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# On 3-triangulation of toroids

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**Abstract.** As toroid (polyhedral torus) could not be convex, it is questionable if it is possible to 3-triangulate them (i.e. divide into tetrahedra with the original vertices). Here, we will discuss some examples of toroids to show that for each vertex number $n \geq 7$, there exists a toroid for which triangulation is possible. Also we will study the necessary number of tetrahedra for the minimal triangulation.

## 1. Introduction

It is known that there is a possibility to divide any polygon with $n$ vertices by $n-3$ diagonals into $n-2$ triangles without gaps and overlaps. This division is called triangulation. To do the triangulation, many different practical applications are made that require computer programs. Examples of such algorithms are given by Seidel [13], Edelsbrunner [8] and Chazelle [4]. The most interesting aspect of the problem is to design algorithms that are as optimal as possible.

Generalization of this process to higher dimensions is also called triangulation. It consists of dividing polyhedra (polytop) into tetrahedra (simplices) with the original vertices. Within higher dimensions, new problems arise besides the fastness of algorithm. It is proved that there is no possibility to triangulate some of non-convex polyhedra [11, 12] in a three-dimensional space, and it is also proved that different triangulations of the same polyhedron may have different numbers of tetrahedra [1], [9], [14]. Considering the smallest and the largest number of tetrahedra in triangulation (the minimal and the maximal triangulation), the authors obtained values, which linearly, resp. squarely depend on the number of vertices. Interesting triangulations are described in the papers of Edelsbrunner, Preparata, West [9] and Sleator, Tarjan, Thurston [14]. Some characteristics of triangulation in a three-dimensional space are given by Chin, Fung, Wang [6], Develin [7] and Stojanović [20, 21], and in $n$-dimensional space by Lee [10]. Algorithms for investigating triangulation in three-dimensional space are given in [22, 23]. This problem is also related to the problems of triangulation of a set of points in a three-dimensional space [1, 9] and rotation distance between pair of trees [14].

By the term "polyhedron" we usually mean a simple polyhedron, topologically equivalent to sphere. But there are classes of polyhedra topologically equivalent to torus. Torus like polyhedra are considered e.g. in [2, 3, 5, 17–19]. Following the definition of Szilassi [18] for such polyhedra, we will use terms toroids. Since toroids are not convex, it is questionable if it is possible to 3-triangulate them. The toroid with the smallest number of vertices is Császár polyhedron [2, 3, 5, 17–19]. It has 7 vertices and is known to be triangulable with 7 tetrahedra. It is obtained as an example of polyhedron without diagonals [5, 15, 16].

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In this paper, 3-triangulations of other toroids will be considered. Some characteristic polyhedra will be described in section 2. In section 3, we will consider 3-triangulation of toroids.

## 2. Some Characteristic Examples of Polyhedra and Their 3-triangulation

**2.1** It is possible to triangulate all convex polyhedra, but this is not the case with non-convex ones. The first example of a non-convex polyhedron, which is impossible to triangulate, was given by Schönhardt [12] and referred to in [11]. This polyhedron is obtained in the following way: triangulate the lateral faces of a trigonal prism $A_1B_1C_1A_2B_2C_2$ by the diagonals $A_1B_2$, $B_1C_2$ and $C_1A_2$ (Fig. 1). Then "twist" the top face $A_2B_2C_2$ by a small amount in the positive direction. In such a polyhedron, none of tetrahedra with vertices in the set $\{A_1, B_1, C_1, A_2, B_2, C_2\}$ is inner, so the triangulation is not possible.

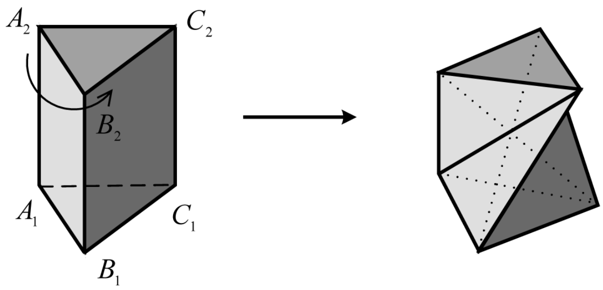


Figure 1: Schönhardt polyhedron

**2.2** Let us now consider triangulations of a bipyramid with a triangular basis $ABC$, and apices $V_1$ and $V_2$ (Fig. 2). There are two different triangulations of this kind. The first is into two tetrahedra $V_1ABC$ and $V_2ABC$, and the second is into three: $V_1V_2AB$, $V_1V_2BC$ and $V_1V_2CA$. So, it is obvious that some 3-triangulable polyhedra is possible to triangulate with different numbers of tetrahedra. That is the reason to introduce terms of minimal and maximal triangulation of a given polyhedron.

**2.3** It is proved that the smallest possible number of tetrahedra in the triangulation of a polyhedron with $n$ vertices is $n-3$. But, it is not possible to triangulate each polyhedron into $n-3$ tetrahedra; for example, all triangulations of an octahedron (6 vertices) give 4 tetrahedra. Here we will mention some examples of polyhedra, triangulable with $n-3$ tetrahedra.

The pyramids with $n-1$ vertices in the basis (i.e., a total of $n$ vertices) are triangulable by doing any 2-triangulation of the basis into $(n-1)-2 = n-3$ triangles. Each of these triangles makes with the apex one of tetrahedra in 3-triangulation. If the basis of a "pyramid" is a space polygon, then it is possible to triangulate it in a similar way without taking care about convexity. For example, if we 2-triangulate lateral sides of trigonal prism $A_1B_1C_1A_2B_2C_2$ by the diagonals $B_1A_2$, $C_1A_2$ and $C_1B_2$ (Fig. 3) then, it is obvious that 3-triangulation is possible with 3 tetrahedra: $A_1B_1C_1A_2$, $B_1C_1A_2B_2$ and $A_2B_2C_1C_2$. Here, we may assume that the basis of the trigonal pyramid is space pentagon $A_1B_1B_2C_2C_1$.

**2.4** Let us return to the two methods of triangulating a bipyramid, but this time with $n-2$ vertices in the basis (which can also be a space polygon). If we divide it into two pyramids and triangulate each of them with taking care of a common 2-triangulation of the basis, then we will obtain $2(n-4)$ tetrahedra. In the second method, each of $n-2$ tetrahedra has a common edge joining the apices of the bipyramid, and moreover, each of them contains a pair of the neighbour vertices of the basis (i.e., one of the edges of the basis). For $n = 5$ (a bipyramid with a triangle basis), it has been found that the first method is "better", i.e., it gives a smaller number of tetrahedra. For $n = 6$ (the octahedron), both methods give 4 tetrahedra and for $n \geq 7$, the second method is "better". In figure Fig. 4 triangulations of a bipyramid with a pentagonal basis

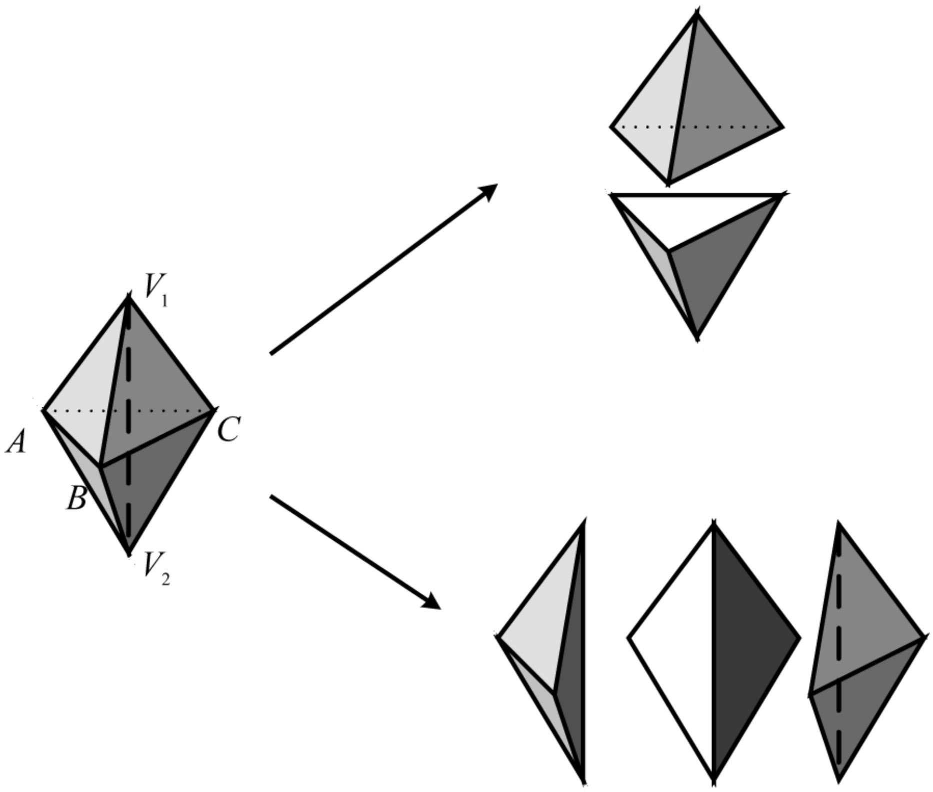


Figure 2: Triangulations of trigonal bipyramids

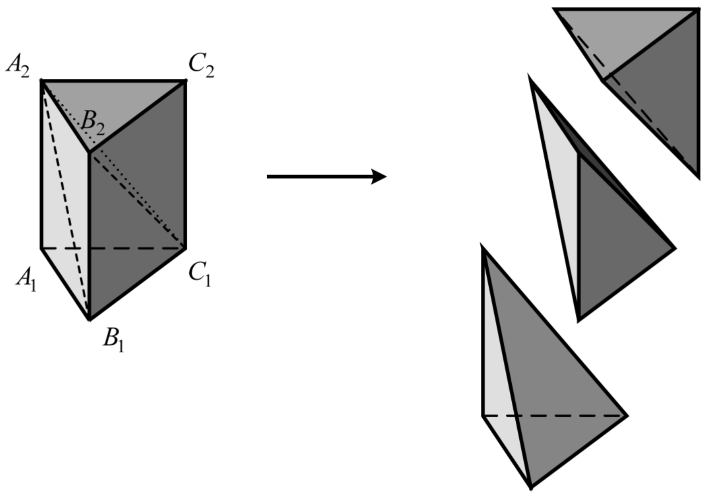


Figure 3: Triangulation of trigonal prism with 3 tetrahedra

(i.e. $n = 7$) are given. Dividing bipyramid into two pyramids leads to triangulation with 6 tetrahedra, and dividing it around the axis $V_1V_2$ gives triangulation with 5 tetrahedra.

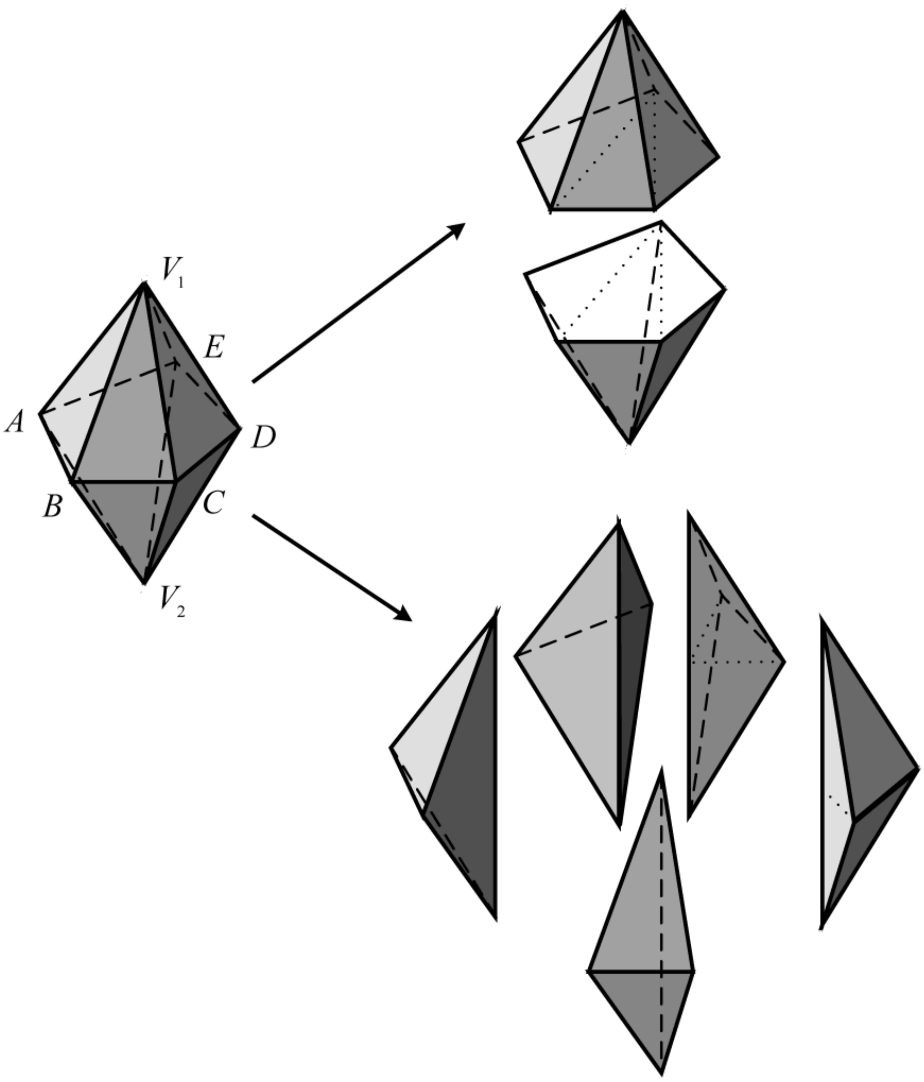


Figure 4: Triangulations of pentagonal bipyramids

**2.5** In [18] Szilassi introduced term toroid:

**Definition 1.** *An ordinary polyhedron is called a* toroid *if it is topologically torus-like (i.e. it can be converted to a torus by continuous deformation) and its faces are simply polygons.*

A toroid with the smallest number of vertices is the Császár polyhedron (Fig. 5). It has 7 vertices and no diagonals, i.e. each vertex is connected to other six by edges. In [2] Bokowski and Eggert proved that Császár polyhedron has four essentially different versions. It is to be noted that in topological terms the various versions of Császár polyhedron are isomorphic – there is only one way to draw the full graph with seven vertices on the torus. Császár polyhedron is possible to 3-triangulate with 7 tetrahedra, as it is shown by Szilassi from Wolfram Demonstrations Project [19].

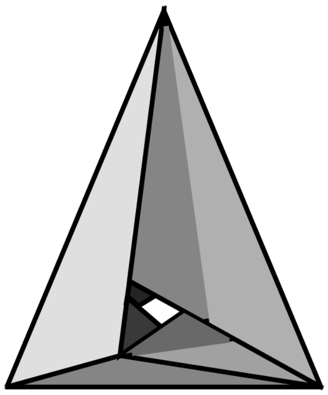

Figure 5: Császár polyhedron

## 3. 3-triangulation of Toroids

**3.1** In order to consider 3-triangulability of toroids, let us introduce the following definitions.

**Definition 2.** *Polyhedron is* piecewise convex *if it is possible to divide it into convex polyhedra* $P_i$, $i = 1, \dots, n$, *with disjunct interiors. A pair of polyhedra* $P_i$, $P_j$ *is said to be* neighbours *if they have common face called* contact face. *If polyhedra* $P_i$ *and* $P_j$ *are not neighbours, they may have a common edge e or a common vertex v only if there is a sequence of neighbours polyhedra* $P_i, P_{i+1}, \dots, P_{i+k} \equiv P_j$ *such that the edge e, or the vertex v belongs to each contact face* $f_l$ *common to* $P_l$ *and* $P_{l+1}$, $l \in \{i, \dots, i+k-1\}$. *Otherwise, polyhedra* $P_i$ *and* $P_j$ *have not common points.*

One example of piecewise convex polyhedron is given in Fig. 6. The figure is shoving a toroid with $n = 19$ vertices, whose pieces are of two kinds, introduced later on as ″elementary polyhedra″ of two types $A$ and $B$.

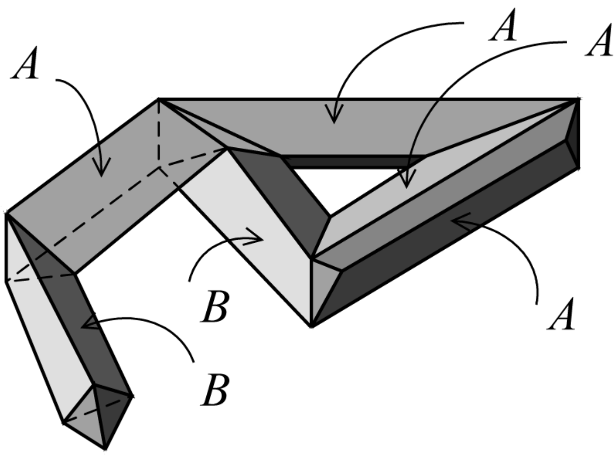


Figure 6: Piecewise convex polyhedron $T_{19}$ with 19 vertices

**Definition 3.** *Toroid is* cyclically piecewise convex *if it is possible to divide it into cycle of convex polyhedra* $P_i$, $i = 1, \dots, n$, *such that* $P_i$ *and* $P_{i+1}$, $i = 1, \dots, n-1$ *and* $P_n$ *and* $P_1$ *are neighbours.*

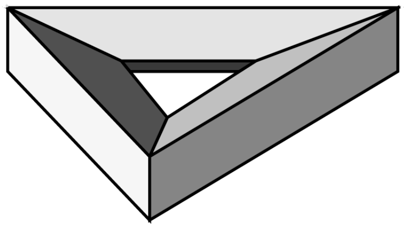

Figure 7: Cyclically piecewise convex polyhedron $T_9$ with 9 vertices

An example of cyclically piecewise convex polyhedron with $n = 9$ vertices composed of three pieces of type $A$ is given on Fig. 7.

Note that division of polyhedra to convex pieces is not always unique. For example in toroid $T_{19}$, two pieces of type $A$ on the right side of the toroid together build a new convex polyhedron. So, we can replace that two pieces with the new one. On the other hand, since it is always possible to 3-triangulate convex polyhedra, the same property holds for piecewise convex toroid (and also for other piecewise convex polyhedra). That will be used in the proofs of the following lemma.

**Lemma 1.** *For each $n \geq 9$, there exists a toroid which is possible to 3-triangulate.*

**Proof.** For each $n$, we shall construct cyclically piecewise convex toroid with $n$ vertices composed of "elementary polyhedra" of two types $A$ and $B$ (Fig. 8). Polyhedra of type $A$ are topologically triangular prisms. Cyclically connecting $k \geq 3$ polyhedra of type $A$ would be possible if we transform two lateral faces of each triangular prism from rectangle to trapeze. Then contact faces of such polyhedron $A$ are its triangular bases. Such new-built toroid have $3k$ vertices. Since for 3-triangulation of each triangular prism, so as of polyhedron $A$, 3 tetrahedra are necessary, constructed toroid is possible to 3-triangulate by $3k$ tetrahedra. The toroid $T_9$ in Fig. 7 is starting example in this series with $k = 3$, and $n = 9$.

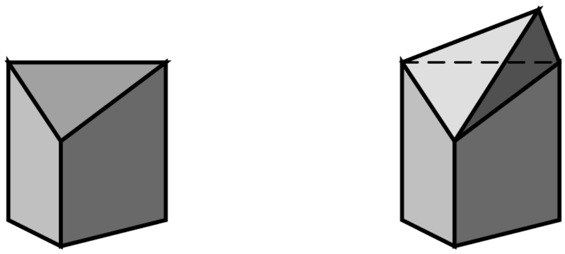

Figure 8: Elementary polyhedra of type $A$ and $B$

For constructing 3-triangulable toroid with $3k + 1$ or $3k + 2$ vertices, we will exchange respectively one or two polyhedra of type $A$ in cycle with one or two polyhedra of type $B$. Elementary polyhedron of type $B$ is built by "gluing" a tetrahedron with two faces congruent to triangular basis of a polyhedron $A$ onto the same polyhedron $A$: one is used for gluing to $A$, and other one serves as a new contact face of $B$.

By this procedure, for each $n \geq 9$, a toroid is built which is cyclically piecewise convex, and we need $n$ tetrahedra for its 3-triangulation. □

**Lemma 2.** *There exists 3-triangulable toroid with $n = 8$ vertices.*

**Proof.** By gluing a tetrahedron onto one of the faces of Császár polyhedron, we will build a new toroid with 8 vertices and 8 tetrahedra in its triangulation. □

Summarizing results of Lemma 1 and Lemma 2 with regard to Császár polyhedron we conclude:

**Theorem 1.** *For each $n \geq 7$, there exists a toroid which is possible to 3-triangulate.*

**3.2** Regarding the minimal triangulation, we shall independently consider the cases $n \geq 9$ and $n = 8$. As mentioned before for case $n = 7$ result is still known, i.e. Császár polyhedron is 3-triangulable only with 7 tetrahedra.

**Lemma 3.** *If it is possible to 3-triangulate toroid with $n \geq 9$ vertices, then the minimal number of tetrahedra necessary for that triangulation is $T_{min} \geq n$.*

Each 3-triangulable polyhedron can be considered as a collection of connected tetrahedra, so it is piecewise convex. Let us form *graph of connection* for convex pieces of toroid in such a way that *nodes* represents convex piece polyhedra, while *edges* represents contact faces between them. For each 3-triangulable toroid $\bar{P}$, such a graph has cyclical part with eventually added branches. So, $\bar{P}$ is possible to decompose to cyclically piecewise convex toroid $P$ built of elements $P_i$, $i = 1, \dots, k$ and eventually additional branches, which are simple piecewise convex polyhedra. Graphs of connection for the toroids $T_9$ and $T_{19}$ (Fig. 7 and Fig. 6) are shown on the figures Fig. 9 and Fig. 10. The first graph for the toroid $T_{19}$ has cyclical part and two branches, while the second one has cyclical part with only one branch. Since "left" branch is composed of elements $A$ and $B$ in such a way that polyhedron is not convex, making cyclical graph for $T_{19}$ is impossible.

If we consider the minimal triangulations $\bar{T}_{min}$ and $\bar{t}^j_{min}$, $j = 1, \dots, l$ of whole polyhedron $\bar{P}$ and its pieces $\bar{P}_j$, it would be expectable that

$$\bar{T}_{min} = \sum_{j=1}^{l} \bar{t}^j_{min}, \tag{1}$$

but it may happen that holds

$$\bar{T}_{min} < \sum_{j=1}^{l} \bar{t}^j_{min}.$$

For example, if two pyramids with 5 vertices in the basis are neighbors, we need 3 tetrahedra to triangulate each of them (Fig. 4). But, these two pyramids together build bipyramid with 7 vertices, and as it was noted, it is possible to triangulate it with 5 tetrahedra. Of course, then we may transform graph of connection and have one node for bipyramid instead of two nodes for two pyramids. In this way, the minimal triangulations of $\bar{P}$ and $\bar{P}_j$ would be harmonized. It means that with the proper choice of graph of connection (i.e. proper choice of pieces of polyhedra), we would have equality (1).

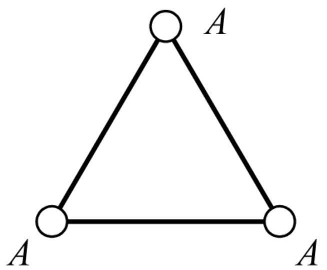


Figure 9: Graph of connection for the toroid $T_9$

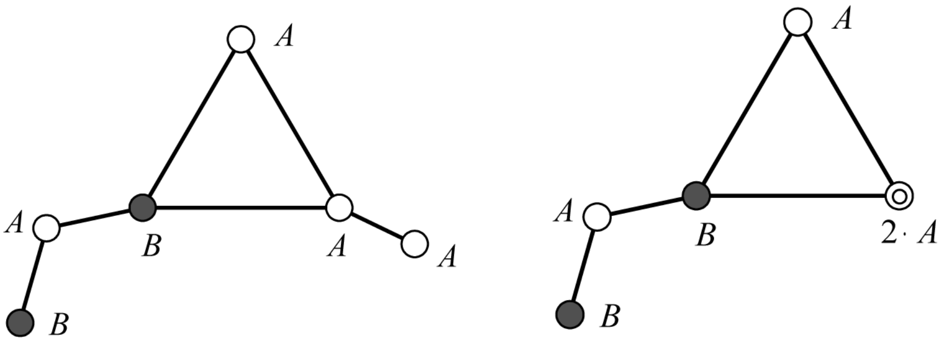


Figure 10: Two graphs of connection for the toroid $T_{19}$

**Proof of Lemma 3.** Let us first consider the minimal number of tetrahedra $T_{min}$ necessary for 3-triangulation of the toroid $P$. Since the minimal 3-triangulation of $P_i$ with $n_i$ vertices has $t_{min} \geq n_i - 3$ tetrahedra, it follows that

$$T_{min} = \sum_{i=1}^{k} t^i_{min} \geq \sum_{i=1}^{k} (n_i - 3) = \sum_{i=1}^{k} n_i - 3k = t.$$

On the other hand, the number of vertices of the toroid $P$ is

$$n = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} m_i,$$

where $m_i$ is the number of vertices for the contact faces $f_i$ between $P_i$ and $P_{i+1}$, $i = 1, \ldots, k-1$, and $f_k$ between $P_k$ and $P_1$. Since $m_i \geq 3$, it follows that

$$n \leq \sum_{i=1}^{k} n_i - 3k = t \leq T_{min}.$$

If the whole toroid $\bar{P}$ has additional branches $\bar{P}_j$, $j = 1, \ldots, l$ with respectively $\bar{n}_j$ vertices and the minimal triangulations with $\bar{t}^j_{min} \geq \bar{n}_j - 3$ tetrahedra, and if the contact faces $\bar{f}_j$ of the toroid $P$ and the branches $\bar{P}_j$ have $\bar{m}_j$ vertices then,

$$\bar{T}_{min} = T_{min} + \sum_{j=1}^{l} \bar{t}^j_{min} \geq T_{min} + \sum_{j=1}^{l} \bar{n}_j - 3l = T$$

and

$$\bar{n} = n + \sum_{j=1}^{l} \bar{n}_j - \sum_{j=1}^{l} \bar{m}_j \leq n + \sum_{j=1}^{l} \bar{n}_j - 3l \leq T_{min} + \sum_{j=1}^{l} \bar{n}_j - 3l = T \leq \bar{T}_{min}$$

where $\bar{n}$ is the number of vertices and $\bar{T}_{min}$ is the number of tetrahedra in the minimal 3-triangulation of $\bar{P}$. □

**Lemma 4.** *If it is possible to 3-triangulate toroid with $n = 8$ vertices, then the minimal number of tetrahedra necessary for the triangulation is $T_{min} \geq 8$.*

**Proof.** In proof of Lemma 2 it is shown that there exists a toroid with $n = 8$ vertices and $T^8_{min} = 8$. Let us suppose that there exists a toroid $P_8$ with $n = 8$ and $T^8_{min} < 8$. Gluing a tetrahedron to $P_8$ gives a toroid

$P_9$ with $n = 9$ vertices and minimal triangulation $T^9_{min} \leq T^8_{min} + 1 < 8 + 1 = 9$. But, by Lemma 3 that is not possible. So, for any 3-triangulable toroid with $n = 8$ vertices $T_{min} \geq 8$. □

So, the next theorem stands:

**Theorem 2.** *If it is possible to 3-triangulate toroid with $n \geq 7$ vertices, then the minimal number of tetrahedra necessary for that triangulation is $T_{min} \geq n$.*

## Summary

Concept of the piecewise convex polyhedron is useful in considering 3-triangulation of non-convex polyhedra especially of toroids. We can do that using graph of connection of these polyhedra. In this paper it was discussed the problems of 3-triangulable toroids existence, and of the minimal number of tetrahedra necessary for the 3-triangulation. In the similar way, it would be possible to investigate the same problems for polyhedra topologically equivalent to sphere with $p$ handles but with more possible cases of graphs of connection. Consequently, this investigation is more complicated and it would be left out for some future work.

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