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Monodromy Groupoid of an Internal Groupoid in Topological Groups with Operations

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Abstract. In this paper, the monodromy groupoids of internal groupoids in the topological groups with operations are studied and a monodromy principle for internal groupoids in groups with operations is obtained.

Introduction

One form of the monodromy principle was enunciated by Chevalley in [16, Theorem 2, Chapter 2]. The general idea is that of extending a local morphism f on a topological structure G, or extending a restriction of f, not to G itself but to some simply connected cover of G. A form of this for topological groups was given in [16, Theorem 3], and developed in [23] for Lie groups. We refer the readers to the introduction of an earlier paper [30] on monodromy and monodromy groupoids. As stated there the notion of monodromy groupoid was indicated by J. Pradines in [36] as part of his grand scheme announced in [36–39] to generalise the standard construction of a simply connected Lie group from a Lie algebra to a corresponding construction of a Lie groupoid from a Lie algebroid (see also [26, 27, 34]).

Let G be a topological groupoid such that the stars $\mathsf{St}_G x$'s, the fibres of initial point map of the groupoid, are path connected and have universal covers. Let $\mathrm{Mon}(G)$ be the disjoint union of the universal covers of the stars $\mathsf{St}_G x$'s at the base points identities of the groupoid G. Then there is a groupoid structure on $\mathrm{Mon}(G)$ defined by the concatenation composition of the paths in the stars $\mathsf{St}_G x$. So there is a projection map $p\colon \mathrm{Mon}(G) \to G$. In [28], the star topological groupoid and topological groupoid structures of $\mathrm{Mon}(G)$ are studied under some suitable local conditions (see [13] and [14] for the smooth groupoid case including topological groupoids). We call $\mathrm{Mon}(G)$, the *monodromy groupoid* of G.

In the locally trivial case, Mackenzie [26, p.67-70] gives a non-trivial direct construction of the topology on $\mathrm{Mon}(G)$ and proves also that $\mathrm{Mon}(G)$ satisfies the monodromy principle on the globalisation of continuous local morphisms on G.

In the case where G is a connected topological group satisfying the usual local conditions of covering space theory, the monodromy groupoid $\mathrm{Mon}(G)$ is the universal covering group, while if G is the topological groupoid $X \times X$, for a topological space X, the monodromy groupoid $\mathrm{Mon}(G)$ is, again under suitable local

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conditions, the fundamental groupoid πX. Thus the monodromy groupoid generalizes the concepts of universal covering group and the fundamental groupoid. For further discussion, see [11].

A *group-groupoid* is a group object in the category of groupoids [15]; equivalently, it is an internal category and hence an internal groupoid in the category of groups [35]. An alternative name, quite generally used, is "2-group", see for example [5]. Recently the notion of monodromy for topological group-groupoids was introduced and investigated in [30], and normality and quotients in group-groupoids were developed in [31].

In [15, Theorem 1] Brown and Spencer proved that the category of internal categories within the groups, i.e., group-groupoids, is equivalent to the category of crossed modules of groups. Then in [35, Section 3], Porter proved that a similar result holds for certain algebraic categories C, introduced by Orzech [33], which definition was adapted by him and called category of groups with operations. Applying Porter's result, the study of internal category theory in C was continued in the works of Datuashvili [19] and [20]. Moreover, she developed cohomology theory of internal categories, equivalently, crossed modules, in categories of groups with operations [18, 21].

In a similar way, the results of [15] and [35] enabled us to develop the results of [30] for internal groupoids in topological groups with operations. So in this paper we aim to prove that the results of [30] can be generalized to a wide class of algebraic categories, which include categories of topological groups, rings, associative algebras, associative commutative algebras, Lie algebras, Leibniz algebras, alternative algebras and others. These are conveniently handled by working in a category $\mathsf{Top}^{\mathsf{C}}$.

The organization of the paper is as follows: In section 1, we recall some preliminary concepts on groupoids, star topological groupoids and topological groupoids. In Section 2 we give a brief construction of the monodromy groupoid Mon(G) for a topological groupoid G such that the stars, the fibres of the initial point map of groupoid, have universal covers. In Section 3, we prove that if G is an internal groupoid in topological groups with operations such that the stars of G have universal coves, then the monodromy groupoid Mon(G) becomes an internal groupoid in groups with operations. Finally in Section 4 we explain other construction of monodromy groupoid Mon(G, W) using free groupoid notions, identify the monodromy groupoids Mon(G) and Mon(G, W) when the stars of G are simply connected and give a week monodromy principle for internal groupoids in the groups with operations.

The main results of this paper form some parts of Ph.D thesis of second author at Erciyes University.

1. Preliminary notions on groupoids

A *groupoid* is a small category in which each morphism is an isomorphism (see for example [7] and [26]). So a groupoid G has a set G of morphisms, which we call just *elements* of G, a set G_0 of *objects* together with *initial* and *final* point maps $s, t\colon G \to G_0$ and *object inclusion* map $\epsilon\colon G_0 \to G$ such that $s\epsilon = t\epsilon = 1_{G_0}$. There exists a partial composition defined by $G_t \times_s G \to G, (g, h) \mapsto g \circ h$, where $G_t \times_s G$ is the pullback of t and s. Here if $g, h \in G$ and $t(g) = s(h)$, then the *composite* $g \circ h$ exists such that $s(g \circ h) = s(g)$ and $t(g \circ h) = t(h)$. Further, this partial composition is associative, for $x \in G_0$ the element $\epsilon(x)$ acts as the identity, and each element g has an inverse g^{-1} such that $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g \circ g^{-1} = \epsilon(s(g))$, $g^{-1} \circ g = \epsilon(t((g))$. The map $G \to G$, $g \mapsto g^{-1}$ is called the *inversion*. In a groupoid G, the initial and final points, the object inclusion, the composite and inversion maps are called *structural maps*. An example of a groupoid is fundamental groupoid of a topological space X, where the objects are points of X and morphisms are homotopy classes of the paths relative to the end points. A group is a groupoid with one object.

In a groupoid G for $x, y \in G_0$ we write $G(x, y)$ for $s^{-1}(x) \cap t^{-1}(y)$. The difference map $\delta\colon G \times_s G \to G$ is given by $\delta(g, h) = g^{-1} \circ h$, and is defined on the double pullback of G by s. If $x \in G_0$, and $W \subseteq G$, we write $\mathsf{St}_W x$ for $W \cap s^{-1}(x)$, and call $\mathsf{St}_W x$ the *star* of W at x. Especially we write $\mathsf{St}_G x$ for $s^{-1}(x)$ and call *star* of G at x. We denote the set of inverses of the morphisms in W by W^{-1}. The set of all morphisms from x to x is a group, called object group at x, and denoted by $G(x) = G(x, x)$.

Let G and H be groupoids. A *morphism* from H to G is a pair of maps $f\colon H \to G$ and $f_0\colon H_0 \to G_0$ such that $sf = f_0 s$, $tf = f_0 t$ and $f(g \circ h) = f(g) \circ f(h)$ for all $(g, h) \in H_t \times_s H$. For such a morphism we simply write $f\colon H \to G$.

A *star topological groupoid* is a groupoid in which the stars $\mathsf{St}_G x$'s have topologies such that for each $g \in G(x, y)$ the left (and hence right) translation

$$L_g \colon \mathsf{St}_G y \to \mathsf{St}_G x, h \mapsto g \circ h$$

is a homeomorphism and G is the topological sum of the $\mathsf{St}_G x$'s. A topological groupoid G defined in Definition 1.1 may be retopologized as the topological sum of its stars to become a star topological groupoid.

A subset W of a star topological groupoid G is called *star connected* (resp. *star simply connected*) if for each $x \in G_0$, the star $\mathsf{St}_W x$ of W at x is connected (resp. simply connected). So a star topological groupoid G is *star connected* (resp. *star simply connected*) if each star $\mathsf{St}_G x$ of G is connected (resp. simply connected).

We adapt the following definition from [26]. See also [9] and [10] for some earlier works on topological groupoids.

Definition 1.1. Let G be a groupoid on G_0. If the set G of morphisms and the set $X = G_0$ of objects have both topologies such that the source and target maps $s, t \colon G \to G_0$, the difference map $\delta \colon (g, h) \mapsto g^{-1} \circ h$ defined on the double pullback $G_t{\times}_s G$ and the unit map $\epsilon \colon G_0 \to G, x \mapsto \epsilon(x)$ are continuous, then G is called a *topological groupoid*. □

Recall that a covering map $p \colon \widetilde{X} \to X$ of connected spaces is called *universal* if it covers every covering of X in the sense that if $q \colon \widetilde{Y} \to X$ is another covering of X then there exists a map $r \colon \widetilde{X} \to \widetilde{Y}$ such that $p = qr$ (hence r becomes a covering). A covering map $p \colon \widetilde{X} \to X$ is called *simply connected* if $\widetilde{X}$ is simply connected. So a simply connected covering is a universal covering.

Let X be a topological space admitting a simply connected cover. A subset U of X is called *liftable* if U is open, path-connected and the inclusion $U \to X$ maps each fundamental group of U trivially. If U is liftable, and $q \colon Y \to X$ is a covering map, then for any $y \in Y$ and $x \in U$ such that $qy = x$, there is a unique map $\hat{\imath} \colon U \to Y$ such that $\hat{\imath}x = y$ and $q\hat{\imath}$ is the inclusion $U \to X$. A space X is called *semi-locally simply connected* if each point has a liftable neighborhood and *locally simply connected* if it has a base of simply connected sets. So a locally simply connected space is also semi-locally simply connected.

Let X be a topological space such that each path component of X admits a simply connected covering space. It is standard that if πX is the fundamental groupoid of X, topologised as in [8], and $x \in X$, then the target map $t \colon \mathsf{St}_{\pi X} x \to X$ is the universal covering map of X based at x (see also Brown [7, Chapter 9]).

The following theorem is proved in [8, Theorem 1]. We give a sketch proof since we need some details of the proof in Theorem 3.11. An alternative but equivalent construction of the topology is in [7, 10.5.8].

Theorem 1.2. *If X is a locally path connected and semi-locally simply connected space, then the fundamental groupoid πX may be given a topology making it a topological groupoid.*

Proof: Let $\mathcal{U}$ be the open cover of X consisting of all liftable subsets. For each U in $\mathcal{U}$ and $x \in U$ define a map $\lambda_x \colon U \to \pi X$ by choosing for each $x' \in U$ a path in U from x to x' and letting $\lambda_x(x')$ be the homotopy class of this path. By the condition on U the map λ_x is well defined. Let $\widetilde{U}_x = \lambda_x(U)$. Then the sets $\widetilde{U}_x^{-1} \alpha \widetilde{V}_y$ for all $\alpha \in \pi X(x, y)$ form a base for a topology such that πX is a topological groupoid with this topology. □

2. A review of monodromy groupoid Mon(G)

Let G be a star topological groupoid such that each star $St_G x$ has a universal cover. The groupoid Mon(G) is defined from the universal covers of stars $\mathsf{St}_G x$'s at the base points identities as follows: As a set, Mon(G) is the union of the stars $\mathsf{St}_{\pi(\mathsf{St}_G x)} \epsilon(x)$. The object set X of Mon(G) is the same as that of G. The initial point map $s \colon$ Mon(G) $\to X$ maps all of star $\mathsf{St}_{\pi(\mathsf{St}_G x)} \epsilon(x)$ to x, while the target point map $t \colon$ Mon(G) $\to X$ is defined on each $\mathsf{St}_{\pi(\mathsf{St}_G x)} \epsilon(x)$ as the composition of the two target maps

$$\mathsf{St}_{\pi(\mathsf{St}_G x)} \epsilon(x) \xrightarrow{t} \mathsf{St}_G x \xrightarrow{t} X.$$

As explained in Mackenzie [26, p.67] there is a multiplication on Mon(G) defined by

$$[a] \bullet [b] = [a \square (a(1) \circ b)]$$

where $\square$ inside the bracket denotes the usual composition of paths and $\circ$ denotes the composition in the groupoid. Here $a(1) \circ b$ is the path defined by $(a(1) \circ b)(t) = a(1) \circ b(t)$ $(0 \leqslant t \leqslant 1)$. Here we point that since G is a star topological groupoid, the left translation given in (1) is a homeomorphism. Hence the path $a(1) \circ b$, which is a left translation of b by $a(1)$, is defined when b is a path. So the path $a \square (a(1) \circ b)$ is defined by

$$(a \square (a(1) \circ b))(t) = \begin{cases} a(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ a(1) \circ b(2t-1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Here if a is a path in $\mathsf{St}_G x$ from $\epsilon(x)$ to $a(1)$, where $t(a(1)) = y$, say, and b is a path in $\mathsf{St}_G y$ from $\epsilon(y)$ to $b(1)$, then for each $t \in [0,1]$ the composition $a(1) \circ b(t)$ is defined in $\mathsf{St}_G y$, yielding a path $a(1) \circ b$ from $a(1)$ to $a(1) \circ b(1)$. It is straightforward to prove that in this way a groupoid is defined on Mon(G) and that the final map of paths induces a morphism of groupoids $p\colon \mathrm{Mon}(G) \to G$.

If each star $\mathsf{St}_G x$ admits a simply connected cover at $\epsilon(x)$, then we may topologise each star of Mon(G) so that it is the universal cover of $\mathsf{St}_G x$ based at $\epsilon(x)$, and then Mon(G) becomes a star topological groupoid. We call Mon(G) the *monodromy groupoid* or *star universal cover* of G.

Let Gpd be the category of groupoids and TopGpd the category of topological groupoids. Let sTopGpd be the full subcategory of TopGpd on those topological groupoids whose stars have universal covers. Then we have a functor

$$\mathrm{Mon}\colon \mathsf{sTopGpd} \to \mathsf{Gpd}$$

assigning the monodromy groupoid Mon(G) to each topological groupoid G such that the stars have universal covers.

Theorem 2.1. [30, Theorem 2.1] *For the topological groupoids G and H such that the stars have universal covers, the monodromy groupoids* $\mathrm{Mon}(G \times H)$ *and* $\mathrm{Mon}(G) \times \mathrm{Mon}(H)$ *are isomorphic.*

Example 2.2. Let G be a topological group which can be thought as a topological groupoid with only one object. If G has a simply connected cover, then the monodromy groupoid Mon(G) of G is just the universal cover of G. □

Example 2.3. [13, Theorem 6.2] If X is a topological space, then $G = X \times X$ becomes a topological groupoid on X. Here a pair (x, y) is a morphism from x to y with inverse morphism (y, x). The groupoid composition is defined by $(x, y) \circ (u, z) = (x, z)$ whenever $y = u$. If X has a simply connected cover, then the monodromy groupoid Mon(G) of G is isomorphic to the fundamental groupoid πX. □

Therefore the monodromy groupoid concept generalises both the fundamental groupoid of a topological space and the universal covering group of a topological group.

In the following theorem which is the main result of [30, Theorem 3.10], the monodromy groupoid for a topological group-groupoid has been developed. In Theorem 3.13 we prove a more general result and develop the monodromy groupoid for an internal groupoid in topological groups with operations.

Theorem 2.4. *Let G be a topological group-groupoid such that each star $\mathsf{St}_G x$ has a universal cover. Then the monodromy groupoid* Mon G *is a group-groupoid.*

3. Monodromy groupoid for an internal groupoid in topological groups with operations

The idea of the definition of categories of groups with operations comes from Higgins [25] and Orzech [33]; and the definition below is from Porter [35] and Datuashvili [22, p.21], which is adapted from Orzech [33].

Definition 3.1. Let C be a category of groups with a set of operations Ω and with a set E of identities such that E includes the group laws, and the following conditions hold: If Ω_i is the set of i-ary operations in Ω, then

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) The group operations written additively $0, -$ and $+$ are the elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $\star \in \Omega_2'$, then $\star^\circ$ defined by $a \star^\circ b = b \star a$ is also in Ω_2'. Also assume that $\Omega_0 = \{0\}$;

(c) For each $\star \in \Omega_2'$, E includes the identity $a \star (b + c) = a \star b + a \star c$;

(d) For each $\omega \in \Omega_1'$ and $\star \in \Omega_2'$, E includes the identities $\omega(a + b) = \omega(a) + \omega(b)$ and $\omega(a) \star b = \omega(a \star b)$.

Then the category C satisfying the conditions (a)-(d) is called a *category of groups with operations*. □

From now on C will be a category of groups with operations.

A *morphism* between any two objects of C is a group homomorphism, which preserves the operations of Ω_1' and Ω_2'.

Remark 3.2. The set Ω_0 contains exactly one element, the group identity; hence for instance the category of associative rings with unit is not a category of groups with operations.

Example 3.3. The categories of groups, rings generally without identity, R-modules, associative, associative commutative, Lie, Leibniz, alternative algebras are examples of categories of groups with operations. □

The category of topological groups with operations are defined in [1] (see also [29]) as follows:

Definition 3.4. A category $\mathsf{Top}^{\mathsf{C}}$ of topological groups with a set Ω of continuous operations and with a set E of identities such that E includes the group laws such that the conditions (a)-(d) of Definition 3.1 are satisfied, is called a *category of topological groups with operations*. □

In the rest of the paper $\mathsf{Top}^{\mathsf{C}}$ will denote the category of topological groups with operations.

A *morphism* between any two objects of $\mathsf{Top}^{\mathsf{C}}$ is a continuous group homomorphism, which preserves the operations in Ω_1' and Ω_2'.

The categories of topological groups, topological rings and topological R-modules are examples of categories of topological groups with operations.

Definition 3.5. *An internal category* C in C is a category in which the initial and final point maps $s, t\colon C \rightrightarrows C_0$, the object inclusion map $\epsilon\colon C_0 \to C$ and the partial composition $\circ\colon C_t \times_s C \to C, (a, b) \mapsto a \circ b$ are the morphisms in the category C. □

Note that since ϵ is a morphism in C, $\epsilon(0) = 0$ and that the composition $\circ$ being a morphism implies that for all $a, b, c, d \in C$ and $\star \in \Omega_2$

$$(a \star c) \circ (b \star d) = (a \circ b) \star (c \circ d) \tag{1}$$

whenever one side makes sense. This is called the *interchange law* [35].

As it was pointed out in [35] by an easy application it follows that any internal category C in C is an internal groupoid since given $a \in C$,

$$a^{-1} = \epsilon(t(a)) - a + \epsilon(s(a)) \tag{2}$$

satisfies $a \circ a^{-1} = \epsilon(s(a))$ and $a^{-1} \circ a = \epsilon(t(a))$; and the map $G \to G, a \mapsto a^{-1}$ is also a morphism in C. So we use the term *internal groupoid* rather than internal category and write G for an internal groupoid.

In particular if C is the category of groups, then an internal groupoid G in C becomes a group object in the category of groupoids, which is quite often called 2-group, see for example [5], *group-groupoid* or $\mathcal{G}$-groupoid [15]. In the case where C is the category of rings, an internal groupoid is a ring object in the category of groupoids [32] (see also [2] and [3] for topological R-module case).

Remark 3.6. [1, 3.7] The following are immediate from Definition 3.5:

(i) By Definition 3.5 we know that in an internal groupoid G in C, the initial and final point maps s and t, the object inclusion map ϵ are the morphisms in C and the interchange law (1) is satisfied. Therefore in an internal groupoid G, the unary operations are endomorphisms of the underlying groupoid of G and the binary operations are morphisms from the underlying groupoid of $G \times G$ to the one of G.

(ii) Let G be an internal groupoid in C and $0 \in G_0$ the identity element. Then $\mathrm{Ker}\, d_0 = \mathsf{St}_G 0$, called in [7] *transitivity component* or *connected component* of 0, is also an internal groupoid which is also an ideal of G. □

Let H and G be two internal groupoids in C. A *morphism* of internal groupoids is a morphism $f\colon H \to G$ of underlying groupoids which is also a morphism of groups with operations. A morphism $f\colon H \to G$ of internal groupoids is called *covering* (resp. *universal covering*) if it is a covering (resp. universal covering) morphism on the underlying groupoids.

The following proposition is known for group-groupoids in [12, Proposition 2.1]. In internal groupoid case the proof is similar by using the interchange law.

Proposition 3.7. *Let G be an internal groupoid in C. Let $g, h \in G$ with $g \in G(x, y)$ and $s(h) = t(g)$. Then*

(1) $g \circ h = g - \epsilon(y) + h$

(2) $g + h - g = \epsilon(x) + h - \epsilon(x)$ *for* $h \in G(0)$

Definition 3.8. An internal groupoid in the category $\mathsf{Top}^{\mathsf{C}}$ of topological groups with operations is called a *topological internal groupoid*.

So a topological internal groupoid is a topological groupoid G in which the set of morphisms and the set G_0 of objects are objects of $\mathsf{Top}^{\mathsf{C}}$ and all structural maps of G, i.e., the source and target maps $s, t\colon G \to G_0$, the object inclusion map $\epsilon\colon G_0 \to G$ and the composition map $\circ\colon G_t \times_s G \to G$, are morphisms of $\mathsf{Top}^{\mathsf{C}}$. From Proposition 3.7 we can see that in an internal groupoid the continuities of the group operation and the object inclusion map imply the continuity of the groupoid composite.

If $\mathsf{Top}^{\mathsf{C}}$ is the category of topological groups, then an internal groupoid in $\mathsf{Top}^{\mathsf{C}}$ becomes a topological group-groupoid.

Example 3.9. A topological group with operations which is abelian according to all binary operations $\star \in \Omega_2$ can be thought as an internal groupoid in topological groups with operations. □

Example 3.10. Let X be an object of $\mathsf{Top}^{\mathsf{C}}$. Then the groupoid $G = X \times X$ defined in Example 2.3 is an internal groupoid in $\mathsf{Top}^{\mathsf{C}}$: The binary operations in G are defined by $(x, y)\widetilde{\star}(u, v) = (x \star u, y \star v)$ for $\star \in \Omega_2$ and the unary operations by $\widetilde{\omega}(x, y) = (\omega(x), \omega(y))$ for $\omega \in \Omega_1$. For the interchange law if $g = (x, y), h = (y, z)$, $k = (u, v)$ and $l = (v, w)$ are the morphisms in G so that the compositions $g \circ h$ and $k \circ l$ are defined, then we have $(g \circ h)\widetilde{\star}(k \circ l) = (x \star u, z \star w)$ and $(g\widetilde{\star}k) \circ (h\widetilde{\star}l) = (x \star u, z \star w)$ and therefore we have the interchange law

$$(g\widetilde{\star}k) \circ (h\widetilde{\star}l) = (g \circ h)\widetilde{\star}(k \circ l).$$

Theorem 3.11. *Let X be an object of $\mathsf{Top}^{\mathsf{C}}$ such that the underlying space is locally path connected and semi-locally simply connected. Then the fundamental groupoid πX is an internal groupoid in $\mathsf{Top}^{\mathsf{C}}$.*

Proof: Let X be a topological group with operations as assumed. By Theorem 1.2, πX has a topology such that it is a topological groupoid. We know by [8, Proposition 3] that when X and Y are endowed with such topologies, for a continuous map $f\colon X \to Y$, the induced morphism $\pi(f)\colon \pi X \to \pi Y$ is also continuous. Hence the continuous binary operations $\star\colon X \times X \to X$ for $\star \in \Omega_2$ and the unary operations $\omega\colon X \to X$ for $\omega \in \Omega_1$ respectively induce continuous binary operations $\widetilde{\star}\colon \pi X \times \pi X \to \pi X$ and unary operations $\widetilde{\omega}\colon \pi X \to \pi X$. So the set of morphisms becomes a topological group with operations. The groupoid structural maps are morphisms of groups with operations, i.e., preserve the operations. Therefore πX becomes an internal groupoid in $\mathsf{Top}^{\mathsf{C}}$. □

Let $\mathsf{sTop}^{\mathsf{C}}$ be the full subcategory of $\mathsf{Top}^{\mathsf{C}}$ on those objects whose underlying spaces are locally path connected and semi-locally simply connected; and let $\mathsf{Cat}(\mathsf{Top}^{\mathsf{C}})$ be the category of internal groupoids in topological groups with operations. Then we have a functor

$$\pi\colon \mathsf{sTop}^{\mathsf{C}} \to \mathsf{Cat}(\mathsf{Top}^{\mathsf{C}}).$$

Theorem 3.12. *Let X and Y be the objects of $\mathsf{sTop}^{\mathsf{C}}$. Then $\pi(X \times Y)$ and $\pi X \times \pi Y$ are isomorphic as internal groupoids in $\mathsf{Top}^{\mathsf{C}}$.*

Proof: From [30, Theorem 3.8] we know that the topological groupoids $\pi(X\times Y)$ and $\pi X\times\pi Y$ are isomorphic. There, in the detail of the proof, it was proved that the morphism

$$f\colon \pi(X \times Y) \to \pi X \times \pi Y, f([a]) = ([p_1 a], [p_2 a])$$

is an isomorphism of topological groupoids. In addition to these, it is immediate to see that f preserves the binary operations and the unary operations, i.e., $f([a] \star [b]) = f([a]) \star f([b])$ for $\star \in \Omega_2$ and $f(\omega a) = \omega f(a)$ for $\omega \in \Omega_1$. □

Let X be topological group with operations. We know from Example 3.10 that $G = X \times X$ is an internal groupoid in $\mathsf{Top}^{\mathsf{C}}$ and from Example 2.3 that the monodromy groupoid of G is the fundamental groupoid πX which is also an internal groupoid in C. Therefore it can be thought that if G is an internal groupoid in $\mathsf{Top}^{\mathsf{C}}$ such that the stars $\mathsf{St}_G x$'s have universal covers, then the monodromy groupoid Mon(G) is also an internal groupoid or not in C. In the following theorem we reply this question.

Theorem 3.13. *Let G be an internal groupoid in $\mathsf{Top}^{\mathsf{C}}$ such that each star $\mathsf{St}_G x$ has a universal cover. Then the monodromy groupoid* Mon(G) *becomes an internal groupoid in* C.

Proof: Let G be an internal groupoid in topological groups with operations as assumed. Therefore the set G_0 of objects and the set G of morphisms are topological groups with operations. Then we define the unary operations by

$$\widetilde{\omega}\colon \mathrm{Mon}(G) \to \mathrm{Mon}(G), [a] \mapsto [\omega(a)]$$

for $\omega \in \Omega_1$ and the binary operations on Mon(G) by

$$\widetilde{\star}\colon \mathrm{Mon}(G) \times \mathrm{Mon}(G) \to \mathrm{Mon}(G), ([a], [b]) \mapsto [a \star b]$$

for $\star \in \Omega_2$. Here for each $\star \in \Omega_2$, the product $a \star b$ is induced by that of G by $(a \star b)(t) = a(t) \star b(t)$ for $t \in [0, 1]$. So if a is a path in $\mathsf{St}_G x$ from $\epsilon(x)$ to $a(1)$ and b is a path in $\mathsf{St}_G y$ from $\epsilon(y)$ to $b(1)$, then $a \star b$ is a path in $\mathsf{St}_G(x \star y)$ from $\epsilon(x \star y)$ to $a(1) \star b(1)$. These operations defined in these ways are well defined: If $a_1 \in [a]$, there is a homotopy $F\colon I \times I \to G$ such that $F(s, 0) = a_1(s)$ and $F(s, 1) = a(s)$ $(0 \leqslant s \leqslant 1)$. Similarly if $b_1 \in [b]$, there is a homotopy $H\colon I \times I \to G$ such that $H(s, 0) = b_1(s)$ and $H(s, 1) = b(s)$. Then we define a homotopy $K\colon I \times I \to G$ by $K(s, t) = F(s, t) \star G(s, t)$. Here $K(s, 0) = a_1 \star b_1$ and $K(s, 1) = a \star b$; and since the binary operations $\star$'s in $\in \Omega_2$ are continuous, K is continuous. So $a_1 \star b_1$ and $a \star b$ are homotopic and therefore the binary operations defined on Mon(G) are well defined. Further since the unary operations $\omega \in \Omega_1$ are continuous, the induced unary operations

$$\widetilde{\omega}\colon \mathrm{Mon}(G) \to \mathrm{Mon}(G), [a] \mapsto [\omega(a)]$$

are well defined. The other details can be checked to prove that Mon(G) is a group with operations in C.

We now prove that the interchange law

$$[(a \star c) \bullet (b \star d)] = [(a \bullet b) \star (c \bullet d)]$$

in Mon(G) is satisfied when $a \bullet b$ and $c \bullet d$ are defined. If these $a \bullet b$ and $c \bullet d$ are defined, then we have the following compositions of the paths in Mon(G):

$$(a \star c) \bullet (b \star d)(t) = \begin{cases} (a \star c)(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ (a \star c)(1) \circ (b \star d)(2t-1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

and hence

$$(a \star c) \bullet (b \star d) = (a \star c)\square((a \star c)(1) \circ (b \star d)). \tag{3}$$

On the other hand

$$(a \bullet b)(t) = \begin{cases} a(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ a(1) \circ b(2t-1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

$$(c \bullet d)(t) = \begin{cases} c(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ c(1) \circ d(2t-1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

$$(a \bullet b) \star (c \bullet d)(t) = \begin{cases} (a \star c)(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ (a(1) \circ b(2t-1)) \star (c(1) \circ d(2t-1)), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

and hence

$$(a \bullet b) \star (c \bullet d) = (a \star c)\square((a(1) \circ b) \star (c(1) \circ d))). \tag{4}$$

By the interchange law in G we have that

$$(a \star c)(1)) \circ (b \star d) = (a(1) \circ b) \star (c(1) \circ d).$$

Therefore comparing the equalities (3) and (4) we obtain that

$$(a \star c) \bullet (b \star d) = (a \bullet b) \star (c \bullet d)$$

which insures the interchange law in Mon(G).

Other necessary details are straightforward and hence Mon(G) becomes an internal groupoid in C. □

In Theorem 3.13 in particularly if we choose $\mathsf{Top}^{\mathsf{C}}$ as the category of topological groups, then an internal in $\mathsf{Top}^{\mathsf{C}}$ becomes a topological group-groupoid and therefore we obtain Theorem 2.4.

Let $\mathsf{sCat}(\mathsf{Top}^{\mathsf{C}})$ be the full sub category of $\mathsf{Cat}(\mathsf{Top}^{\mathsf{C}})$, the category of internal groupoids in topological groups with operations, on those internal groupoids whose stars have simply connected covers. Let $\mathsf{Cat}(\mathsf{C})$ be the category of internal groupoids in C. Therefore we have a functor

$$\mathrm{Mon}\colon \mathsf{sCat}(\mathsf{Top}^{\mathsf{C}}) \longrightarrow \mathsf{Cat}(\mathsf{C})$$

assigning the monodromy groupoid Mon(G) as the internal groupoid in C to each internal groupoid G in $\mathsf{Top}^{\mathsf{C}}$ such that the stars have simply connected covers.

Hence we can now restate Theorem 2.1 for internal groupoids in $\mathsf{Top}^{\mathsf{C}}$ as follows.

Theorem 3.14. For the internal groupoids G and H in $\mathsf{Top}^{\mathsf{C}}$ such that the stars have universal covers, the monodromy groupoids Mon($G \times H$) and Mon(G) × Mon(H) as internal groupoids in C are isomorphic.

Proof: In the detail of the proof of Theorem 2.1, it is proved that the morphism

$$f\colon \mathrm{Mon}(G\times H)\to \mathrm{Mon}(G)\times \mathrm{Mon}(H), f([a])=([p_1a],[p_2a])$$

is an isomorphism of groupoids. In addition to this, we prove that f is a morphism of internal groupoids.

$$\begin{aligned} f([a]\star[b]) &= ([p_1(a\star b)],[p_2(a\star b)]) \\ &= ([p_1a\star p_1b],[p_2a\star p_2b]) \\ &= ([p_1a],[p_2a])\star([p_1b],[p_2b]) \\ &= f([a])\star f([b]). \end{aligned}$$

for $[a],[b]\in \mathrm{Mon}(G\times H)$ and $\star\in\Omega_2$; and

$$\begin{aligned} f(\omega[a]) &= ([p_1(\omega a],[p_2(\omega a)]) \\ &= ([\omega(p_1a)],[\omega(p_2a)]) \\ &= (\omega[(p_1a)],\omega[(p_2a)]) \\ &= (\omega([(p_1a)],[(p_2a)]) \\ &= \omega f([a]). \end{aligned}$$

for $\omega\in\Omega_1$. Hence f becomes a morphism of $\mathsf{Cat}(\mathsf{C})$ as required. □

4. A monodromy principle for internal groupoids in groups with operations

In this section we recall another construction $\mathrm{Mon}(G,W)$ of the monodromy groupoid from [13] (sea also [4] and [14]) and give a monodromy principle for internal groupoids in C.

The construction here is a generalization to the groupoid case of a construction for groups by Douady and Lazard in [23]. Let G be a star topological groupoid, and let W be any subset of G containing $X=G_0$, e.i., W containing all the identity morphisms and such that $W=W^{-1}$. Then W obtains the structure of *pregroupoid*: this means that W has the structure of maps $s,t\colon W\to X$, $\epsilon\colon X\to W$ with $s\epsilon=t\epsilon=1_X$, and further there is a partial multiplication on W in which if uv is defined then $t(u)=s(v)$, $u(\epsilon t(u))=\epsilon s(u)u=u$, and each $u\in W$ has an inverse u^{-1} such that $u\circ u^{-1}=\epsilon s(u)$, $u^{-1}\circ u=\epsilon t(u)$. For further discussion of this, see for example Crowell and Smythe [17]. For our purposes, we do not need this, since we know already that W is embeddable in a groupoid.

There is a standard construction $\mathrm{Mon}(G,W)$ associating to a pregroupoid W a morphism $\tilde{\imath}\colon W\to \mathrm{Mon}(G,W)$ to a groupoid $\mathrm{Mon}(G,W)$ and which is universal for pregroupoid morphisms to a groupoid. First form the free groupoid $F(W)$ on the graph W, and denote the inclusion $W\to F(W)$ by $u\mapsto[u]$. Let N be the normal subgroupoid (Higgins [25], Brown [7]) of $F(W)$ generated by the elements $[u][v]\circ[u\circ v]^{-1}$ for all $u,v\in W$ such that $u\circ v$ is defined and belongs to W. Then $\mathrm{Mon}(G,W)$ is defined to be the quotient groupoid $F(W)/N$. The composition $W\to F(W)\to \mathrm{Mon}(G,W)$ is written $\tilde{\imath}$, and is the required universal morphism.

In the case W is the pregroupoid arising from a subset W of a groupoid G, there is a unique morphism of groupoids $p\colon \mathrm{Mon}(G,W)\to G$ such that $p\tilde{\imath}$ is the inclusion $i\colon W\to G$. It follows that $\tilde{\imath}$ is injective. Clearly, p is surjective if and only if W generates G. In this case, we call $\mathrm{Mon}(G,W)$ the *monodromy groupoid* of (G,W).

The Lie versions of the following results are given in [13, Theorem 4.2].

Theorem 4.1. *Suppose that G is a star connected star topological groupoid and W is an open neighbourhood of* $\mathrm{Ob}(G)$ *satisfying the condition:*

($\star$) W is star path-connected and W^2 is contained in a star path-connected neighbourhood V of $\mathrm{Ob}(G)$ *such that for all $x\in\mathrm{Ob}(G)$, V_x is liftable.*

Then there is an isomorphism over G of star topological groupoids $\mathrm{Mon}(G,W)\to\mathrm{Mon}(G)$, *and hence the morphism* $\mathrm{Mon}(G,W)\to G$ *is a star universal covering map.*

As a result of Theorem 4.1 and Theorem 3.13 we obtain the following Corollary.

Corollary 4.2. *Let G be an internal groupoid in* $\mathsf{Top}^{\mathsf{C}}$ *such that the stars have universal covers and W an open neighbourhood of* $\mathrm{Ob}(G)$ *in G satisfying the condition ($\star$) in Theorem 4.1. Then the monodromy groupoid* $\mathrm{Mon}(G, W)$ *is an internal groupoid in* C.

Proof: By Theorem 3.13, $\mathrm{Mon}(G)$ is an internal groupoid in C and by Theorem 4.1 the groupoids $\mathrm{Mon}(G, W)$ and $\mathrm{Mon}(G)$ are isomorphic as star topological groupoids. So $\mathrm{Mon}(G, W)$ also becomes an internal groupoid in C. □

Let $q\colon E \to X$ be a surjective function and let the *symmetry groupoid* S_q of q be the groupoid over X of bijections $E_x \to E_y$ for all fibres $E_x = q^{-1}(x)$ of q, and all $x, y \in X$. The following theorem is stated in [16, Theorem 2, Chapter 2] as the Monodromy Principle.

Theorem 4.3. *Let X be a connected and simply-connected space, let W be a connected neigbourhood of the diagonal of $X \times X$ such that each section $W_x = \{y \in X : (x, y) \in W\}$ is connected. Let $\phi\colon W \to S_q$ be a morphism of pregroupoids. Suppose $e_0 \in E$ is given. Then there is a unique function $\psi\colon X \to E$ which assigns to every $x \in X$ an element $\psi(x) \in E_x$ such that $(\psi q)(e_0) = e_0$ and $\psi(y) = \phi(x, y)\psi(x)$ whenever $\phi(x, y)$ is defined.*

As related to Theorem 4.3 we now state a monodromy principle for internal groupoids in C, which we call "weak" because it involves no continuity conditions on maps.

Theorem 4.4. (Weak Monodromy Principle) *Let G be an internal groupoid in* $\mathsf{Top}^{\mathsf{C}}$ *and let W be an open subset of G containing O_G and W is star connected. Suppose that G is star simply connected. Let H be an internal groupoid over O_G in* C *and let $\phi\colon W \to H$ be a morphism of pregroupoids which is the identity on O_G and preserves the group operations. Then ϕ extends uniquely to a morphism $\tilde{\phi}\colon G \to H$ of internal groupoids in* C.

Proof: By [13, Proposition 2.5] $\mathrm{Mon}(G, W)$ is star connected, and by [13, Proposition 2.3], $p\colon \mathrm{Mon}(G, W) \to G$ is, when restricted to stars, a covering map of connected spaces. Since G is star simply connected, it follows that p is an isomorphism. By the universal property of $\mathrm{Mon}(G, W)$ the pregroupoid morphism $\phi\colon W \to H$ extends uniquely to a morphism $\tilde{\phi}\colon G \to H$ of groupoids and which preserves the group operations by the interchange rule. Hence $\tilde{\phi}\colon G \to H$ becomes a morphism of internal groupoids in C. □

Note that in Theorem 4.4 and Theorem 4.3 there is no topology given on H or S_q and there are no assumptions of continuity of ϕ.

5. Conclusion

To explain and clarify the relations between graph and free object in internal groupoids we first recall a fact on semi-abelian categories: The notion of semi-abelian category as proposed in [24] (see also [40] and [41]) has typical categorical properties such as possessing finite products, coproducts, a zero object and hence kernels, pullbacks of monomorphisms and coequalizers of kernel pairs. Groups, rings, algebras and all abelian categories are semi-abelian, say.

In [6] for a certain algebraic theory the term 'algebraic model' is used for the objects of the semi-abelian category. Let $\mathbb{T}$ be an algebraic theory whose category is semi-abelian. A *topological model* of $\mathbb{T}$ is a model of the theory of $\mathbb{T}$ with a topology which makes all the operations of the theory continuous. The category $\mathsf{Top}^{\mathbb{T}}$, for a semi-abelian theory $\mathbb{T}$, is generally no longer semi-abelian because it is not Bar exact. But in [6] the category $\mathsf{Top}^{\mathbb{T}}$ of the topological models $\mathbb{T}$ is studied and some classical results in topological groups is generalized to this category $\mathsf{Top}^{\mathbb{T}}$.

Hence this paper might be useful to allow for the notion of free objects, and in particular free groupoids on graph objects in internal groupoids. Therefore it could be possible to obtain $\mathrm{Mon}(G, W)$ as an internal groupoid dealing graph and free object terms in a similar way to that in [6]. It could also be possible to have a topology on $\mathrm{Mon}(G, W)$, may be using holonomy theorem in [4, Theorem 2.1], such that $\mathrm{Mon}(G, W)$ is an internal groupoid in $\mathsf{Top}^{\mathsf{C}}$ and obtain a strong monodromy principle for this type of internal groupoids.

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