))$$

After taking the limit of both sides as $\lambda \to 1^+$, we get

$$\limsup_{x\to\infty}\Big(s(x)-\sigma_p(x)\Big) \leq 0. \tag{12}$$

On the other hand, from Lemma 3.1 (ii), we have

$$\begin{aligned} s(x)-\sigma_p(x) &= \frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x)) + \frac{1}{p(x)-p(\lambda x)}\int_{\lambda x}^{x} p'(t)(s(x)-s(t))dt\\ &\geq \frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x)) + \frac{1}{p(x)-p(\lambda x)}\int_{\lambda x}^{x} p'(t)\min_{\lambda x\leq t\leq x}(s(x)-s(t))dt\\ &\geq \frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x)) + \min_{\lambda x\leq t\leq x}(s(x)-s(t))\end{aligned}$$

After taking lim inf of both sides as $x \to \infty$, we have

$$\begin{aligned}\liminf_{x\to\infty}\Big(s(x)-\sigma_p(x)\Big) &\geq \liminf_{x\to\infty}\left(\frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x))+\min_{\lambda x\leq t\leq x}(s(x)-s(t))\right)\\ &\geq \liminf_{x\to\infty}\left(\frac{p(\lambda x)}{p(x)-p(\lambda x)}(\sigma_p(x)-\sigma_p(\lambda x))\right)+\liminf_{x\to\infty}\left(\min_{\lambda x\leq t\leq x}(s(x)-s(t))\right)\end{aligned}$$

Since $\sigma_p(x) \to s$ as $x \to \infty$, by the condition (7), the first term on the right-hand side of the equality above vanishes and we obtain

$$\liminf_{x\to\infty}\Big(s(x)-\sigma_p(x)\Big) \geq \liminf_{x\to\infty}\left(\min_{\lambda x\leq t\leq x}(s(x)-s(t))\right)$$

After taking the limit of both sides as $\lambda \to 1^-$, we get

$$\liminf_{x\to\infty}\Big(s(x)-\sigma_p(x)\Big) \geq 0. \tag{13}$$

Combining (12) and (13), we have $\lim_{x\to\infty} s(x) = \lim_{x\to\infty} \sigma_p(x)$. □

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