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Overview and Comparative Analysis of the Properties of the Hodge-De Rham and Tachibana Operators

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Abstract. In the present paper we consider two natural, elliptic, self-adjoint second order differential operators acting on exterior differential forms on Riemannian manifolds. These operators are the well-known Hodge-de Rham and little-known Tachibana operators. Basic properties of these operators are very similar, or vice versa are dual with respect to each other. We review the results (partly obtained by the authors) on the geometry of these operators and demonstrate the comparative analysis of their properties.

1. Introduction and notations

1.1. In the present paper we make overview and comparative analysis of the properties of the well-known Hodge-de Rham and little-known Tachibana operators acting on exterior differential forms on Riemannian manifolds.

The paper is based on our report at the International Conference "XVIII Geometrical Seminar" (Serbia, May 25-28, 2014).

1.2. Let (M, g) be an n-dimensional Riemannian manifold with the Levi-Civita connection ∇ and $\Omega^r(M)$ be *the space of smooth r-forms* on M for all $r = 1, ..., n-1$. If (M, g) is compact we can define *the Hodge product* for $\omega, \theta \in \Omega^r(M)$ by the formula $\langle \omega, \theta \rangle = \int_M \frac{1}{r!} g(\omega, \theta) dVol$ where $g(\omega, \theta)$ is the point-wise inner product. Using the Hodge product, we can define the adjoint *exterior codifferential* d^*: $\Omega^{r+1}(M) \to \Omega^r(M)$ to the *exterior differential* d: $\Omega^r(M) \to \Omega^{r+1}(M)$ via the formula $\langle d\omega, \theta \rangle = \langle \omega, d^*\theta \rangle$ whenever $\omega \in \Omega^{r-1}(M)$, $\theta \in \Omega^{r+1}(M)$

2. A specification of the Bourguignons result and conformal Killing forms

2.1. J.P. Bourguignon (see [3]) considered the space of *natural* (with respect to isometric diffeomorphisms) *first-order differential operators* on $\Omega^r(M)$ with values in the space of homogeneous tensors on M. He proved the existence of a basis of this space which consists of three operators $\{D_1 = d; D_2 = d^*; D_3\}$.

As for the third operator D_3, J.P. Bourguignon said that D_3 does not have any geometric interpretation for $r > 1$. It was also pointed out that in the case $r = 1$ the kernel of D_3 consists of *infinitesimal conformal*

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transformations of (M, g). We remark here that they are also called *conformal Killing vector fields* (see [14]; [36, pp. 53-54]).

Improving Bourguignons result, we showed that

$$D_1 = \frac{1}{r+1}d\ ; D_2 = \frac{1}{n-r+1}g \bigwedge d^*\ ; D_3 = \nabla - \frac{1}{r+1}d - \frac{1}{n-r+1}g \bigwedge d^*$$

and proved that the kernel of the third basis operator D_3 consists of *conformal Killing r-forms* (see [23]; [24]).

Conformal Killing r-forms, $1 \leqslant r \leqslant n-1$, or conformal Killing-Yano tensors of rank r are a generalization of *conformal Killing vector fields* (see [14]; [36, pp. 53-54]; [13, p. 309]). For $r = 1$, such forms are dual to conformal Killing vector fields. The vector space of these r-forms will be denoted by $\mathbf{T}^r(M, \mathbb{R})$.

Remark We emphasize here that the conformal Killing r-forms, $1 < r < n$, considered in the monograph [36] have nothing in common (except the name) with those considered in the modern literature and, in particular, in this paper; the notion was introduced by K. Yano as a generalization conformal Killing vector fields but has not been developed further.

2.2. The vector space $\mathbf{K}^r(M, \mathbb{R})$ of co-closed conformal Killing r-forms is defined as $\mathbf{K}^r(M, \mathbb{R}) = \mathbf{T}^r(M, \mathbb{R}) \cap \mathbf{F}^r(M, \mathbb{R})$ where $\mathbf{F}^r(M, \mathbb{R})$ is a vector space of co-closed r-forms. For $r = 1$, such forms are dual to *Killing vector fields* (see [36, pp. 37-38]; [13, p. 237]). Therefore, these forms were called *Killing forms* or *Killing tensors* (see [36, pp. 65-66]). In turn, the space $\mathbf{P}^r(M, \mathbb{R})$ of closed conformal Killing r-forms (or, *planar r-forms*) is defined by the equation $\mathbf{P}^r(M, \mathbb{R}) = \mathbf{T}^r(M, \mathbb{R}) \cap \mathbf{D}^r(M, \mathbb{R})$ where $\mathbf{D}^r(M, \mathbb{R})$ is a vector space of closed r-forms. For $r = 1$, such forms are dual to *concircular vector field*s (see [14]).

Take a local orientation of (M, g) and the corresponding the Hodge star operator $* : \Omega^r(M) \to \Omega^{n-r}(M)$ which is a linear isomorphism mapping any r-form on (M, g) to $(n-r)$-form for an arbitrary r with $0 \leq r \leq n$ (see [2, p. 33]; [17, p. 203]). Then the spaces $\mathbf{P}^r(M, \mathbb{R})$ and $\mathbf{K}^{n-r}(M, \mathbb{R})$ are isomorphic to each other with respect to the Hodge star operator $*$ (see [23]).

Remark During the whole almost half-century history, beginning with Tachibana and Kashiwada papers (see [31], [12]) and ending with recent works (see [19]; [18]; [27]), conformal Killing forms have caused extensive interest of researchers, partly because of their numerous physical applications (see, for example, [23], [1], [9, pp. 414,426], [35], [25] and etc.). A survey of results on the geometry of conformal Killing, co-closed conformal Killing and closed conformal Killing forms and applications of these forms to General Relativity can be found in the introduction of our paper [26].

3. Two elliptic operators on forms and a brief review of their basic properties

3.1. Using the basis operators $D_1 = d; D_2 = d^*$ and D_3, we can define the well-known *Hodge-de Rham Laplacian* (see [2, p. 34]; [17])

$$\Delta = d^* \circ d + d \circ d^* : \Omega^r(M) \to \Omega^r(M)$$

and the Tachibana operator $\square = r(r+1)D_3^* D_3 : \Omega^r(M) \to \Omega^r(M)$ such that (see [26])

$$D_3^* \circ D_3 = \frac{1}{r(r+1)}\left(\bar{\Delta} - \frac{1}{r+1}d^* \circ d - \frac{1}{n-r+1}d \circ d^*\right)$$

where D_3^* is an adjoint operator to D_3 and $\bar{\Delta} = \nabla^* \circ \nabla$ is the *Bochner rough Laplacian*.

The exterior differential r-form ω on a compact (M, g) is called *harmonic* if $\Delta\omega = 0$. Harmonic r-forms on (M, g) constitute a vector space denoted by $\mathbf{H}^r(M, \mathbb{R})$. By the Hodge theory (see [10]), the r-th Betti number equals the dimension of $\mathbf{H}^r(M, \mathbb{R})$: $b_r(M) = dim\mathbf{H}^r(M, \mathbb{R})$ for all $r = 1, ..., n-1$. In particular, if (M, g) is an n-dimensional compact flat Riemannian torus T^n then $b_r(T^n) = \dfrac{n!}{r!(n-r)!}$.

An important property of the Hodge-de Rham Laplacian Δ is that it commutes $*\Delta = \Delta*$. In particular, if ω is a harmonic r-form, then $*\omega$ is a harmonic $(n-r)$-form too, i.e., $\mathbf{H}^r(M, \mathbb{R}) \to \mathbf{H}^{n-r}(M, \mathbb{R})$ is an isomorphism.

This implies the following equation $b_r(M) = b_{n-r}(M)$ which is well known as the *Poincare duality theorem* for Betti numbers. We remark further that the Betti numbers $b_0(M), b_1(M), ..., b_n(M)$ are topological invariants of a compact manifold M.

Remark The results of the first section of this Paragraph were found in the 1940s by Weyl, Hodge, de Rham and Kodaira.

3.2. On the other hand, the Tachibana operator $\square$ has the same properties as the Hodge-de Rham operator Δ. Namely, the following theorem holds (see [26]; [21]; [28]).

Theorem 3.1. *Let (M, g) be an n-dimensional compact oriented Riemannian manifold and $\square$ the Tachibana operator. Then*

1. *$\square$ is an elliptic and self-adjoint operator;*
2. *the kernel of $\square$ consists of conformal Killing r-forms;*
3. *$*\square = \square*$ and if ω is a conformal Killing form, so is $*\omega$.*

From the above properties we conclude that $dim\mathbf{T}^r(M, \mathbb{R}) = t_r(M) < \infty$ and $t_{n-r}(M) = t_r(M)$ for all numbers $t_1(M), ..., t_{n-1}(M)$. These numbers have been named in [20] as the *Tachibana numbers* of (M, g). Moreover, we proved in [22] that these numbers are conformal invariants of (M, g). In turn, we can state that the following equalities $dim\mathbf{K}^r(M, \mathbb{R}) = k_r(M) < \infty$ and $dim\mathbf{P}^r(M, \mathbb{R}) = p_r(M) < \infty$ are true. The numbers $k_r(M)$ and $p_r(M)$ we have called the *Killing* and *planarity numbers* of a compact Riemannian manifold (M, g), respectively (see [22]). These numbers satisfy the following duality property $p_r(M) = k_{n-r}(M)$ for all $r = 1, ..., n-1$ and are scalar projective invariant of (M, g) (see [23] and [22]). In particular, we proved the equality $k_r(S^n) = \dfrac{(n+1)!}{(r+1)!(n-r)!}$ for the standard sphere S^n in Euclidian space $\mathbb{R}^{n+1}$ (see [20]; [30]). As a consequence of this statement, we obtain the following equalities $p_r(S^n) = k_{n-r}(S^n) = \dfrac{(n+1)!}{(n-r+1)!r!}$. In addition, we recall that any two connected, simply connected complete Riemannian manifolds of constant sectional curvature are isometric to each other (see [8, p. 265]). Therefore, the following equalities $k_r(M) = \dfrac{(n+1)!}{(r+1)!(n-r)!}$ and $p_r(M) = \dfrac{(n+1)!}{(n-r+1)!r!}$ are true on an arbitrary connected, simply connected compact Riemannian manifold (M, g) of positive constant sectional curvature.

4. The vanishing theorems of the Tachibana numbers

4.1. Let $\bar{R}$: $\Omega^2(M) \to \Omega^2(M)$ be the well-known symmetric *Riemannian curvature operator* (see [4]; [17]). Namely, the curvature operator $\bar{R}$ at any $x \in M$ is the linear symmetric bilinear map $\bar{R}$: $\Lambda^2(T_xM) \to \Lambda^2(T_xM)$ characterized by the identity $g(\bar{R}(X \wedge Y), V \wedge Z) = g(R(X, Y)Z, V)$ for orthonormal vectors $X, Y, Z, V \in T_xM$ and the Riemannian curvature tensor R. Since $\bar{R}$ is symmetric, it makes sense to talk about the positivity and the nonnegativity of $\bar{R}$. In this case the following proposition holds (see [17, p. 212]): If (M, g) is compact oriented and its curvature operator $\bar{R} \geq 0$ then $b_r(M) \leq b_r(T^n) = \dfrac{n!}{r!(n-r)!}$ where $r = 1, ..., n-1$. In addition, if $\bar{R} > 0$ then $b_r(M) = 0$ for all $r = 1, ..., n-1$. This is known as the *Mayer vanishing theorem* of the Betti numbers. In particular, for a compact Riemannian manifold of positive constant sectional curvature, or for a compact conformally flat oriented Riemannian manifold with the positive-definite Ricci tensor *Ric*, we have $b_r(M) = 0$ for all $r = 1, ..., n-1$ (see [36, pp. 77-78]). On the other hand, we proved in [20] the following proposition.

Theorem 4.1. *The Tachibana numbers $t_r(M) = 0$, $r = 1, ..., n-1$, vanish for an n-dimensional compact oriented Riemannian manifold (M, g) with the negative-definite curvature operator $\bar{R}$. In addition, if $\bar{R} \leq 0$, then $t_r(M) \leq t_r(T^n) = \dfrac{n!}{r!(n-r)!}$.*

The above theorem is an analogue of the "Mayer vanishing theorem" of the Betti numbers. In particular, this theorem implies that for a compact Riemannian manifold of constant negative sectional curvature, or for a compact conformally flat oriented Riemannian manifold with the negative-definite Ricci tensor *Ric*, we have $t_r(M) = 0$ for all $r = 1, ..., n-1$.

4.2. We denote by $S_0^2 M$ the vector space of covariant traceless symmetric 2-tensors over (M, g). From the symmetry properties of the Riemannian curvature tensor R we see that the curvature tensor actually defines a symmetric bilinear map $\overset{\circ}{R}: S_0^2(T_x M) \to S_0^2(T_x M)$ at every point $x \in M$, such as (see [4]; [2, pp. 51-52])

$$g(\overset{\circ}{R}(X \circ Y), X \circ Y) = 2g(R(X, Y)Y, X)$$

where $X \circ Y = 2^{-1}(X \otimes Y + Y \otimes X)$ for any orthogonal vectors $X, Y \in T_x M$. The scalar product on the left-hand side of the identity is the induced unity at the level of the vector space $S^2(T_x M)$ of symmetric bilinear two-forms. This relation defines an algebraic symmetric operator which is called the *curvature operator of the second kind* (see [11]). Since $\overset{\circ}{R}$ is symmetric, it makes sense to talk about the positivity and the nonnegativity of $\overset{\circ}{R}$. In this case the following proposition holds (see [33]): If (M, g) is compact oriented and its curvature operator $\overset{\circ}{R} > 0$ then $b_r(M) = 0$ for all $r = 1, ..., n-1$. This is known as the *Tachibana vanishing theorem* of the Betti numbers. On the other hand, we proved in [30] the following dual proposition.

Theorem 4.2. *The Tachibana numbers $t_r(M)$, $r = 1, ..., n-1$, vanish for an n-dimensional compact oriented Riemannian manifold (M, g) with the negative-definite curvature operator $\overset{\circ}{R}$.*

5. A relationship between the Betti and Tachibana numbers

5.1. Sh. Tachibana has proved in [32] the following proposition: If (M, g) is a $2r$-dimensional compact conformally flat Riemannian manifold with the constant scalar curvature $s > 0$ then the orthogonal decomposition $\mathbf{T}^r(M, \mathbb{R}) = \mathbf{P}^r(M, \mathbb{R}) \oplus \mathbf{K}^r(M, \mathbb{R})$ holds. In this case we can conclude that $t_r(M) = p_r(M) + k_r(M)$ and $b_r(M) = 0$. On the other hand, we have proved in [26] and [21] the following theorem that generalizes Tachibana's proposition.

Theorem 5.1. *Let (M, g) be a connected compact oriented Riemannian n-manifold. Suppose that the Betti number $b_r(M)$ vanishes for some $1 \le r \le n-1$ and the corresponding Tachibana number $t_r(M)$ and Killing number $k_r(M)$ are related by $t_r(M) > k_r(M) > 0$. Then the difference $t_r(M) - k_r(M)$ is equal to the planarity number $p_r(M)$ of the given manifold.*

We can amplify this proposition by the following statement (see [21]).

Corollary 5.2. *Suppose that, for a connected compact oriented n-dimensional Riemannian manifold (M, g), the Betti, Tachibana, and Killing numbers satisfy the conditions $b_1(M) = 0$ and $t_1(M) > k_1(M) > 0$. Then $t_1(M) = \frac{1}{2}(n+1)(n+2)$ and $b_2(M) = ... = b_{n-1}(M) = 0$ because the manifold (M, g) is conformally diffeomorphic to the standard sphere S^n in Euclidian space $\mathbb{R}^{n+1}$.*

The duality properties of the Betti, Killing and planarity numbers imply the following proposition.

Corollary 5.3. *Suppose that, for a connected compact oriented n-dimensional Riemannian manifold (M, g), the Betti number $b_{n-1}(M)$ vanishes and the Killing number $k_{n-1}(M)$ does not vanish. Then $t_1(M) = \frac{1}{2}(n+1)(n+2)$ and $b_2(M) = ... = b_{n-1}(M) = 0$ because the manifold (M, g) is conformally diffeomorphic to the standard sphere S^n in Euclidian space $\mathbb{R}^{n+1}$.*

5.2. Many propositions are concerned with the dimensions of vector spaces of conformal Killing and Killing r-forms, $1 \le r \le n-1$, and vector fields in a neighborhood of an arbitrary point of the Riemannian manifold (M, g). We investigated the problem of the existence of compact Riemannian manifold with nonzero Tachibana numbers $t_r(M)$ for all $r = 1, ..., n-1$ in our paper [26]; [21]; [30] and [29]. In the theorems of this section we discuss the dimensions of the vector spaces of conformal Killing, Killing and planarity r-forms, $1 \le r \le n-1$, globally defined on an n-dimensional compact Riemannian manifold (see [27] and [21]).

Theorem 5.4. *Let (M, g) be a connected compact oriented n-dimensional Riemannian manifold. Suppose that one of the following two conditions holds: 1. the Ricci tensor Ric is nonpositive and the Tachibana number $t_1(M) = p \neq 0$; 2. the Ricci tensor Ric is nonnegative and the Betti number $b_1(M) = p \neq 0$. Then the Tachibana numbers satisfy the relations $t_r(M) = b_r(M) \geq \frac{p!}{r!(p-r)!}$ for all $r \leq p$. If $Ric \leq 0$ and $t_1(M) = n$ or $Ric \geq 0$ and $b_1(M) = n$, then (M, g) is a flat Riemannian n-torus.*

Finally, in this paragraph we consider the Tachibana numbers $t_r(M)$, $1 \leq r \leq n-1$, of an n-dimensional connected compact conformally flat Riemannian manifolds (M, g).

First, we known (see [14]; [2, p. 62]) that an n-dimensional compact and a simply connected Riemannian manifold which is locally conformally flat must be globally conformally equivalent with the standard sphere S^n in Euclidian space $\mathbb{R}^{n+1}$ and, therefore, $b_1(M) = ... = b_{n-1}(M) = 0$. Moreover, we recall that a compact connected manifold is simply connected if and only if its first Betti number $b_1(M) = 0$.

Second, from our Theorem 5.1 we conclude that for the standard sphere S^n in Euclidian space $\mathbb{R}^{n+1}$ the equalities $t_r(S^n) = k_r(S^n) + p_r(S^n) = \frac{(n+2)!}{(r+1)!(n-r+1)!}$ are true. Then for an n-dimensional compact simply connected conformally flat Riemannian manifold (M, g) we have $t_r(M) = t_r(S^n) = \frac{(n+2)!}{(r+1)!(n-r+1)!}$ because the Tachibana number $t_r(M)$ is a conformal invariant of (M, g) for all $r = 1, ..., n-1$.

In conclusion, we can state the "existence theorem" of the Tachibana numbers (see [21] and [30]).

Theorem 5.5. *If the first Betti number $b_1(M)$ of a connected compact conformally flat n-dimensional Riemannian manifold (M, g) vanishes, then $t_r(M) = \frac{(n+2)!}{(r+1)!(n-r+1)!}$ and $b_2(M) = ... = b_{n-1}(M) = 0$.*

We further remark that there are other interesting examples of compact Riemannian manifolds with non-zero Tachibana numbers (see [26]).

6. Eigenvalues of the Hodge-de Rham and Tachibana operators and their boundary theorems

6.1. A real number λ^r for which there is an r-form ω which is not identically zero such that $\Delta\omega = \lambda^r\omega$ is called an *eigenvalue* of Δ and the corresponding r-form ω an *eigenform* of Δ corresponding to λ^r . The eigenforms corresponding to a fixed λ^r form a subspace of $\Omega^r(M)$, namely the *eigenspace* of λ^r . In addition, we denote by $m(\lambda^r) = dimV_{\lambda^r}$ the *multiplicity of the eigenvalue* λ^r of the Hodge-de Rham operator Δ. The following statements about eigenvalues of Δ and their corresponding forms are valid (see [5, p. 334-343]; [7, p. 273-321]).

1. The Laplacian Δ has a positive eigenvalue λ^r and in fact a whole sequence of eigenvalues which diverge to $+\infty$.
2. $\lambda^r = \lambda^{n-r}$ for all $r = 1, ..., n$.
3. The eigenspaces of λ^r are finite dimensional.
4. The eigenforms corresponding to distinct eigenvalues are orthogonal.

Let (M, g) be a compact oriented manifold. If, in addition, the curvature operator $\bar{R}$ satisfies the inequality $\bar{R} \geq \delta$ for some positive number δ at every point $x \in M$, then $\lambda^r \geq inf\,\{r(n-r+1)\delta; (n-r)(r+1)\delta\}$ and the Betti numbers $b_r(M) = 0$ for all $r = 1, ..., n-1$ (see [5, pp. 342-343]; [34]).

For $n < 2r$ the equality $\lambda_1^r = (r+1)(n-r)\delta$ is attained for some eigenform $\omega \in \mathbf{K}^r(S^n)$ on the Euclidian n-sphere S^n of the constant curvature δ . In this case we have the inequality $m(\lambda^r) \leq k_r(S^n) = \frac{(n+1)!}{(r+1)!(n-r)!}$. On the other hand, for $n > 2r$ the equality $\lambda_1^r = r(n-r+1)\delta$ is attained for some eigenform $\omega \in \mathbf{P}^r(S^n)$ on the Euclidian n-sphere S^n of the constant curvature δ . In this case we have the inequality $m(\lambda^r) \leq p_r(S^n) = \frac{(n+1)!}{(n-r+1)!r!}$. Finally, for $n = 2r$ the equality $\lambda_1^r = r(r+1)\delta$ holds for some eigenform

$\omega \in \mathbf{T}^r(S^{2r})$ on the Euclidian $2r$-sphere S^{2r} of the constant curvature δ . In this case we have the inequality $m(\lambda^r) \le t_r(S^{2r}) = 2\frac{(2r+1)!}{(r+1)!r!}$.

It is obvious that in the case $n = 2r$ the Hodge-de Rham operator Δ has the form $\Delta = (r+1)\bar{\Delta} - (r+1)\Box$. In this case we have proved in [26] the following theorem.

Theorem 6.1. *Let (M, g) be a $2r$-dimensional compact oriented conformally flat Riemannian manifold with positive scalar curvature s. Then the eigenvalue λ of the Hodge Laplacian Δ on r-forms satisfies the inequality $\lambda \ge (r+1)(2r(2r-1))^{-1}s_0$ where $s_0 = inf_{x \in M} s(x)$. If there exists an eigen r-form corresponding to $\lambda = (r+1)(2r(2r-1))^{-1}s_0$ then it is a conformal Killing form and the multiplicity of the eigenvalue λ^r satisfies the inequality $m(\lambda^r) \le 2\frac{(2r+1)!}{(r+1)!r!}$.*

6.2. The number $\lambda^r \in \mathbb{R}$ is called an *eigenvalue* if there exists some non-zero $\omega \in \Omega^r(M)$, such as $\Box\omega = \lambda^r\omega$ (see [26]). We denote by $m(\lambda^r) := \dim V_{\lambda^r}$ the *multiplicity of the eigenvalue* λ^r of the Tachibana operator $\Box$.

Using the general theory of elliptic operators it can be proved that the r-spectrum of $\Box : \Omega^r(M) \to \Omega^r(M)$ denoted by $Spec^{(r)}\Box$ lies on the non-negative real half line and is pure discrete, i.e., is given by a sequence $Spec^{(r)}\Box = \{0 \le \lambda_1^r \le \lambda_2^r \le ... \to \infty\}$, where all eigenvalues are of finite multiplicity (see [5]; [7]). This sequence begins with zero if and only if the Tachibana number $t_r(M) \ne 0$. We have the following theorem (see [28]).

Theorem 6.2. *Let $\Box$ be the Tachibana operator on $\Omega^r(M)$ for a compact n-dimensional Riemannian manifold (M, g). Then the following four statements hold.*

1. *The r-spectrum $Spec^{(r)}\Box$ of $\Box$ has the form $Spec^{(r)}\Box = \{0 \le \lambda_1^r \le \lambda_2^r \le ... \to \infty\}$.*
2. *$Spec^{(r)}\Box = Spec^{(n-r)}\Box$.*
3. *All eigenspaces of $\Box$ are finite dimensional.*
4. *Eigenspaces for different eigenvalues are orthogonal.*

Remark The above theorem is an analogue of the well known theorem on the spectrum $Spec^{(r)}\Delta$ of the Hodge-de Rham Laplacian $\Delta : \Omega^r(M) \to \Omega^r(M)$ (see [5]; [7]).

6.3. We say that the curvature operator $\mathring{R}$ is negative and bounded from above if $\mathring{R}$ satisfies the inequality $g(\mathring{R}(\varphi, \varphi) \le -\varepsilon\ g(\varphi, \varphi)$ for any traceless symmetric 2-tensor φ and a positive number ε. In view of the above, we can formulate the theorem (see [28]).

Theorem 6.3. *Let (M, g) be a compact Riemannian manifold with the negative curvature operator $\mathring{R}$ bounded above by some negative number $-\varepsilon$. Then $t_r(M) = 0$ and the first eigenvalue λ_1^r of the Tachibana operator $\Box$ satisfies the inequality $\lambda_1^r \ge r(n-r)\varepsilon > 0$. If (M, g) admits an eigenform ω that corresponds to $\lambda_1^r = r(n-r)\varepsilon$ then ω is harmonic and the multiplicity of the first eigenvalue λ_1^r satisfies the inequality $m(\lambda^r) \le b_r(M)$ for the Betti number $b_r(M)$ of (M, g) and all $r = 1, ..., n-1$.*

It is obvious that in the case $n = 2r$ the Tachibana operator $\Box$ has the form $\Box = \bar{\Delta} - \frac{1}{(r+1)}\Delta$. In this case we have proved in [28] the following theorem.

Theorem 6.4. *Let (M, g) be a $2r$-dimensional compact oriented conformally flat Riemannian manifold with negative scalar curvature s. Then the eigenvalue λ of any eigen r-form of $\Box$ satisfies the inequality $\lambda^r \ge (2(2r-1))^{-1}s_0$ where $s_0 = inf_{x \in M}|s(x)|$. If there exists an eigen r-form corresponding to the eigenvalue $\lambda^r = (2(2r-1))^{-1}s_0$ then it is harmonic and the multiplicity of the eigenvalue λ^r satisfies the inequality $m(\lambda^r) \le b_r(M)$ for the Betti number $b_r(M)$ of (M, g).*

Finally, in this paragraph we consider an n-dimensional compact hyperbolic space $\mathbb{H}^n$ with standard metric g_0 having constant sectional curvature equal to -1. In this case, we obtain from Theorem 6.4 that $\lambda_1^r \ge r(n-r)$ for the first eigenvalue λ_1^r of the Tachibana operator $\Box$. At the same time it is well known (see [8]) that L^2-harmonic r-forms appear on a simply connected complete hyperbolic manifold (M, g) of constant sectional curvature -1 if and only if $n = 2r$. Therefore, if (M, g) is a compact hyperbolic space $(\mathbb{H}^n, g_0)$ then the equality $\lambda_1^r = r(n-r)$ is attained if and only if $n = 2r$. In this case the multiplicity of λ^r is equal to the Betti number $b_r(\mathbb{H}^{2r})$.

7. Table

In the following table we list the summary of basic properties of the Hodge-de Rham and Tachibana operators.

Elliptic operators on forms from the basis d, d^*, D	
The Hodge-de Rham Laplacian $\Delta = d\,d^* + d\,d^*$	The Tachibana operator $\square = r(r+1)D * D$
Kernels of operators and their scalar invariants	
The Hodge theorem: $\mathbf{H}^r(M, \mathbb{R}) = Ker\ \Delta$ $dim\ \mathbf{H}^r(M, \mathbb{R}) = b_r(M) < \infty$	$\mathbf{T}^r(M, \mathbb{R}) = Ker\ \square$ $dim\ \mathbf{T}^r(M, \mathbb{R}) = t_r(M) < \infty$ $t_r(M)$ is a conformal invariant
Duality properties	
The Poincare duality properties: $b_r(M) = b_{n-r}(M)$	$t_r(M) = t_{n-r}(M)$
Vanishing theorems for Betti and Tachibana numbers	
Meyer and Tachibana theorems: $\bar{R} > 0$ or $\mathring{R} > 0 \Rightarrow b_r(M) = 0$	$\bar{R} < 0$ or $\mathring{R} < 0 \Rightarrow t_r(M) = 0$
First eigenvalues of operators	
The Meyer theorem: $\bar{R} > \delta > 0 \Rightarrow$ $\lambda_1^r \geq inf\{r(n-r+1)\delta, (r+1)(n-r)\delta\}$	$\mathring{R} < -\delta < \mathbf{0} \Rightarrow \lambda_1^r \geq r(n-r)\delta$
Multiplicities of Eigenvalues	
The Tachibana theorem: $\lambda_1^r = inf\{r(n-r+1)\delta, (r+1)(n-r)\delta\}$ $\Rightarrow$ ω is conformal Killing and $m(\lambda^r) \leq t_r(M)$	$\lambda_1^r = r(n-r)\delta \Rightarrow$ $\Rightarrow$ ω is harmonic and $m(\lambda^r) \leq b_r(M)$

8. Appendix

The theory of natural (with respect to isometric diffeomorphisms) first and second order differential operators acting on exterior differential forms maybe extended to generalized Riemannian manifolds (see [6], [15] and [16]). This is our future work.

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