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Fixed Points of Integral Type Contractions in Uniform Spaces

Aris Aghanians[a], Kourosh Nourouzi[a]

[a]*Faculty of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran*

Abstract. In this paper, we discuss the existence of fixed points for integral type contractions in uniform spaces endowed with both a graph and an E-distance. We also give two sufficient conditions under which the fixed point is unique. Our main results generalize some recent metric fixed point theorems.

1. Introduction and preliminaries

In [9], Branciari discussed the existence and uniqueness of fixed points for mappings from a complete metric space (X, d) into itself satisfying a general contractive condition of integral type. The result therein is a generalization of the Banach contraction principle in metric spaces. In fact, Branciari considered mappings $T : (X, d) \to (X, d)$ satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t)\mathrm{d}t \le \alpha \int_0^{d(x,y)} \varphi(t)\mathrm{d}t \qquad (x, y \in X),$$

where $\alpha \in (0, 1)$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable function on $[0, +\infty)$ whose Lebesgue-integral is finite on each compact subset of $[0, +\infty)$, and satisfies $\int_0^\varepsilon \varphi(t)dt > 0$ for all $\varepsilon > 0$. Recently, an integral version of Ćirić's contraction was given in [14].

In 2008, Jachymski [12] generalized the Banach contraction principle in metric spaces endowed with a graph. This idea was followed by the authors in uniform and modular spaces (see [3, 5–7]). In [1], the concept of an E-distance was introduced in uniform spaces as a generalization of a metric and a w-distance and then many different nonlinear contractions were generalized from metric to uniform spaces (see, e.g., [2, 4, 13]).

The aim of this paper is to study the existence and uniqueness of a fixed point for integral type contractions in uniform spaces endowed with both a graph and an E-distance. Our results generalize Theorem 2.1 in [9] as well as Corollary 3.1 in [12] by replacing metric spaces with uniform spaces endowed with a graph and by considering a weaker contractive condition. We also prove an integral version of [12, Theorems 3.2 and 3.3].

We begin with notions in uniform spaces that are needed in this paper. For more detailed discussion, the reader is referred to, e.g., [15].

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Corresponding author: Kourosh Nourouzi
Email addresses: a.aghanians@dena.kntu.ac.ir (Aris Aghanians), nourouzi@kntu.ac.ir (Kourosh Nourouzi)

By a uniform space $(X, \mathcal{U})$, shortly denoted here by X, it is meant a nonempty set X together with a uniformity $\mathcal{U}$. For instance, if d is a metric on a nonempty set X, then it induces a uniformity, called the uniformity induced by the metric d, in which the members of $\mathcal{U}$ are all the supersets of the sets

$$\Big\{(x, y) \in X \times X : d(x, y) < \varepsilon\Big\},$$

where $\varepsilon > 0$.

It is well-known that a uniformity $\mathcal{U}$ on a nonempty set X is separating if the intersection of all members of $\mathcal{U}$ is equal to the diagonal of the Cartesian product $X \times X$, that is, the set $\{(x, x) : x \in X\}$ which is often denoted by $\Delta(X)$. If $\mathcal{U}$ is a separating uniformity on a nonempty set X, then the uniform space X is said to be separated.

We next recall the definition of an E-distance on a uniform space X as well as the notions of convergence, Cauchyness and completeness with E-distances.

Definition 1.1 ([1]). *Let X be a uniform space. A function $p : X \times X \to [0, +\infty)$ is called an E-distance on X if*

i) for each member V of $\mathcal{U}$, there exists a $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $(x, y) \in V$ for all $x, y, z \in X$;

ii) the triangular inequality holds for p, that is,

$$p(x, y) \leq p(x, z) + p(z, y) \qquad (x, y, z \in X).$$

Let p be an E-distance on a uniform space X. A sequence $\{x_n\}$ in X is said to be p-convergent to a point $x \in X$, denoted by $x_n \xrightarrow{p} x$, if it satisfies the usual metric condition, that is, $p(x_n, x) \to 0$ as $n \to \infty$, and similarly, p-Cauchy if it satisfies $p(x_m, x_n) \to 0$ as $m, n \to \infty$. The uniform space X is called p-complete if every p-Cauchy sequence in X is p-convergent to some point of X.

In the next lemma, an important property of E-distances in separated uniform spaces is formulated.

Lemma 1.2 ([1]). *Let p be an E-distance on a separated uniform space X and $\{x_n\}$ and $\{y_n\}$ be two arbitrary sequences in X. If $x_n \xrightarrow{p} x$ and $x_n \xrightarrow{p} y$, then $x = y$. In particular, if $x, y \in X$ and $p(z, x) = p(z, y) = 0$ for some $z \in X$, then $x = y$.*

Finally, we recall some concepts about graphs. For more details on graph theory, see, e.g., [8].

Let X be a uniform space and consider a directed graph G without any parallel edges such that the set $V(G)$ of its vertices is X, that is, $V(G) = X$ and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta(X)$. So the graph G can be simply denoted by $G = (V(G), E(G))$. By $\widetilde{G}$, it is meant the undirected graph obtained from G by ignoring the direction of the edges of G, that is,

$$V(\widetilde{G}) = X \quad \text{and} \quad E(\widetilde{G}) = \Big\{(x, y) \in X \times X : \text{either } (x, y) \text{ or } (y, x) \text{ belongs to } E(G)\Big\}.$$

A subgraph H of G is itself a directed graph such that $V(H)$ and $E(H)$ are contained in $V(G)$ and $E(G)$, respectively, and $(x, y) \in E(H)$ implies $x, y \in V(H)$ for all $x, y \in X$.

We need also a few notions about connectivity of graphs. Suppose that x and y are two vertices in $V(G)$. A finite sequence $(x_i)_{i=0}^N$ consisting of $N + 1$ vertices of G is a path in G from x to y if $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$. The graph G is weakly connected if there exists a path in $\widetilde{G}$ between each two vertices of $\widetilde{G}$.

2. Main results

In this section, we consider the Euclidean metric on $[0, +\infty)$ and denote by λ the Lebesgue measure on the Borel σ-algebra of $[0, +\infty)$. For a Borel set $E = [a, b]$, we will use the notation $\int_a^b \varphi(t)\mathrm{d}t$ to show the Lebesgue integral of a function φ on E. We employ a class Φ consisting of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

(Φ1) φ is Lebesgue-integrable on $[0,+\infty)$;

(Φ2) The value of the Lebesgue integral $\int_0^\varepsilon \varphi(t)\mathrm{d}t$ is positive and finite for all $\varepsilon > 0$.

The next lemma embodies some important properties of functions of the class Φ which we need in the sequel.

Lemma 2.1. *Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a function in the class Φ and $\{a_n\}$ be a sequence of nonnegative real numbers. Then the following statements hold:*

1. *If $\int_0^{a_n} \varphi(t)\mathrm{d}t \to 0$ as $n \to \infty$, then $a_n \to 0$ as $n \to \infty$.*
2. *If $\{a_n\}$ is monotone and converges to some $a \geq 0$, then $\int_0^{a_n} \varphi(t)\mathrm{d}t \to \int_0^a \varphi(t)\mathrm{d}t$ as $n \to \infty$.*

Proof. 1. Let $\int_0^{a_n} \varphi(t)\mathrm{d}t \to 0$ and suppose first on the contrary that $\limsup_{n\to\infty} a_n = \infty$. Then $\{a_n\}$ contains a subsequence $\{a_{n_k}\}$ which diverges to ∞. By passing to a subsequence if necessary, one may assume without loss of generality that $\{a_{n_k}\}$ is a nondecreasing subsequence of $\{a_n\}$. Because the sequence $\{\int_0^{a_{n_k}} \varphi(t)\mathrm{d}t\}$ of nonnegative numbers increases to zero, so $a_{n_k} = 0$ for all $k \geq 1$. This is a contradiction and therefore the sequence $\{a_n\}$ is bounded.

Next, if $\limsup_{n\to\infty} a_n = \varepsilon > 0$, then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $a_{n_k} \to \varepsilon$. Pick an integer $k_0 > 0$ so that the strict inequality $a_{n_k} > \frac{\varepsilon}{2}$ holds for all $k \geq k_0$. Therefore,

$$0 < \int_0^{\frac{\varepsilon}{2}} \varphi(t)\mathrm{d}t \leq \int_0^{a_{n_k}} \varphi(t)\mathrm{d}t \to 0,$$

which is again a contradiction. So $\limsup_{n\to\infty} a_n = 0$, and consequently,

$$0 \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n = 0,$$

that is, $a_n \to 0$.

2. Let $\{a_n\}$ be nondecreasing. If for sufficiently large indices n we have $a_n = a$, then there is nothing to prove. Otherwise, put $E_n = [0, a_n]$ for all $n \geq 1$. Then each E_n is a Borel subset of $[0,+\infty)$ and we have $E_1 \subseteq E_2 \subseteq \cdots$ and $\bigcup_{n=1}^\infty E_n = [0, a]$. Because the function $E \overset{\mu}{\longmapsto} \int_E \varphi \mathrm{d}\lambda$ is a Borel measure on $[0,+\infty)$, using the continuity of μ from below we get

$$\int_0^a \varphi(t)\mathrm{d}t = \mu\Big(\bigcup_{n=1}^\infty E_n\Big) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \int_0^{a_n} \varphi(t)\mathrm{d}t.$$

A similar argument is true if $\{a_n\}$ is nonincreasing since each E_n defined above is of finite μ-measure by (Φ2). $\square$

Let T be a mapping from a uniform space X endowed with a graph G into itself. We denote as usual the set of all fixed points for T by $\mathrm{Fix}(T)$, and by C_T, we mean the set of all $x \in X$ such that $(T^n x, T^m x)$ is an edge of $\widetilde{G}$ for all $m, n \geq 0$. Clearly, $\mathrm{Fix}(T) \subseteq C_T$.

Definition 2.2. *Let p be an E-distance on a uniform space X endowed with a graph G. We say that a mapping $T : X \to X$ is an integral type p-G-contraction if*

IC 1) T preserves the egdes of G, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

IC 2) there exists a $\varphi \in \Phi$ and a constant $\alpha \in (0, 1)$ such that the contractive condition

$$\int_0^{p(Tx,Ty)} \varphi(t)\mathrm{d}t \leq \alpha \int_0^{p(x,y)} \varphi(t)\mathrm{d}t$$

holds for all $x, y \in X$ with $(x, y) \in E(G)$.

Now, we give some examples of integral type p-G-contractions.

Example 2.3. *Let p be an E-distance on a uniform space X endowed with a graph G and x_0 be a point in X such that $p(x_0, x_0) = 0$. Since $E(G)$ contains the loop (x_0, x_0), it follows that the constant mapping $T = x_0$ preserves the edges of G, and since $p(x_0, x_0) = 0$, (IC2) holds trivially for any arbitrary $\varphi \in \Phi$ and $\alpha \in (0, 1)$. Therefore, T is an integral type p-G-contraction. In particular, each constant mapping on X is an integral type p-G-contraction if and only if $p(x, x) = 0$ for all $x \in X$.*

Example 2.4. *Let (X, d) be a metric space and $T : X \to X$ a mapping satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)\mathrm{d}t \leq \alpha \int_0^{d(x,y)} \varphi(t)\mathrm{d}t \qquad (x, y \in X),$$

where $\varphi \in \Phi$ and $\alpha \in (0, 1)$. If we consider X as a uniform space with the uniformity induced by the metric d, then T is an integral type d-G_0-contraction, where G_0 is the complete graph with the vertices set X, that is, $V(G_0) = X$ and $E(G_0) = X \times X$. The existence and uniqueness of fixed point for these kind of contractions were considered by Branciari in [9].

Example 2.5. *Let $\leq$ and p be a partial order and an E-distance on a uniform space X, respectively, and consider the poset graphs G_1 and G_2 by*

$$V(G_1) = X \quad \textit{and} \quad E(G_1) = \Big\{(x, y) \in X \times X : x \leq y\Big\},$$

and

$$V(G_2) = X \quad \textit{and} \quad E(G_2) = \Big\{(x, y) \in X \times X : x \leq y \vee y \leq x\Big\}.$$

Then integral type p-G_1-contractions are precisely the ordered integral type p-contractions, that is, nondecreasing mappings $T : X \to X$ which satisfy (IC2) for all $x, y \in X$ with $x \leq y$ and for some $\varphi \in \Phi$ and $\alpha \in (0, 1)$. And integral type p-G_2-contractions are those mappings $T : X \to X$ which are order preserving and satisfy (IC2) for all comparable $x, y \in X$ and for some $\varphi \in \Phi$ and $\alpha \in (0, 1)$.

Remark 2.6. *Let T be a mapping from an arbitrary uniform space X into itself. If X is endowed with the complete graph G_0, then the set C_T coincides with X.*

If $\leq$ is a partial order on X and X is endowed with either G_1 or G_2, then a point $x \in X$ belongs to C_T if and only if $T^n x$ is comparable to $T^m x$ for all $m, n \geq 0$. In particular, if T is monotone, then each $x \in X$ satisfying $x \leq Tx$ or $Tx \leq x$ belongs to C_T.

Example 2.7. *Let p be any arbitrary E-distance on a uniform space X endowed with a graph G and define a function $\varphi : [0, +\infty) \to [0, +\infty)$ by the rule $\varphi(t) = t^\beta$ for all $t \geq 0$, where $\beta \geq 0$ is constant. It is clear that φ is Lebesgue-integrable on $[0, +\infty)$ and $\int_0^\varepsilon \varphi(t)\mathrm{d}t = \frac{\varepsilon^{1+\beta}}{1+\beta}$ which is positive and finite for all $\varepsilon > 0$, that is, $\varphi \in \Phi$. Now, let a mapping $T : X \to X$ satisfy $p(Tx, Ty) \leq \alpha p(x, y)$ for all $x, y \in X$ with $(x, y) \in E(G)$, where $\alpha \in (0, 1)$. Then T satisfies (IC2) for the function φ defined as above and the number $\alpha^{1+\beta} \in (0, 1)$. In fact, if $x, y \in X$ and $(x, y) \in E(G)$, then*

$$\int_0^{p(Tx,Ty)} \varphi(t)\mathrm{d}t = \frac{p(Tx, Ty)^{1+\beta}}{1+\beta} \leq \alpha^{1+\beta} \cdot \frac{p(x, y)^{1+\beta}}{1+\beta} = \alpha^{1+\beta} \int_0^{p(x,y)} \varphi(t)\mathrm{d}t.$$

Therefore, our contraction generalizes Banach's contraction with E-distances in uncountably many ways. In particular, if T is a Banach G-p-contraction (i.e., the Banach contraction in uniform spaces endowed with an E-distance and a graph), then T is an integral type p-G-contraction for uncountably many functions $\varphi \in \Phi$.

To prove the existence of a fixed point for an integral type p-$\widetilde{G}$-contraction, we need the following two lemmas:

Lemma 2.8. *Let p be an E-distance on a uniform space X endowed with a graph G and $T : X \to X$ be an integral type p-G-contraction. Then $p(T^n x, T^n y) \to 0$ as $n \to \infty$, for all $x, y \in X$ with $(x, y) \in E(G)$.*

Proof. Let $x, y \in X$ be such that $(x, y) \in E(G)$. According to Lemma 2.1, it suffices to show that $\int_0^{p(T^n x, T^n y)} \varphi(t)\mathrm{d}t \to 0$, where $\varphi \in \Phi$ is as in (IC2). To this end, note that because T preserves the edges of G, we have $(T^n x, T^n y) \in E(G)$ for all $n \geq 0$, and so by (IC2), we find

$$\int_0^{p(T^n x, T^n y)} \varphi(t)\mathrm{d}t \leq \alpha \int_0^{p(T^{n-1} x, T^{n-1} y)} \varphi(t)\mathrm{d}t \leq \cdots \leq \alpha^n \int_0^{p(x,y)} \varphi(t)\mathrm{d}t \qquad (n \geq 1),$$

where $\alpha \in (0, 1)$ is as in (IC2). Since, by (Φ2), $\int_0^{p(x,y)} \varphi(t)\mathrm{d}t$ is finite (even possibly zero), it follows immediately that $\int_0^{p(T^n x, T^n y)} \varphi(t)\mathrm{d}t \to 0$. □

Lemma 2.9. *Let p be an E-distance on a uniform space X endowed with a graph G and $T : X \to X$ be an integral type p-$\widetilde{G}$-contraction. Then the sequence $\{T^n x\}$ is p-Cauchy for all $x \in C_T$.*

Proof. Suppose on the contrary that $\{T^n x\}$ is not p-Cauchy for some $x \in C_T$. Then there exist an $\varepsilon > 0$ and positive integers m_k and n_k such that

$$m_k > n_k \geq k \quad \text{and} \quad p(T^{m_k} x, T^{n_k} x) \geq \varepsilon \qquad k = 1, 2, \ldots .$$

If the integer n_k is kept fixed for sufficiently large indices k (say, $k \geq k_0$), then using Lemma 2.8, one may assume without loss of generality that $m_k > n_k$ is the smallest integer with $p(T^{m_k} x, T^{n_k} x) \geq \varepsilon$, that is,

$$p(T^{m_k - 1} x, T^{n_k} x) < \varepsilon \qquad (k \geq k_0).$$

Hence we have

$$\begin{aligned} \varepsilon &\leq p(T^{m_k} x, T^{n_k} x) \\ &\leq p(T^{m_k} x, T^{m_k - 1} x) + p(T^{m_k - 1} x, T^{n_k} x) \\ &< p(T^{m_k} x, T^{m_k - 1} x) + \varepsilon \end{aligned}$$

for each $k \geq k_0$. Since $x \in C_T$, it follows that $(Tx, x) \in E(\widetilde{G})$ and by Lemma 2.8, we have $p(T^{m_k} x, T^{m_k - 1} x) \to 0$. Thus, letting $k \to \infty$ yields $p(T^{m_k} x, T^{n_k} x) \to \varepsilon$. On the other hand, we have

$$p(T^{m_k + 1} x, T^{n_k + 1} x) \leq p(T^{m_k + 1} x, T^{m_k} x) + p(T^{m_k} x, T^{n_k} x) + p(T^{n_k} x, T^{n_k + 1} x)$$

for all $k \geq 1$. Letting $k \to \infty$, since $(Tx, x), (x, Tx) \in E(\widetilde{G})$, it follows by Lemma 2.8 that

$$\limsup_{k \to \infty} p(T^{m_k + 1} x, T^{n_k + 1} x) \leq \varepsilon.$$

Moreover, the inequality

$$p(T^{m_k + 1} x, T^{n_k + 1} x) \geq p(T^{m_k} x, T^{n_k} x) - p(T^{m_k} x, T^{m_k + 1} x) - p(T^{n_k + 1} x, T^{n_k} x)$$

holds for all $k \geq 1$. Thus, similarly we have

$$\liminf_{k \to \infty} p(T^{m_k + 1} x, T^{n_k + 1} x) \geq \varepsilon.$$

Hence, $p(T^{m_k + 1} x, T^{n_k + 1} x) \to \varepsilon$. By passing to two subsequences with the same choice function if necessary, one may assume without loss of generality that both $\{p(T^{m_k} x, T^{n_k} x)\}$ and $\{p(T^{m_k + 1} x, T^{n_k + 1} x)\}$ are monotone. Therefore, using Lemma 2.1 twice, we have

$$\int_0^{\varepsilon} \varphi(t)\mathrm{d}t = \lim_{k \to \infty} \int_0^{p(T^{m_k + 1} x, T^{n_k + 1} x)} \varphi(t)\mathrm{d}t \leq \alpha \lim_{k \to \infty} \int_0^{p(T^{m_k} x, T^{n_k} x)} \varphi(t)\mathrm{d}t = \alpha \int_0^{\varepsilon} \varphi(t)\mathrm{d}t,$$

where $\varphi \in \Phi$ and $\alpha \in (0, 1)$ are as in (IC2). Therefore, $\int_0^{\varepsilon} \varphi(t)\mathrm{d}t = 0$, which is a contradiction. Consequently, the sequence $\{T^n x\}$ is p-Cauchy for all $x \in C_T$. □

In 1971 Ćirić [10] introduced the following two notions (see also [11]).

Definition 2.10 ([10]). *Let (X, τ) be a topological space and $T : X \to X$ be an operator. The operator T is said to be orbitally continuous if $T^{n_i}x \to p$, then $T(T^{n_i}x) \to Tp$ as $i \to \infty$.*

Definition 2.11 ([10]). *Let (X, d) be a metric space and $T : X \to X$ be an operator. The metric space (X, d) is said to be T-orbitally complete if any Cauchy sequence of the form $\{T^{n_i}x\}_{i=1}^{\infty}$, $x \in X$, converges in X.*

Jachymski [12, Definition 2.4] generalized these notions by introducing the notion of an orbitally G-continuous mapping in metric spaces endowed with a graph G.

Now we shall generalize the above notion of orbitally continuity to orbitally p-G-continuity.

Definition 2.12. *Let p be an E-distance on a uniform space X endowed with a graph G and T be a mapping from X into itself. We say that*

i) T is orbitally p-G-continuous on X if for all $x, y \in X$ and all sequences $\{a_n\}$ of positive integers with $(T^{a_n}x, T^{a_{n+1}}x) \in E(G)$ for $n = 1, 2, \ldots$, $T^{a_n}x \xrightarrow{p} y$ as $n \to \infty$, implies $T(T^{a_n}x) \xrightarrow{p} Ty$ as $n \to \infty$.

ii) T is a p-Picard operator if T has a unique fixed point $u \in X$ and $T^n x \xrightarrow{p} u$ for all $x \in X$.

iii) T is a weakly p-Picard operator if $\{T^n x\}$ is p-convergent to a fixed point of T for all $x \in X$.

Example 2.13. *Let X be any arbitrary uniform space with more than one point equipped with an E-distance p. Choose a nonempty proper subset A of X and pick a and b from A and A^c, respectively. Then the mapping $T : X \to X$ defined by $Tx = a$ if $x \in A$, and $Tx = b$ if $x \notin A$ is a weakly p-Picard operator which fails to be p-Picard. In fact, we have* $\mathrm{Fix}(T) = \{a, b\}$. *Therefore, a weakly p-Picard operator is not necessarily p-Picard.*

Now, we are ready to prove our main theorems. The first result guarantees the existence of a fixed point when an integral type p-$\widetilde{G}$-contraction is orbitally p-$\widetilde{G}$-continuous on X or the triple (X, p, G) has a certain property.

Theorem 2.14. *Let p be an E-distance on a separated uniform space X endowed with a graph G such that X is p-complete, and $T : X \to X$ be an integral type p-$\widetilde{G}$-contraction. Then $T\mid_{C_T}$ is a weakly p-Picard operator if one of the following statements holds:*

i) T is orbitally p-$\widetilde{G}$-continuous on X;

ii) The triple (X, p, G) satisfies the following property:

($*$) If a sequence $\{x_n\}$ in X is p-convergent to an $x \in X$ and satisfies $(x_n, x_{n+1}) \in E(\widetilde{G})$ for all $n \geq 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(\widetilde{G})$ for all $k \geq 1$.

In particular, having been held (i) or (ii), $\mathrm{Fix}(T) \neq \emptyset$ *if and only if $C_T \neq \emptyset$.*

Proof. If $C_T = \emptyset$, then there is nothing to prove. Otherwise, note first that since T preserves the edges of $\widetilde{G}$, it follows that C_T is T-invariant, that is, T maps C_T into itself. Now, let $x \in C_T$ be given. Then $(T^n x, T^{n+1}x) \in E(\widetilde{G})$ for all $n \geq 0$. Moreover, by Lemma 2.9, the sequence $\{T^n x\}$ is p-Cauchy in X, and because X is p-complete, there exists a $u \in X$ (depends on x) such that $T^n x \xrightarrow{p} u$.

To prove the existence of a fixed point for T, suppose first that T is orbitally p-$\widetilde{G}$-continuous. Then $T^{n+1}x \xrightarrow{p} Tu$ and because X is separated, Lemma 1.2 ensures that $Tu = u$, that is, u is a fixed point for T.

On the other hand, if Property (∗) holds, then $\{T^n x\}$ contains a subsequence $\{T^{n_k}x\}$ such that $(T^{n_k}x, u) \in E(\widetilde{G})$ for all $k \geq 1$. Since $p(T^{n_k}x, u) \to 0$, by passing to a subsequence if necessary, one may assume without loss of generality that $\{p(T^{n_k}x, u)\}$ is monotone. Hence by Lemma 2.1, we have

$$\int_0^{p(T^{n_k+1}x,Tu)} \varphi(t)\mathrm{d}t \leq \alpha \int_0^{p(T^{n_k}x,u)} \varphi(t)\mathrm{d}t \to 0 \quad \text{as} \quad k \to \infty,$$

where $\alpha \in (0, 1)$ is as in (IC2). Using Lemma 2.1 once more, one obtains $p(T^{n_k+1}x, Tu) \to 0$ and since X is separated, Lemma 1.2 guarantees that $Tu = u$, that is, u is a fixed point for T.

Finally, $u \in \mathrm{Fix}(T) \subseteq C_T$, and so $T|_{C_T}$ is a weakly p-Picard operator. □

Setting $G = G_0$ in Theorem 2.14, we have the following result, which is a generalization of [9, Theorem 2.1] to uniform spaces equipped with an E-distance.

Corollary 2.15. *Let p be an E-distance on a separated uniform space X such that X is p-complete. Let $T : X \to X$ satisfy*

$$\int_0^{p(Tx,Ty)} \varphi(t)\mathrm{d}t \leq \alpha \int_0^{p(x,y)} \varphi(t)\mathrm{d}t \qquad (x, y \in X),$$

where $\varphi \in \Phi$ and $\alpha \in (0, 1)$. Then T is a p-Picard operator.

Proof. By Theorem 2.14, the mapping T is a weakly p-Picard operator. To complete the proof, it suffices to show that T has a unique fixed point. To this end, let x and y be two fixed points for T. Then

$$\int_0^{p(x,y)} \varphi(t)\mathrm{d}t = \int_0^{p(Tx,Ty)} \varphi(t)\mathrm{d}t \leq \alpha \int_0^{p(x,y)} \varphi(t)\mathrm{d}t,$$

which is impossible unless $p(x, y) = 0$. Similarly, one can show that $p(x, x) = 0$ and since X is separated, it follows by Lemma 1.2 that $x = y$. □

Because $\widetilde{G_1} = \widetilde{G_2} = G_2$, setting $G = G_1$ or $G = G_2$ in Theorem 2.14, we obtain the ordered version of Branciari's result as follows:

Corollary 2.16. *Let p be an E-distance on a partially ordered separated uniform space X such that X is p-complete and a mapping $T : X \to X$ satisfy*

$$\int_0^{p(Tx,Ty)} \varphi(t)\mathrm{d}t \leq \alpha \int_0^{p(x,y)} \varphi(t)\mathrm{d}t$$

for all comparable elements x and y of X, where $\varphi \in \Phi$ and $\alpha \in (0, 1)$. Assume that there exists an $x \in X$ such that $T^m x$ and $T^n x$ are comparable for all $m, n \geq 0$. Then T is a weakly p-Picard operator if one of the following statements holds:

- *T is orbitally p-G_2-continuous on X;*
- *X satisfies the following property:*

  If a sequence $\{x_n\}$ in X with successive comparable terms is p-convergent to an $x \in X$, then x is comparable to x_n for all $n \geq 1$.

Next, we are going to prove two theorems on uniqueness of the fixed points for integral type p-$\widetilde{G}$-contractions.

Theorem 2.17. *Let p be an E-distance on a separated uniform space X endowed with a graph G such that X is p-complete, and let $T : X \to X$ be an integral type p-$\widetilde{G}$-contraction such that the function φ in (IC2) satisfies*

$$\int_0^{a+b} \varphi(t)\mathrm{d}t \leq \int_0^a \varphi(t)\mathrm{d}t + \int_0^b \varphi(t)\mathrm{d}t \tag{1}$$

for all $a, b \geq 0$. If G is weakly connected and C_T is nonempty, then there exists a unique $u \in X$ such that $T^n x \xrightarrow{p} u$ for all $x \in X$. In particular, T is a p-Picard operator if and only if $\mathrm{Fix}(T)$ *is nonempty.*

Proof. Let x and y be two arbitrary elements of X. Since G is weakly connected, there exists a path $(x_i)_{i=0}^N$ in $\widetilde{G}$ from x to y. Since T preserves the edges of $\widetilde{G}$, it follows that $(T^n x_{i-1}, T^n x_i) \in E(\widetilde{G})$ for all $n \geq 0$ and $i = 1, \ldots, N$. Therefore, by (1) and (IC2) we have

$$\begin{aligned}
\int_0^{p(T^n x, T^n y)} \varphi(t)\mathrm{d}t &\leq \int_0^{\sum_{i=1}^N p(T^n x_{i-1}, T^n x_i)} \varphi(t)\mathrm{d}t \\
&\leq \sum_{i=1}^N \int_0^{p(T^n x_{i-1}, T^n x_i)} \varphi(t)\mathrm{d}t \\
&\leq \alpha \sum_{i=1}^N \int_0^{p(T^{n-1} x_{i-1}, T^{n-1} x_i)} \varphi(t)\mathrm{d}t \\
&\vdots \\
&\leq \alpha^n \sum_{i=1}^N \int_0^{p(x_{i-1}, x_i)} \varphi(t)\mathrm{d}t
\end{aligned}$$

for all $n \geq 0$, where $\varphi \in \Phi$ and $\alpha \in (0, 1)$ are as in (IC2). Since, by (Φ2), $\sum_{i=1}^N \int_0^{p(x_{i-1}, x_i)} \varphi(t)\mathrm{d}t$ is finite (possibly zero), it follows immediately that $\int_0^{p(T^n x, T^n y)} \varphi(t)\mathrm{d}t \to 0$. Hence by Lemma 2.1, $p(T^n x, T^n y) \to 0$.

Now, pick a point $x \in C_T$. By Lemma 2.9, the sequence $\{T^n x\}$ is p-Cauchy in X and since X is p-complete, there exists a $u \in X$ such that $T^n x \xrightarrow{p} u$. If y is an arbitrary point in X, then

$$0 \leq p(T^n y, u) \leq p(T^n y, T^n x) + p(T^n x, u) \to 0 \quad \text{as} \quad n \to \infty.$$

So $T^n y \xrightarrow{p} u$. The uniqueness of u follows immediately from Lemma 1.2. □

Theorem 2.18. *Let p be an E-distance on a separated uniform space X endowed with a graph G and $T : X \to X$ be an integral type p-$\widetilde{G}$-contraction. If the subgraph of G with the vertices* $\mathrm{Fix}(T)$ *is weakly connected, then T has at most one fixed point in X.*

Proof. Let x and y be two fixed points for T. Then there exists a path $(x_i)_{i=0}^N$ in $\widetilde{G}$ from x to y such that $x_1, \ldots, x_{N-1} \in \mathrm{Fix}(T)$. Since $E(\widetilde{G})$ contains all loops, we can assume without loss of generality that the length of this path, that is, the integer N is even. Now, by (IC2) we have

$$\int_0^{p(x_{i-1}, x_i)} \varphi(t)\mathrm{d}t = \int_0^{p(Tx_{i-1}, Tx_i)} \varphi(t)\mathrm{d}t \leq \alpha \int_0^{p(x_{i-1}, x_i)} \varphi(t)\mathrm{d}t \qquad i = 1, \ldots, N,$$

where $\varphi \in \Phi$ and $\alpha \in (0, 1)$, which is impossible unless $\int_0^{p(x_{i-1}, x_i)} \varphi(t)\mathrm{d}t = 0$ or equivalently, $p(x_{i-1}, x_i) = 0$ for $i = 1, \ldots, N$. Because $E(\widetilde{G})$ is symmetric, a similar argument yields $p(x_i, x_{i-1}) = 0$ for $i = 1, \ldots, N$. Since N is even, using Lemma 1.2 finitely many times, we get $x = x_0 = x_2 = \cdots = x_N = y$. Consequently, T has at most one fixed point in X. □

Remark 2.19. *Theorem 2.18 guarantees that in a separated uniform space X endowed with a graph G and an E-distance p, if* $(x, y) \in E(G)$*, then both x and y cannot be a fixed point for any integral type p-*$\widetilde{G}$*-contraction T. In other words, each weakly connected component of G intersects* $\mathrm{Fix}(T)$ *in at most one point. So in partially ordered separated uniform spaces equipped with an E-distance p, no ordered integral type p-contraction has two comparable fixed points.*

Remark 2.20. *Since the Riemann integral (proper and improper) is subsumed in the Lebesgue integral, it follows that one may replace Lebesgue-integrability with Riemann-integrability of* φ *on* $[0, +\infty)$ *in* $(\Phi 1)$*, where the value of the integral on* $[0, +\infty)$ *is allowed to be* ∞*. Facing with Riemann integrals, we should assume that the function* φ *is bounded. Therefore, all of the results of this paper can be restated and reproved for Riemann integrals instead of Lebesgue integrals. A similar remark holds for Riemann-Stieltjes integrable functions with respect to any fixed nondecreasing function on* $[0, +\infty)$*.*

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