Multiple Orthogonality in the Space of Trigonometric Polynomials of Semi–Integer Degree

Marija P. Stanić[a], Tatjana V. Tomović[a]

[a]Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Kragujevac, Serbia

Abstract.

In this paper we consider multiple orthogonal trigonometric polynomials of semi–integer degree, which are necessary for constructing of an optimal set of quadrature rules with an odd number of nodes for trigonometric polynomials in Borges' sense [Numer. Math. 67 (1994) 271–288]. We prove that such multiple orthogonal trigonometric polynomials satisfy certain recurrence relations and present numerical method for their construction, as well as for construction of mentioned optimal set of quadrature rules. Theoretical results are illustrated by some numerical examples.

1. Introduction

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy orthogonality conditions with respect to more than one measure. Such polynomials in the algebraic case arise in the theory of simultaneous rational approximation, in particular in Hermite–Padé approximation of a system of $r \in \mathbb{Z}^+$ Markov functions (see [14, 15]). For more details about multiple orthogonal algebraic polynomials see, e.g., [1–3, 6, 8, 11, 12, 14, 16–18, 20–22].

A generalization of orthogonal trigonometric polynomials of semi–integer degree in the sense that they satisfy orthogonality conditions spread over $p \in \mathbb{N}$ different measures leads to the concept of multiple orthogonal trigonometric polynomials of semi–integer degree, which were introduced in [13], where also their main properties were proved. In this paper we restrict our attention to the multiple orthogonal trigonometric polynomials of semi–integer degree with respect to $p \in \mathbb{N}$ even weight functions.

The paper is organized as follows. Some basic facts about the type I and type II multiple orthogonal trigonometric polynomials of semi–integer degree from [13] are repeated in Section 2. In Section 3 we consider multiple orthogonality with respect to a set of even weight functions. Section 4 is devoted to nearly diagonal multi–indices and the corresponding recurrence relations, while Section 5 is devoted to applications in numerical integration, where one numerical example is given, too.

2. Multiple orthogonal trigonometric polynomials of semi–integer degree

For nonnegative integer m by $\mathcal{T}_m^{1/2}$ we denote the linear space of all trigonometric polynomials of semi–integer degree less than or equal to $m + 1/2$, i.e., the linear span of the set $\{\cos(k + 1/2)x, \sin(k + 1/2)x : k = 0, 1, \dots, m\}$. By $\mathcal{T}^{1/2}$ we denote the linear space of all trigonometric polynomials of semi–integer degree, and by $\mathcal{P}_m$ the space of all algebraic polynomials of degree less than or equal to m.

Let p be a positive integer and let $\mathbf{n} = (n_1, n_2, \dots, n_p)$ be a *multi–index*, i.e., a vector of p nonnegative integers, with length $|\mathbf{n}| = n_1 + n_2 + \cdots + n_p$. We introduce a partial order on multi–indices in the following way: $\mathbf{m} \preceq \mathbf{n} \Leftrightarrow m_\nu \le n_\nu$ for every $\nu = 1, 2, \dots, p$.

Let $W = (w_1, w_2, \dots, w_p)$ be a vector of p weight functions, which are integrable and nonnegative on some interval E of length 2π, vanishing there only on a set of a measure zero. In what follows, we always assume that interval E is closed on the left and open on the right, i.e., that interval E is of the form $[L, 2\pi + L)$, for $L \in \mathbb{R}$. We introduce the following inner products

$$\langle f, g \rangle_\nu = \int_E f(x) g(x) w_\nu(x) \, dx, \quad \nu = 1, 2, \dots, p, \quad f, g \in \mathcal{T}^{1/2}. \tag{1}$$

There are two types of multiple orthogonal trigonometric polynomials of semi–integer degree (see [13]).

$1°$ Type I multiple orthogonal trigonometric polynomials of semi–integer degree with respect to W are collected in a vector $(A_{\mathbf{n},1}^{1/2}, A_{\mathbf{n},2}^{1/2}, \dots, A_{\mathbf{n},p}^{1/2})$ of trigonometric polynomials of semi–integer degree, where $A_{\mathbf{n},\nu}^{1/2}$ has semi–integer degree $n_\nu - 1/2$, $\nu = 1, 2, \dots, p$, such that the following orthogonality conditions hold:

$$\sum_{\nu=1}^p \left\langle A_{\mathbf{n},\nu}^{1/2}(x), \cos\left(k + \frac{1}{2}\right)x \right\rangle_\nu = 0, \qquad k = 0, 1, 2, \dots, |\mathbf{n}| - 2, \tag{2}$$

$$\sum_{\nu=1}^p \left\langle A_{\mathbf{n},\nu}^{1/2}(x), \sin\left(k + \frac{1}{2}\right)x \right\rangle_\nu = 0, \qquad k = 0, 1, 2, \dots, |\mathbf{n}| - 2,$$

with the normalizations:

$$\sum_{\nu=1}^p \left\langle A_{\mathbf{n},\nu}^{1/2}(x), \cos\left(|\mathbf{n}| - \frac{1}{2}\right)x \right\rangle_\nu = 1, \tag{3}$$

$$\sum_{\nu=1}^p \left\langle A_{\mathbf{n},\nu}^{1/2}(x), \sin\left(|\mathbf{n}| - \frac{1}{2}\right)x \right\rangle_\nu = 1.$$

$2°$ Type II multiple orthogonal trigonometric polynomial of semi–integer degree with respect to W is trigonometric polynomial $T_{\mathbf{n}}^{1/2}$ of semi–integer degree $|\mathbf{n}| + 1/2$ which satisfies the following orthogonality conditions

$$\left\langle T_{\mathbf{n}}^{1/2}(x), \cos\left(k_\nu + \frac{1}{2}\right)x \right\rangle_\nu = 0, \qquad k_\nu = 0, 1, \dots, n_\nu - 1, \tag{4}$$

$$\left\langle T_{\mathbf{n}}^{1/2}(x), \sin\left(k_\nu + \frac{1}{2}\right)x \right\rangle_\nu = 0, \qquad k_\nu = 0, 1, \dots, n_\nu - 1,$$

for $\nu = 1, 2, \dots, p$.

For $p = 1$ multiple orthogonal trigonometric polynomials reduce to the ordinary orthogonal trigonometric polynomials of semi–integer degree (see [5, 9, 10, 19]).

The orthogonality conditions (4) give system of $2(n_1 + n_2 + \cdots + n_p) = 2|\mathbf{n}|$ linear equations for the $2|\mathbf{n}| + 2$ unknown coefficients $a_k, b_k, k = 0, 1, \dots, |\mathbf{n}|$, of trigonometric polynomial

$$T_{\mathbf{n}}^{1/2}(x) = \sum_{k=0}^{|\mathbf{n}|} \left(a_k \cos\left(k + \frac{1}{2}\right)x + b_k \sin\left(k + \frac{1}{2}\right)x\right) \in \mathcal{T}_{|\mathbf{n}|}^{1/2}.$$

Thus, we have to fix two coefficients in advance and we choose to fix the leading coefficients $a_{|\mathbf{n}|}$ and $b_{|\mathbf{n}|}$, $(a_{|\mathbf{n}|}, b_{|\mathbf{n}|}) \neq (0,0)$. For the following two special choices of the leading coefficients, $(a_{|\mathbf{n}|}, b_{|\mathbf{n}|}) \in \{(1,0),(0,1)\}$, we introduce the notation

$$T_{\mathbf{n}}^{C,1/2}(x) = \cos\left(\left(|\mathbf{n}| + \frac{1}{2}\right)x\right) + \sum_{k=0}^{|\mathbf{n}|-1}\left(c_k^{(\mathbf{n})}\cos\left(k + \frac{1}{2}\right)x + d_k^{(\mathbf{n})}\sin\left(k + \frac{1}{2}\right)x\right), \tag{5}$$

$$T_{\mathbf{n}}^{S,1/2}(x) = \sin\left(\left(|\mathbf{n}| + \frac{1}{2}\right)x\right) + \sum_{k=0}^{|\mathbf{n}|-1}\left(f_k^{(\mathbf{n})}\cos\left(k + \frac{1}{2}\right)x + g_k^{(\mathbf{n})}\sin\left(k + \frac{1}{2}\right)x\right).$$

We call $T_{\mathbf{n}}^{C,1/2}(x)$ and $T_{\mathbf{n}}^{S,1/2}(x)$ the monic cosine and the monic sine multiple orthogonal trigonometric polynomial of semi–integer degree, respectively.

If the system (4) has a unique solution, then the multi–index $\mathbf{n}$ is normal (see [13]). If all multi–indices are normal, then we have a *perfect system*.

The matrix of coefficients of systems (4) can be singular, thus we need to impose some additional conditions on the p weight functions to provide the uniqueness of multiple orthogonal trigonometric polynomials of semi–integer degree. The uniqueness is guaranteed if the following set of functions

$$\{w_\nu \cos(k_\nu + 1/2)x, w_\nu \sin(k_\nu + 1/2)x : \nu = 1,2,\ldots,p, k_\nu = 0,1,\ldots,n_\nu - 1\},$$

form a Chebyshev system on E for the multi–index $\mathbf{n}$. Such $W = (w_1, w_2, \ldots, w_p)$ is called trigonometric AT system (TAT system) of weight functions for multi–index $\mathbf{n}$.

The following theorem about the zeros of type II multiple orthogonal trigonometric polynomial of semi–integer degree was proved in [13].

Theorem 2.1. *Suppose that $\mathbf{n}$ is a multi–index such that $W = (w_1, w_2, \ldots, w_p)$ is a TAT system of weight functions for all multi–indices $\mathbf{m}$ such that $\mathbf{m} \preceq \mathbf{n}$. Type II multiple orthogonal trigonometric polynomial of semi–integer degree $T_{\mathbf{n}}^{1/2}(x)$ with respect to W has exactly $2|\mathbf{n}| + 1$ simple zeros on E.*

3. TAT system of even weight functions

Let $W = (w_1, w_2, \ldots, w_p)$ be a TAT system for multi–index $\mathbf{n}$ on the interval $E = [-\pi, \pi)$, such that each $w_k, k = 1,2,\ldots,p$, is an even function on $(-\pi, \pi)$.

Theorem 3.1. *Let $\mathbf{n}$ be a multi–index and $W = (w_1, w_2, \ldots, w_p)$ be a TAT system with respect to $\mathbf{n}$ on the interval $E = [-\pi, \pi)$. If all the weight functions from W are even on the interval $(-\pi, \pi)$, then all coefficients $d_k^{(\mathbf{n})}$ and $f_k^{(\mathbf{n})}$, $k = 0,1,\ldots,|\mathbf{n}| - 1$, in (5) are equal to zero, i.e., the monic multiple orthogonal trigonometric polynomials of semi–integer degree reduce to*

$$T_{\mathbf{n}}^{C,1/2}(x) = \cos\left(\left(|\mathbf{n}| + \frac{1}{2}\right)x\right) + \sum_{k=0}^{|\mathbf{n}|-1} c_k^{(\mathbf{n})}\cos\left(k + \frac{1}{2}\right)x \tag{6}$$

and

$$T_{\mathbf{n}}^{S,1/2}(x) = \sin\left(\left(|\mathbf{n}| + \frac{1}{2}\right)x\right) + \sum_{k=0}^{|\mathbf{n}|-1} g_k^{(\mathbf{n})}\sin\left(k + \frac{1}{2}\right)x. \tag{7}$$

Proof. Let us consider the monic cosine multiple orthogonal trigonometric polynomial of semi–integer degree $T_{\mathbf{n}}^{C,1/2}(x)$. Since all the weight functions from W are even, the system (4) reduces to the following two independent systems

$$\sum_{k=0}^{|\mathbf{n}|-1} c_k^{(\mathbf{n})} \left\langle \cos\left(k+\frac{1}{2}\right)x, \cos\left(k_\nu+\frac{1}{2}\right)x \right\rangle_\nu = -\left\langle \cos\left(|\mathbf{n}|+\frac{1}{2}\right)x, \cos\left(k_\nu+\frac{1}{2}\right)x \right\rangle_\nu, \quad k_\nu = 0,1,\ldots,n_\nu-1, \ \nu = 1,2,\ldots,p,$$

and

$$\sum_{k=0}^{|\mathbf{n}|-1} d_k^{(\mathbf{n})} \left\langle \sin\left(k+\frac{1}{2}\right)x, \sin\left(k_\nu+\frac{1}{2}\right)x \right\rangle_\nu = 0, \quad k_\nu = 0,1,\ldots,n_\nu-1, \ \nu = 1,2,\ldots,p, \tag{8}$$

with unknown coefficient $c_k^{(\mathbf{n})}$ and $d_k^{(\mathbf{n})}$, $k = 0,1,\ldots,|\mathbf{n}|-1$, respectively. Since W is a TAT system with respect to $\mathbf{n}$ on the interval $E = [-\pi,\pi)$, system (4) has the unique solution, so the previous two systems have the unique solutions. Therefore, system (8) has only the trivial solution $d_k^{(\mathbf{n})} = 0$, $k = 0,1,\ldots,|\mathbf{n}|-1$. Due to uniqueness, we conclude that $T_{\mathbf{n}}^{C,1/2}(x)$ depends only on cosine functions.

The statement for $T_{\mathbf{n}}^{S,1/2}(x)$ can be proved analogously. □

The following corollary follows immediately from (6) and (7).

Corollary 3.2. *Suppose that $\mathbf{n}$ is a multi–index such that $W = (w_1, w_2, \ldots, w_p)$ is a TAT system of weight functions with respect to $\mathbf{n}$ on interval $[-\pi, \pi)$. If all weight functions from W are even on the interval $(-\pi, \pi)$, then*

$$T_{\mathbf{n}}^{C,1/2}(-\pi) = 0 \quad and \quad T_{\mathbf{n}}^{S,1/2}(0) = 0.$$

Theorem 3.3. *Let $\mathbf{n} = (n_1, n_2, \ldots, n_p)$ be a multi–index such that $W = (w_1, w_2, \ldots, w_p)$ is a TAT system with respect to $\mathbf{n}$ on the interval $[-\pi, \pi)$. If all weight functions from W are even on the interval $(-\pi, \pi)$, then we have*

$$\int_{-1}^{1} C_{\mathbf{n}}(x) C_{k_\nu}(x) \sqrt{\frac{1+x}{1-x}} w_\nu(\arccos x) = 0, \quad k_\nu = 0,1,\ldots,n_\nu-1,$$

and

$$\int_{-1}^{1} S_{\mathbf{n}}(x) S_{k_\nu}(x) \sqrt{\frac{1-x}{1+x}} w_\nu(\arccos x) = 0, \quad k_\nu = 0,1,\ldots,n_\nu-1,$$

for $\nu = 1,2,\ldots,p$, where $C_{k_\nu}, S_{k_\nu} \in \mathcal{P}_{k_\nu}$, $C_{\mathbf{n}}, S_{\mathbf{n}} \in \mathcal{P}_{|\mathbf{n}|}$, are algebraic polynomials given by

$$C_{\mathbf{n}}(x) = \sum_{k=0}^{|\mathbf{n}|} c_k^{(\mathbf{n})} \Big(T_k(x) - (1-x)U_{k-1}(x) \Big)$$

and

$$S_{\mathbf{n}}(x) = \sum_{k=0}^{|\mathbf{n}|} g_k^{(\mathbf{n})} \Big(T_k(x) + (1+x)U_{k-1}(x) \Big),$$

T_k and U_k, $k \in \mathbb{N}_0$, are Chebyshev polynomials of the first and second kind, respectively.

Proof. Since all the weight functions $w_\nu(x)$, $\nu = 1,2,\ldots,p$, are even on the interval $(-\pi,\pi)$, from the orthogonality conditions for $T_{\mathbf{n}}^{C,1/2}$, we conclude that

$$\int_0^\pi T_{\mathbf{n}}^{C,1/2}(x) T_{k_\nu}^{C,1/2}(x) \, w_\nu(x) \, dx = 0, \quad k_\nu = 0,1,\ldots,n_\nu-1, \ \nu = 1,2,\ldots,p.$$

Applying the substitution $x := \arccos x$, we get

$$\int_{-1}^{1} T_{\mathbf{n}}^{C,1/2}(\arccos x) T_{k_\nu}^{C,1/2}(\arccos x) \frac{w_\nu(\arccos x)}{\sqrt{1 - x^2}} \, dx = 0. \tag{9}$$

By using elementary trigonometric transformations we get

$$\cos\left(\left(k + \frac{1}{2}\right)\arccos x\right) = \sqrt{\frac{1 + x}{2}} T_k(x) - \sqrt{\frac{1 - x}{2}} \sqrt{1 - x^2} U_{k-1}(x),$$

where $T_k(x)$ and $U_{k-1}(x)$, are Chebyshev polynomials of the first and second kind, respectively. Thus,

$$T_{\mathbf{n}}^{C,1/2}(\arccos x) = \sqrt{\frac{1 + x}{2}} \sum_{k=0}^{|\mathbf{n}|} c_k^{(\mathbf{n})}\left(T_k(x) - (1 - x)U_{k-1}(x)\right).$$

By substituting the obtained formulas in (9), after some elementary transformations, we get the first assertion.

According to orthogonality conditions for $T_{\mathbf{n}}^{S,1/2}$ and equality

$$\sin\left(\left(k + \frac{1}{2}\right)\arccos x\right) = \sqrt{\frac{1 - x}{2}} T_k(x) + \sqrt{\frac{1 + x}{2}} \sqrt{1 - x^2} U_{k-1}(x),$$

the second assertion can be proved in the same way. $\square$

From the proof of Theorem 3.3 we conclude that

$$T_{\mathbf{n}}^{C,1/2}(\arccos x) = \sqrt{\frac{1 + x}{2}} C_{\mathbf{n}}(x), \tag{10}$$

where $C_{\mathbf{n}}$ is the type II multiple orthogonal algebraic polynomial with respect to the multi–index $\mathbf{n}$ and weight functions

$$\left(\sqrt{\frac{1 + x}{1 - x}} w_1(\arccos x), \sqrt{\frac{1 + x}{1 - x}} w_2(\arccos x), \dots, \sqrt{\frac{1 + x}{1 - x}} w_p(\arccos x)\right) \tag{11}$$

on the interval $[-1, 1)$. This implies that the zeros the type II multiple orthogonal trigonometric polynomial of semi–integer degree $T_{\mathbf{n}}^{C,1/2}$ can be calculated from the zeros of the corresponding type II multiple orthogonal algebraic polynomial (see [11], [12]). Connection between the zeros of these polynomials is given by the following Lemma.

In what follows we simple say that $W = (w_1, w_2, \dots, w_p)$ is a TAT system of even weight functions on the interval $[-\pi, \pi)$ when all weight functions from W are even on the interval $(-\pi, \pi)$.

Lemma 3.4. *Let $\mathbf{n}$ be a multi–index and let $W = (w_1, w_2, \dots, w_p)$ be a TAT system of even weight functions on the interval $[-\pi, \pi)$ with respect to $\mathbf{n}$. If τ_k, $k = 1, 2, \dots, |\mathbf{n}|$, are the zeros of the type II multiple orthogonal algebraic polynomial of degree $|\mathbf{n}|$ with respect to the vector of weight functions (11) on the interval $[-1, 1)$, then the zeros of the type II multiple orthogonal trigonometric polynomial of semi–integer degree $|\mathbf{n}| + 1/2$ with respect to $(W, \mathbf{n})$ on the interval $[-\pi, \pi)$ are given by:*

$$x_0 = -\pi, \quad x_{2|\mathbf{n}|-k+1} = -x_k = \arccos \tau_k, \quad k = 1, 2, \dots, |\mathbf{n}|.$$

Proof. From Corollary 3.2 we have that $x_0 = -\pi$. From (10) we get:

$$x_{2|\mathbf{n}|-k+1} = -x_k = \arccos \tau_k, \quad k = 1, 2, \dots, |\mathbf{n}|,$$

where τ_k, $k = 1, 2, \dots, |\mathbf{n}|$, are zeros of the type II multiple orthogonal algebraic polynomial with respect to the multi–index $\mathbf{n}$ and the vector of weight functions (11) on the interval $[-1, 1)$. $\square$

Completely similar arguments can be applied for the proof the following Lemma.

Lemma 3.5. *Let* **n** *be a multi–index and let* $W = (w_1, w_2, \ldots, w_p)$ *be a TAT system of even weight functions on the interval* $(-\pi, \pi)$ *with respect to* **n**. *If* τ_k, $k = 1, 2, \ldots, |\mathbf{n}|$, *are the zeros of the type II multiple orthogonal algebraic polynomial of degree* $|\mathbf{n}|$ *with respect to the vector of weight functions*

$$\left(\sqrt{\frac{1-x}{1+x}} w_1(\arccos x), \sqrt{\frac{1-x}{1+x}} w_2(\arccos x), \ldots, \sqrt{\frac{1-x}{1+x}} w_p(\arccos x) \right)$$

on the interval $[-1, 1)$, *then the zeros of the type II multiple orthogonal trigonometric polynomial of semi–integer degree* $|\mathbf{n}| + 1/2$ *with respect to* $(W, \mathbf{n})$ *on the interval* $[-\pi, \pi)$ *are given by:*

$$x_{|\mathbf{n}|} = 0, \quad x_{2|\mathbf{n}|-k} = -x_k = \arccos \tau_{k+1}, \quad k = 0, 1, \ldots, |\mathbf{n}| - 1.$$

4. Recurrence relations in the case of nearly diagonal multi–index

Since multiple orthogonal algebraic polynomials for nearly diagonal multi–index satisfy recurrence relations of order $p + 1$ (see [11], [21]), the natural extension is investigation of recurrence relations for type II multiple orthogonal trigonometric polynomials of semi–integer degree, and we derive the similar recurrence relations in the case when weight functions are even on $(-\pi, \pi)$.

Let $n \in \mathbb{N}$ and write it as $n = \ell p + j$, for $\ell = [n/p]$ and $j \in \{0, 1, \ldots, p-1\}$. The nearly diagonal multi–index $\mathbf{d}(n)$ corresponding to n is given by

$$\mathbf{d}(n) = (\underbrace{\ell + 1, \ell + 1, \ldots, \ell + 1}_{j \text{ times}}, \underbrace{\ell, \ell, \ldots, \ell}_{p-j \text{ times}}).$$

For the corresponding type II monic multiple orthogonal trigonometric polynomials of semi–integer degree with respect to the even weight functions $W = (w_1, w_2, \ldots, w_p)$ we use the following simple notation:

$$T_n^{C,1/2} = T_{\mathbf{d}(n)}^{C,1/2}, \quad T_n^{S,1/2} = T_{\mathbf{d}(n)}^{S,1/2}.$$

Theorem 4.1. *Let* m *be a nonnegative integer and let* $W = (w_1, w_2, \ldots, w_p)$ *be a TAT system of even weight functions on the interval* $[-\pi, \pi)$ *with respect to all nearly diagonal multi–indices* $\mathbf{d}(k) \leq \mathbf{d}(m+1)$, $k \in \mathbb{N}$. *The type II multiple orthogonal trigonometric polynomials of semi–integer degree with nearly diagonal multi–indices* $T_m^{C,1/2}$ *and* $T_m^{S,1/2}$ *satisfy the following recurrence relations:*

$$2 \cos x T_m^{C,1/2}(x) = T_{m+1}^{C,1/2}(x) + \sum_{k=0}^{p} \alpha_{m,p-k} T_{m-k}^{C,1/2}(x), \tag{12}$$

$$2 \cos x T_m^{S,1/2}(x) = T_{m+1}^{S,1/2}(x) + \sum_{k=0}^{p} \beta_{m,p-k} T_{m-k}^{S,1/2}(x), \tag{13}$$

with the initial conditions $T_0^{C,1/2}(x) = \cos(x/2)$, $T_i^{C,1/2}(x) = 0$, $i = -1, -2, \ldots, -p$ *and* $T_0^{S,1/2}(x) = \sin(x/2)$, $T_i^{S,1/2}(x) = 0$, $i = -1, -2, \ldots, -p$, *for relations* (12) *and* (13), *respectively.*

Proof. Due to (6) we have the following simple equality:

$$2 \cos x T_m^{C,1/2}(x) = T_{m+1}^{C,1/2}(x) + \sum_{k=0}^{m} \alpha_{m,k} T_k^{C,1/2}(x). \tag{14}$$

Let us assume that $m = \ell p + j$. Notice that for $m \le p$ equality (14) is of the form (12) with the given initial conditions.

Suppose that $m > p$. We need to prove that $\alpha_{m,rp+i} = 0$ for $r = 0, 1, \ldots, \ell - 2$ and $i = 0, 1, \ldots, p - 1$, and for $r = \ell - 1$ and $i = 0, 1, \ldots, j - 1$, which means that the right hand side of (14) reduces to

$$T_{m+1}^{C,1/2}(x) + \sum_{k=m-p}^{m} \alpha_{m,k} T_k^{C,1/2}(x),$$

i.e., to the right hand side of (12).

For all $r = 0, 1, \ldots, \ell - 2$ we multiply the both hand sides of (14) by $T_r^{C,1/2}(x) w_1(x)$ and integrate on the interval $[-\pi, \pi)$. According to the orthogonality conditions (4) (the first coordinate of the multi–index $\mathbf{d}(m)$ is equal to $\ell + 1$, $\ell \ge r + 2$) the left hand side, i.e.,

$$\int_{-\pi}^{\pi} \cos x\, T_r^{C,1/2}(x) T_m^{C,1/2}(x)\, w_1(x)\, \mathrm{d}x,$$

is equal to zero whenever $m > p$, and the right hand side reduces to

$$\alpha_{m,rp} \int_{-\pi}^{\pi} T_r^{C,1/2}(x) T_{rp}^{C,1/2}(x)\, w_1(x)\, \mathrm{d}x.$$

The previous integral cannot be zero, because if we assume the contrary, then we have one orthogonality condition more which implies that $T_{rp}^{C,1/2}(x) \equiv 0$. Therefore, we obtain $\alpha_{m,rp} = 0$. In order to prove that $\alpha_{m,rp+1} = 0$ we multiply the both hand sides of equality (14) by $T_r^{C,1/2}(x) w_2(x)$ and integrate on $[-\pi, \pi)$. Generally, multiplying (14) with $T_r^{C,1/2}(x) w_{i+1}(x)$ and integrating on the interval $[-\pi, \pi)$, from orthogonality conditions (4) we obtain that $\alpha_{m,rp+i} = 0$, $i = 0, 1, 2, \ldots, p - 1$.

Finally, suppose that $r = \ell - 1$. Multiplying (14) by $T_{\ell-1}^{C,1/2}(x) w_{i+1}(x)$ and integrating on $[-\pi, \pi)$, due to orthogonality, we obtain that $\alpha_{m,(\ell-1)p+i} = 0$ for $i = 0, 1, \ldots, j - 1$.

In a similar way one can obtain the recurrence relation (13) for $T_m^{S,1/2}(x)$. $\quad\square$

According to the previous theorem, the type II multiple orthogonal trigonometric polynomials of semi–integer degree $T_m^{C,1/2}$ and $T_m^{C,1/2}$ can be obtained if we have coefficients of the recurrence relations.

First, we consider the simplest case $p = 2$ for trigonometric polynomials $T_m^{C,1/2}$ (in a similar way one can obtain corresponding coefficients of the recurrence relations for trigonometric polynomials $T_m^{S,1/2}$). In this case we have the multi–indices $\mathbf{d}(m) = (m_1, m_2)$, where $m_1 = [(m + 1)/2]$ and $m_2 = [m/2]$ (obviously, $m_1 + m_2 = m$). Recurrence relation (12) reduces to

$$T_{m+1}^{C,1/2}(x) = (2 \cos x - \alpha_{m,2}) T_m^{C,1/2} - \alpha_{m,1} T_{m-1}^{C,1/2}(x) - \alpha_{m,0} T_{m-2}^{C,1/2}(x), \quad m = 0, 1, 2, \ldots, \tag{15}$$

with the initial conditions $T_0^{C,1/2}(x) = \cos(x/2)$, $T_{-1}^{C,1/2}(x) = T_{-2}^{C,1/2}(x) = 0$. In order to determine the recursion coefficients we use (15) and the orthogonality conditions

$$\left\langle T_m^{C,1/2}, T_i^{C,1/2} \right\rangle_1 = 0 \quad \text{for} \quad i \le \left[\frac{m-1}{2}\right], \quad \left\langle T_m^{C,1/2}, T_i^{C,1/2} \right\rangle_2 = 0 \quad \text{for} \quad i \le \left[\frac{m-2}{2}\right],$$

where $\langle \cdot, \cdot \rangle_i$, $i = 1, 2$, are the inner products given by (1). Since $\left\langle T_1^{C,1/2}, T_0^{C,1/2} \right\rangle_1 = 0$, from (15) for $m = 0$ we get

$$\alpha_{02} = \frac{\left\langle 2 \cos x\, T_0^{C,1/2}, T_0^{C,1/2} \right\rangle_1}{\left\langle T_0^{C,1/2}, T_0^{C,1/2} \right\rangle_1}. \tag{16}$$

By using (15) for $m = 1$, as well as the facts that $\left\langle T_2^{C,1/2}, T_0^{C,1/2} \right\rangle_1 = 0$ and $\left\langle T_2^{C,1/2}, T_0^{C,1/2} \right\rangle_2 = 0$, we obtain

$$\alpha_{11} = \frac{\left\langle 2\cos x\, T_1^{C,1/2}, T_0^{C,1/2} \right\rangle_1}{\left\langle T_0^{C,1/2}, T_0^{C,1/2} \right\rangle_1}, \quad \alpha_{12} = \frac{\left\langle 2\cos x\, T_1^{C,1/2} - \alpha_{1,1} T_0^{C,1/2}, T_0^{C,1/2} \right\rangle_2}{\left\langle T_1^{C,1/2}, T_0^{C,1/2} \right\rangle_2}. \tag{17}$$

In a similar way, (15) for $m = 2$ and the orthogonality conditions $\left\langle T_3^{C,1/2}, T_0^{C,1/2} \right\rangle_1 = 0$, $\left\langle T_3^{C,1/2}, T_0^{C,1/2} \right\rangle_2 = 0$, and $\left\langle T_3^{C,1/2}, T_1^{C,1/2} \right\rangle_1 = 0$

$$\alpha_{20} = \frac{\left\langle 2\cos x\, T_2^{C,1/2}, T_0^{C,1/2} \right\rangle_1}{\left\langle T_0^{C,1/2}, T_0^{C,1/2} \right\rangle_1}, \tag{18}$$

$$\alpha_{21} = \frac{\left\langle 2\cos x\, T_2^{C,1/2} - \alpha_{2,0} T_0^{C,1/2}, T_0^{C,1/2} \right\rangle_2}{\left\langle T_1^{C,1/2}, T_0^{C,1/2} \right\rangle_2},$$

$$\alpha_{2,2} = \frac{\left\langle 2\cos x\, T_2^{C,1/2} - \alpha_{20} T_0^{C,1/2} - \alpha_{2,1} T_1^{C,1/2}, T_1^{C,1/2} \right\rangle_1}{\left\langle T_2^{C,1/2}, T_1^{C,1/2} \right\rangle_1}.$$

In general, continuing this procedure one can prove the following result.

Theorem 4.2. *Let $m = 2\ell + \nu$, where $\ell = [m/2]$ and $\nu \in \{0,1\}$. The recursion coefficients in (15) can be expressed in the form*

$$\alpha_{m,0} = \frac{\left\langle 2\cos x\, T_m^{C,1/2}, T_{[(m-2)/2]}^{C,1/2} \right\rangle_{\nu+1}}{\left\langle T_{m-2}^{C,1/2}, T_{[(m-2)/2]}^{C,1/2} \right\rangle_{\nu+1}}, \tag{19}$$

$$\alpha_{m,1} = \frac{\left\langle 2\cos x\, T_m^{C,1/2} - \alpha_{m,0} T_{m-2}^{C,1/2}, T_{[(m-1)/2]}^{C,1/2} \right\rangle_{\nu}}{\left\langle T_{m-1}^{C,1/2}, T_{[(m-1)/2]}^{C,1/2} \right\rangle_{\nu}},$$

$$\alpha_{m,2} = \frac{\left\langle 2\cos x\, T_m^{C,1/2} - \alpha_{m,0} T_{m-2}^{C,1/2} - \alpha_{m,1} T_{m-1}^{C,1/2}, T_{[m/2]}^{C,1/2} \right\rangle_{\nu+1}}{\left\langle T_m^{C,1/2}, T_{[m/2]}^{C,1/2} \right\rangle_{\nu+1}},$$

where we put $\langle \cdot, \cdot \rangle_{j+2k} = \langle \cdot, \cdot \rangle_j$, $j = 1, 2$, for each $k \in \mathbb{Z}$.

The previous theorem can be extended to $p \in \mathbb{N}$, $p \geq 3$, and even weight functions w_j, $j = 1, 2, \ldots, p$. Taking $\langle \cdot, \cdot \rangle_{j+pk} = \langle \cdot, \cdot \rangle_j$, $k \in \mathbb{Z}$, the following result holds.

Theorem 4.3. *Let m be a nonnegative integer, $W = (w_1, w_2, \ldots, w_p)$ be a TAT system of even weight functions on the interval $[-\pi, \pi)$ with respect to all nearly diagonal multi–indices $\mathbf{d}(k) \preceq \mathbf{d}(m+1)$, $k \in \mathbb{N}$. The type II multiple orthogonal trigonometric polynomials of semi–integer degree with nearly diagonal multi–indices $T_m^{C,1/2}$ and $T_m^{S,1/2}$ satisfy the recurrence relations:*

$$T_{m+1}^{C,1/2}(x) = (2\cos x - \alpha_{m,p}) T_m^{C,1/2}(x) - \sum_{k=0}^{p-1} \alpha_{m,k} T_{m-p+k}^{C,1/2}(x), \tag{20}$$

$$T_{m+1}^{S,1/2}(x) = (2\cos x - \beta_{m,p}) T_m^{S,1/2}(x) - \sum_{k=0}^{p-1} \beta_{m,k} T_{m-p+k}^{S,1/2}(x), \tag{21}$$

with coefficients given by

$$\alpha_{m,0} = \frac{\left\langle 2\cos x T_m^{C,1/2}, T_{[(m-p)/p]}^{C,1/2} \right\rangle_{j+1}}{\left\langle T_{m-p}^{C,1/2}, T_{[(m-p)/p]}^{C,1/2} \right\rangle_{j+1}}, \quad \alpha_{m,k} = \frac{\left\langle 2\cos x T_m^{C,1/2} - \sum\limits_{i=0}^{k-1} \alpha_{m,i} T_{m-p+i}^{C,1/2}, T_{[(m-p+k)/p]}^{C,1/2} \right\rangle_{j+k+1}}{\left\langle T_{m-p+k}^{C,1/2}, T_{[(m-p+k)/p]}^{C,1/2} \right\rangle_{j+k+1}}, \quad k = 1, 2, \ldots, p,$$

and

$$\beta_{m,0} = \frac{\left\langle 2\cos x T_m^{S,1/2}, T_{[(m-p)/p]}^{S,1/2} \right\rangle_{j+1}}{\left\langle T_{m-p}^{S,1/2}, T_{[(m-p)/p]}^{S,1/2} \right\rangle_{j+1}}, \quad \beta_{m,k} = \frac{\left\langle 2\cos x T_m^{S,1/2} - \sum\limits_{i=0}^{k-1} \beta_{m,i} T_{m-p+i}^{S,1/2}, T_{[(m-p+k)/p]}^{S,1/2} \right\rangle_{j+k+1}}{\left\langle T_{m-p+k}^{S,1/2}, T_{[(m-p+k)/p]}^{S,1/2} \right\rangle_{j+k+1}}, \quad k = 1, 2, \ldots, p,$$

for $\ell = [m/p]$ and $j = m - \ell p \in \{0, 1, \ldots, p-1\}$.

All of the necessary inner products can be computed exactly, except for rounding errors, by using the quadrature rules of Gaussian type for trigonometric polynomials with respect to the corresponding weight functions (see [9, 10, 19])

$$\int_{-\pi}^{\pi} g(t)\, w_\nu(t)\, \mathrm{d}t = \sum_{k=0}^{2N} A_{\nu,k}^{(N)} g(\tau_{\nu,k}^{(N)}) + R_{\nu,N}(g), \quad \nu = 1, 2, \ldots, p. \tag{22}$$

Thus, for the numerical construction of the type II multiple orthogonal trigonometric polynomials of semi–integer degree $T_m^{C,1/2}$ and $T_m^{S,1/2}$, $m \geq 0$, for nearly diagonal multi–indices with respect to even weight functions we use only recurrence relations (20) and (21) and quadrature rules of Gaussian type for trigonometric polynomials (22). This procedure is a kind of discretized Stieltjes–Gautschi procedure (see [7]).

5. Applications

Motivated by the paper of Borges [4], Milovanović, Stanić, and Tomović in [13] introduced the following definition of an optimal set of quadrature rules for trigonometric polynomials.

Definition 5.1. *Let **n** be a multi–index and let $W = (w_1, w_2, \ldots, w_p)$ be a TAT system for **n** on interval E. A set of quadrature rules of the form*

$$\int_E f(x) w_\nu(x)\, \mathrm{d}x \approx \sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} f(x_k), \quad \nu = 1, 2, \ldots, p, \tag{23}$$

is an optimal set with respect to $(W, \mathbf{n})$ if and only if the weight coefficients, $A_{\nu,k}$, $\nu = 1, 2, \ldots, p$, $k = 0, 1, \ldots, 2|\mathbf{n}|$, and nodes, x_k, $k = 0, 1, \ldots, 2|\mathbf{n}|$, satisfy the following equations:

$$\sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} = \int_E w_\nu(x)\, \mathrm{d}x, \tag{24}$$

$$\sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} \cos(m_\nu x_k) = \int_E \cos(m_\nu x)\, w_\nu(x)\, \mathrm{d}x, \quad m_\nu = 1, 2, \ldots, |\mathbf{n}| + n_\nu,$$

$$\sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} \sin(m_\nu x_k) = \int_E \sin(m_\nu x)\, w_\nu(x)\, \mathrm{d}x, \quad m_\nu = 1, 2, \ldots, |\mathbf{n}| + n_\nu,$$

for $\nu = 1, 2, \ldots, p$.

For the optimal set of quadrature rules for trigonometric polynomials the following generalization of fundamental theorem of Gaussian rules holds (see [13]).

Theorem 5.2. *Let* **n** *be a multi–index and let* $W = (w_1, w_2, \ldots, w_p)$ *be a TAT system for* **n** *on an interval E. A set of quadrature rules* (23) *is the optimal set with respect to* $(W, \mathbf{n})$ *if and only if:*

1° *all rules are exact for all polynomials from* $\mathcal{T}_{|\mathbf{n}|}$;

2° $T_{\mathbf{n}}^{1/2}(x) = \prod\limits_{k=0}^{2|\mathbf{n}|} \sin((x - x_k)/2)$ *is the type II multiple orthogonal trigonometric polynomial of semi–integer degree* $|\mathbf{n}| + 1/2$ *with respect to* $(W, \mathbf{n})$.

The nodes x_k, $k = 0, 1, \ldots, 2|\mathbf{n}|$, of the optimal set of quadrature rules for trigonometric polynomials can be computed as the zeros of corresponding type II multiple orthogonal trigonometric polynomials of semi–integer degree $T_{\mathbf{n}}^{1/2}$, which, in the case of nearly diagonal multi–indices $\mathbf{d}(n)$, for even weight functions on the interval $E = (-\pi, \pi)$, can be obtained using recurrence relations given in Section 4. Notice that, from Theorem 2.1, the type II multiple orthogonal trigonometric polynomials of semi–integer degree $T_{\mathbf{n}}^{1/2}$ have exactly $2|\mathbf{n}| + 1$ simple zeros on the interval E.

The weights coefficients $A_{\nu,k}$, $\nu = 1, 2, \ldots, p$, $k = 0, 1, \ldots, 2|\mathbf{n}|$, can be computed by requiring that each rule integrates exactly trigonometric polynomials from $\mathcal{T}_{|\mathbf{n}|}$.

Now we give a numerical example to illustrate the obtained theoretical results and proposed numerical procedure.

Example 5.3. Let us construct the optimal set of quadrature rules on $E = [-\pi, \pi)$, for $p = 2$, multi–index $\mathbf{n} = (2, 2)$, and with respect to the even weight functions $w_1(x) = 1 + \cos x$ and $w_2(x) = 1 + \cos 2x$ on the interval $(-\pi, \pi)$.

It is easy to see that the following set

$$\left\{ \cos\frac{x}{2} w_1(x), \sin\frac{x}{2} w_1(x), \cos\frac{3x}{2} w_1(x), \sin\frac{3x}{2} w_1(x), \cos\frac{x}{2} w_2(x), \sin\frac{x}{2} w_2(x), \cos\frac{3x}{2} w_2(x), \sin\frac{3x}{2} w_2(x) \right\}$$

is a Chebyshev system on the interval $[-\pi, \pi)$.

We obtain the type II multiple orthogonal trigonometric polynomial of semi–integer degree $T_{\mathbf{n}}^{C,1/2} \in \mathcal{T}_4^{1/2}$ by using the recurrence relations given in Section 4. From the initial condition $T_0^{C,1/2}(x) = \cos(x/2)$ and (16) we get $\alpha_{0,2} = 4/3$, which with (15) for $m = 0$ gives

$$T_1^{C,1/2}(x) = \cos\frac{3x}{2} - \frac{1}{3}\cos\frac{x}{2}.$$

In the following step, by using (17) we get $\alpha_{1,1} = 5/9$ and $\alpha_{1,2} = 8/3$, and then

$$T_2^{C,1/2}(x) = \cos\frac{5x}{2} - 3\cos\frac{3x}{2} + \cos\frac{x}{2}.$$

In a similar way, by using equations (18) and (15) for $m = 2$, we get $T_3^{C,1/2}(x) = \cos(7x/2)$. Finally, from (19), for $m = 3$, and (15) for $m = 3$, we obtain $T_4^{C,1/2}(x) = \cos(9x/2)$, so the nodes of the optimal set of quadrature rules are:

$$x_0 = -\pi, \ x_1 = -\frac{7\pi}{9}, \ x_2 = -\frac{5\pi}{9}, \ x_3 = -\frac{3\pi}{9}, \ x_4 = -\frac{\pi}{9}, \ x_5 = \frac{\pi}{9}, \ x_6 = \frac{3\pi}{9}, \ x_7 = \frac{5\pi}{9}, \ x_8 = \frac{7\pi}{9}.$$

For each $\nu = 1, 2$, the corresponding weight coefficients $A_{\nu,k}$, $k = 0, 1, \ldots, 8$, are given in Table 1 (numbers in parentheses denote decimal exponents).

k	$A_{1,k}$	$A_{2,k}$
0	$-6.890988372772788(-29)$	1.396263401595464
1	$1.633317908364284(-1)$	$8.193609984127726(-1)$
2	$5.769024031826910(-1)$	$4.210249322138756(-2)$
3	1.047197551196598	$3.490658503988659(-1)$
4	1.354160908374076	1.232931610759035
5	1.354160908374076	1.232931610759035
6	1.047197551196598	$3.490658503988659(-1)$
7	$5.769024031826910(-1)$	$4.210249322138756(-2)$
8	$1.633317908364284(-1)$	$8.193609984127726(-1)$

Table 1: Weight coefficients $A_{\nu,k}$, $\nu = 1, 2$, $k = 0, 1, \ldots, 8$, of the optimal set of quadrature rules with respect to $W = (1 + \cos x, 1 + \cos 2x)$ and $\mathbf{n} = (2, 2)$.

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