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Essential Ideals in Subrings of $C(X)$ that Contain $C^*(X)$

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Abstract. Let $A(X)$ be a subring of $C(X)$ that contains $C^*(X)$. In Redlin and Watson (1987) and in Panman et al. (2012), correspondences $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ are defined between ideals in $A(X)$ and z-filters on X, and it is shown that these extend the well-known correspondences studied separately for $C^*(X)$ and $C(X)$, respectively, to any intermediate ring $A(X)$. Moreover, the inverse map $\mathcal{Z}_A^{-1}$ sets up a one-one correspondence between the maximal ideals of $A(X)$ and the z-ultrafilters on X. In this paper, first, we characterize essential ideals in $A(X)$. Afterwards, we show that $\mathcal{Z}_A^{-1}$ maps essential (resp., free) z-filters on X to essential (resp., free) ideals in $A(X)$ and $\mathfrak{Z}_A^{-1}$ maps essential $\mathfrak{Z}_A$-filters to essential ideals. Similar to $C(X)$ we observe that the intersection of all essential minimal prime ideals in $A(X)$ is equal to the socle of $A(X)$. Finally, we give a new characterization for the intersection of all essential maximal ideals of $A(X)$.

1. Introduction

In this paper, X is assumed to be a completely regular Hausdorff space and all rings are commutative, reduced, and have an identity element. $C(X)$ ($C^*(X)$) stands for the ring of all real valued (bounded) continuous functions on X. A nonzero ideal I in a commutative ring R is called essential if it intersects every nonzero ideal nontrivially. This concept was first introduced in [19] and plays an important role in the structure theory of noncummutative Notherian rings. Essential ideals can also characterize for any commutative ring R the Socle of R, denoted Soc(R), which is the sum of all minimal ideals of R. It is known that the intersection of all essential ideals in any commutative ring R is Soc(R), see [14] or [21].

Essential ideals in $C(X)$ and their intersections were investigated by F. Azarpanah in [3] and [4] (see also [11]). We study essential ideals in subrings of $C(X)$ that contain $C^*(X)$ (as intermediate rings). Intermediate rings of continuous functions have been studied by several authors in different forms and by different names. Each intermediate ring $A(X)$ forms an archimedean lattice-ordered algebra called a Φ-algebra, which has been studied by A.W. Hager, M. Henrksen, J.R. Isbell, D.G. Johnson, P. Nanzetta, and D. Plank in [15]-[17]. Intermediate rings have also been studied as algebras of functions by J.R. Isbell in [18], and as β-subalgebras of $C(X)$ by D. Plank in [23]. Also, they have been investigated as intermediate algebras by H.L. Byun, L. Redlin and S. Watson in [6], [7] and [24], and also [22] and [26]. In 1997, J.M. Dominguez, J. Gomez and M.A. Mulero showed in [8] that intermediate rings can be realized as certain rings of fractions of $C^*(X)$. Some examples and methods of constructing intermediate rings of continuous functions can be found in [9] and [10]. In [2] a description is given for the intersection of the free maximal ideals in such rings.

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In Section 3, we show that an ideal I of $A(X)$ is essential if and only if $\mathcal{Z}_A[I]$ is an essential z-filter on X if and only if $\bigcap \mathcal{Z}_A[I]$ is a nowhere dense subset of X if and only if $\mathfrak{Z}_A[I]$ is an essential z-filter on X. This leads us to characterize essential prime (resp., maximal) ideals in $A(X)$. As in C(X), we also show that every prime ideal of A(X) is either essential or both minimal and maximal among prime ideals. Furthermore, maximal ideals M^p are essential ideals, where $p \in \beta X \setminus I(X)$ and $I(X)$ is the set of isolated points of X. It is shown in [29] that the intersection of all essential minimal prime ideals in $C(X)$ is yet another characterization of Soc($C(X)$). We show that in $A(X)$ the intersection of all essential minimal prime ideals coincides with Soc($C(X)$). The map $\mathcal{Z}_A$ was introduced in [24], and its properties were further investigated in [6], [7] and [26]. In [22], another correspondence $\mathfrak{Z}_A$ between ideals of $A(X)$ and z-filters on X is introduced. Using some properties of $\mathfrak{Z}_A$ and $\mathcal{Z}_A$, we show that $\mathcal{Z}_A^{-1}$, maps essential (resp., free) z-filters on X to essential (resp., free) ideals in $A(X)$ and $\mathfrak{Z}_A^{-1}$ maps essential $\mathfrak{Z}_A$-filters to essential ideals.

In Section 4, we characterize the intersection of all essential maximal ideals in $A(X)$. It is proved that $f \in A(X)$ belongs to the intersection of all essential maximal ideals of $A(X)$ if and only if for each $g \in A(X)$ and $n \in \mathbb{N}$, the set $\{x \in X : |f(x)||g(x)| \geq \frac{1}{n}\}$ is a finite subset of $I(X)$, where $I(X)$ is the set of isolated points of X.

2. Preliminaries

Definition 2.1. *If $f \in A(X)$ and $H \in Z[X]$, then f is called H-regular if there is a $g \in A(X)$ such that $f(x)g(x) = 1$ for all $x \in H$.*

For each $f \in A(X)$, $\mathcal{Z}_A(f) = \{E \in Z[X] : f \text{ is } E^C\text{-regular}\}$. In other words, $\mathcal{Z}_A(f)$ consists of those zero sets such that f is locally invertible in $A(X)$ on their complements. In [24], Redlin and Waston defined $\mathcal{Z}_A[I] = \bigcup_{f \in I} \mathcal{Z}_A(f)$ and $\mathcal{Z}_A^{-1}[\mathcal{F}] = \{f \in A(X) : \mathcal{Z}_A \subseteq \mathcal{F}\}$ for an ideal I of $A(X)$ and a z-filter $\mathcal{F}$ on X. The following lemma is used in the sequel.

Lemma 2.2. *For $f, g \in A(X)$ the following statements hold.*

1. $Z(f) = \bigcap \mathcal{Z}_A(f)$.
2. $\mathcal{Z}_A(f) \cap \mathcal{Z}_A(g) = \mathcal{Z}_A(fg)$.
3. *If $f \in A(X)$, then $|f| \in A(X)$.*

Item (1) is shown in [25, Proposition 2.2]. Item (2) is proved in [22, Lemma 1.5]. For item (3), see [6, Theorem 1.1].

Lemma 2.3. *[24, Theorems 1 and 3] Let $f \in A(X)$, I an ideal of $A(X)$ and $\mathcal{F}$ a z-filter on X.*

1. *$\mathcal{Z}_A(f)$ is a z-filter on X if and only if f is not invertible in $A(X)$.*
2. *$\mathcal{Z}_A[I]$ is a z-filter on X.*
3. *$\mathcal{Z}_A^{-1}[\mathcal{F}]$ is an ideal of $A(X)$.*

P. Panaman, J. Sack and S. Watson defined in [22] and [26] that for each $f \in A(X)$, $\mathfrak{Z}_A(f) = \{E \in Z[X] : f$ is H-regular in $A(X)$ for all zero sets $H \subseteq E^C\}$. They also introduced for an ideal I of $A(X)$, $\mathfrak{Z}_A(I) = \bigcup_{f \in I} \mathfrak{Z}_A(f)$ and for a z-filter $\mathcal{F}$, $\mathfrak{Z}_A^{-1}[\mathcal{F}] = \{f \in A(X) : \mathfrak{Z}_A(f) \subseteq \mathcal{F}\}$.

Lemma 2.4. *The following statements hold.*

1. *If I is an ideal in $A(X)$, then $\mathfrak{Z}_A[I]$ is a z-filter on X.*
2. *If $\mathcal{U}$ is a z-ultrafilter on X, then $\mathcal{Z}_A^{-1}[\mathcal{U}] = \mathfrak{Z}_A^{-1}[\mathcal{U}]$ is a maximal ideal in $A(X)$.*

For the proof of the above lemma, see [22, Theorems 4.3 and 4.7].

3. Essential Ideals in $A(X)$

We begin with the following lemma.

Lemma 3.1. *$Soc(A(X)) = Soc(C(X)) = \{f \in C(X)$: $X \setminus Z(f)$ is finite $\}$.*

Proof. The second equality is proved in [20, Proposition 3.3]. We have $A(X) \subseteq C(X)$. Therefore $\mathrm{Soc}(A(X)) \subseteq \mathrm{Soc}(C(X))$. Let us prove the other inclusion. Consider $f \in \mathrm{Soc}(C(X))$. Then $X \setminus Z(f)$ is finite. Suppose that $X \setminus Z(f) = \{x_1, ..., x_n\}$. Each x_i is an isolated point, so for each $1 \leq i \leq n$, the ideal $A(X)e_i = C(X)e_i \cap A(X)$ is a minimal ideal in $A(X)$, where $e_i(x_i) = 1$ and $e_i(X \setminus \{x_i\}) = 0$. Therefore $f = f(x_1)e_1 + f(x_2)e_2 + ... + f(x_n)e_n \in \mathrm{Soc}(A(X))$. □

Recall from [4], a z-filter $\mathcal{F}$ in a space X is called an essential z-filter if $\mathcal{F} \cap \mathcal{F}' \neq \{X\}$ for every nontrivial z-filter $\mathcal{F}'$ (where the trivial z-filter is the z-filter $\{X\}$). F. Azarpanah showed in [4, Theorem 1.3] that an ideal I of $C(X)$ is essential if and only if $\bigcap Z[I]$ is a nowhere dense subset of X. The following is a counterpart for $A(X)$.

Proposition 3.2. *Let I be a non-zero ideal in $A(X)$. Then the following statements are equivalent.*

1. *I is an essential ideal in $A(X)$.*
2. *$\mathcal{Z}_A[I]$ is an essential z-filter in X.*
3. *$\bigcap \mathcal{Z}_A[I]$ is a nowhere dense subset of X.*
4. *$\mathfrak{Z}_A[I]$ is an essential z-filter on X.*

Proof. (1)⇒(2) Let $\mathcal{F}$ be a non-trivial z-filter. Then $\mathcal{Z}_A^{-1}[\mathcal{F}] \cap I \neq 0$. This implies that $\mathcal{F} \cap \mathcal{Z}_A[I] \neq \{X\}$.

(2)⇒(3) By Lemma 2.2(1), we have

$$\bigcap Z(I) = \bigcap_{f\in I} Z(f) = \bigcap_{f\in I}(\bigcap \mathcal{Z}_A(f)) = \bigcap \bigcup_{f\in I} \mathcal{Z}_A(f) = \bigcap \mathcal{Z}_A[I].$$

Now suppose that $x \in int \bigcap \mathcal{Z}_A[I] = int \bigcap Z[I]$. X being a completely regular space, so there exists $g \in C^*(X)$ such that $x \notin Z(g)$ and $X \setminus int \bigcap Z[I] \subseteq Z(g)$. Therefore $fg = 0$ for all $f \in I$. This implies that $\mathcal{Z}_A(fg) = \mathcal{Z}_A(f) \cap \mathcal{Z}_A(g) = \{X\}$ for all $f \in I$. Thus $\mathcal{Z}_A[I] \cap \mathcal{Z}_A(g) = \{X\}$, but $\mathcal{Z}_A(g)$ is a nontrivial z-filter, a contradiction. So $int \bigcap \mathcal{Z}_A[I] = \emptyset$.

(3)⇒(1) Let $0 \neq f \in A(X)$ and $I \cap (f) = 0$. Then $\bigcap Z[I] \cup Z(f) = X$. This says that $X \setminus Z(f)$ is a nonempty open subset in $\bigcap Z[I] = \bigcap \mathcal{Z}_A[I]$, a contradiction.

(2)⇒(4) It is easily seen that $\mathcal{Z}_A[I] \subseteq \mathfrak{Z}_A[I]$. By essentiality of $\mathcal{Z}_A[I]$, for every nontrivial z-filter $\mathcal{F}$ we have $\mathfrak{Z}_A[I] \cap \mathcal{F} \neq \{X\}$, which completes the proof.

(4)⇒(1) Assume that $0 \neq f \in A(X)$ and $I \cap (f) = 0$. Then $\mathfrak{Z}_A(f) \cap \mathfrak{Z}_A[I] = \{X\}$. To see this, let $E \in \mathfrak{Z}_A(f) \cap \mathfrak{Z}_A[I]$. Then $E \in \mathfrak{Z}_A(f) \cap \mathfrak{Z}_A(g) = \mathfrak{Z}_A(fg)$, for some $g \in I$. But $fg = 0$. Thus $E = X$. On the other hand $\mathfrak{Z}_A(f)$ is a nontrivial z-filter. This shows that $\mathfrak{Z}_A[I]$ is a non-essential z-filter, which is a contradiction. □

Recall from [6] and [24], an ideal I of $A(X)$ is fixed if $\bigcap \mathcal{Z}_A[I] \neq \emptyset$ ($\bigcap Z[I] \neq \emptyset$) and free if $\bigcap \mathcal{Z}_A[I] = \emptyset$.

Corollary 3.3. *An ideal I of $A(X)$ is essential (resp., free) if and only if $IC(X)$ is essential (resp., free) in $C(X)$.*

Proof. It follows from Proposition 3.2 and this fact that $\bigcap \mathcal{Z}[I] = \bigcap Z[I] = \bigcap Z[IC(X)]$. □

Remark 3.4. *Every bounded ideal of $C(X)$ (an ideal with bounded elements such as $Soc(C(X))$, $C_K(X) = \{f \in C(X)$: $cl_X(X \setminus Z(f))$ is compact $\}$ and $C_\psi(X) = \{f \in C(X) : cl_X(X \setminus Z(f))$ is pseudocompact $\}$) is an ideal of $A(X)$. Note that $C_\psi(X)$ is the largest bounded ideal in $C(X)$, see [5]. Since $C_\psi(X)$ is an intersection of essential ideals in $C(X)$, it is also an intersection of essential ideals of $A(X)$, by Corollary 3.3. Therefore $C_\psi(X)$ is essential in $A(X)$ if and only if it is essential in $C(X)$.*

Now, Proposition 3.2 implies the following result.

Corollary 3.5. *Every free ideal in $A(X)$ is an essential ideal.*

Recall from [26], a z-filter $\mathcal{F}$ on X is called a $\mathfrak{Z}_A$-*filter* if $\mathfrak{Z}_A[\mathfrak{Z}_A^{-1}[\mathcal{F}]] = \mathcal{F}$. If $A(X) = C(X)$, then $\mathfrak{Z}_A$-filters are simply z-filters, by [22, Corollary 2.4]. If I is an ideal of $C(X)$, then we can see that $\mathfrak{Z}_A[\mathfrak{Z}_A^{-1}[\mathfrak{Z}_A[I]] = \mathfrak{Z}_A[I]$. Thus $\mathfrak{Z}_A[I]$ is a $\mathfrak{Z}_A$-filter on X. However, not every z-filter is a $\mathfrak{Z}_A$-filter. For more details see [26]. The next results (Proposition 3.4 and Theorem 3.5) show that the essential (resp., free) z-filters behave like the z-ultrafilters.

Proposition 3.6. *Let $\mathcal{F}$ be a z-filter on X and $\mathfrak{Z}_A^{-1}[\mathcal{F}]$ be an ideal of $A(X)$.*

1. *If $\mathcal{F}$ is an essential $\mathfrak{Z}_A$-filter, then $\mathfrak{Z}_A^{-1}[\mathcal{F}]$ is an essential ideal in $A(X)$.*
2. *If $\mathfrak{Z}_A^{-1}[\mathcal{F}]$ is a free ideal, then $\mathcal{F}$ is a free z-filter.*
3. *If $\mathfrak{Z}_A^{-1}[\mathcal{F}]$ is an essential ideal, then $\mathcal{F}$ is an essential z-filter.*

Proof. (1) Assume that $0 \neq f \in A(X)$ and $\mathfrak{Z}_A^{-1}[\mathcal{F}] \cap (f) = 0$. We will show that it follows that $\mathcal{F} \cap \mathfrak{Z}_A(f) = \{X\}$. Since $\mathcal{F}$ is an essential $\mathfrak{Z}_A$-filter, this implies that $\mathfrak{Z}_A(f)$ is a trivial z-filter, thus contradicting the assumption that $0 \neq f$ and concluding the proof of this item. To see this, suppose that $E \in \mathcal{F} \cap \mathfrak{Z}_A(f)$. By hypothesis, $E \in \mathcal{F} = \mathfrak{Z}_A[\mathfrak{Z}_A^{-1}[\mathcal{F}]]$, so there exists $h \in \mathfrak{Z}_A^{-1}[\mathcal{F}]$ such that $E \in \mathfrak{Z}_A(h)$. Since $\mathfrak{Z}_A^{-1}[\mathcal{F}]$ is an ideal, we have $fh \in \mathfrak{Z}_A^{-1}[\mathcal{F}]$. Therefore $fh \in \mathfrak{Z}_A^{-1}[\mathcal{F}] \cap (f) = 0$. Thus $E \in \mathfrak{Z}_A(h) \cap \mathfrak{Z}_A(f) = \mathfrak{Z}_A(fh) = \{X\}$.

(2) It is easy to see that $\mathcal{Z}_A[\mathfrak{Z}_A^{-1}[\mathcal{F}]] \subseteq \mathcal{F}$. By this and hypothesis, $\mathcal{F}$ is a free z-filter.

(3) By Proposition 3.2, the essentiality of the ideal $\mathfrak{Z}_A^{-1}[\mathcal{F}]$ implies the essentiality of the z-filter $\mathfrak{Z}_A[\mathfrak{Z}_A^{-1}[\mathcal{F}]]$. On the other hand $\mathfrak{Z}_A[\mathfrak{Z}_A^{-1}[\mathcal{F}]] \subseteq \mathcal{F}$. Thus for each non-trivial z-filter $\mathcal{F}'$, we have $\mathfrak{Z}_A[\mathfrak{Z}_A^{-1}[\mathcal{F}]] \cap \mathcal{F}' \subseteq \mathcal{F} \cap \mathcal{F}'$. This and the essentiality of $\mathfrak{Z}_A[\mathfrak{Z}_A^{-1}[\mathcal{F}]]$ imply that $\mathcal{F}$ is an essential z-filter. □

In [7], a ring $A(X)$ of continuous functions is called a C-ring if there is a completely regular space Y such that $A(X)$ is isomorphic to $C(Y)$. Clearly $C(X)$ and $C^*(X)$ are C-rings. By the above theorem and [27, Theorem 14] (which states that for each z-filter $\mathcal{F}$ and intermediate C-ring $A(X)$, $\mathfrak{Z}_A^{-1}[\mathcal{F}]$ is an ideal of $A(X)$), if $A(X)$ is a C-ring and $\mathcal{F}$ is an essential $\mathfrak{Z}_A$-filter on X, then $\mathfrak{Z}_A^{-1}[\mathcal{F}]$ is an essential ideal in $A(X)$.

Theorem 3.7. *Let $\mathcal{F}$ be a z-filter on X. Then the following hold.*

1. *$\mathcal{F}$ is a free z-filter if and only if $\mathcal{Z}_A^{-1}[\mathcal{F}]$ is a free ideal in $A(X)$.*
2. *$\mathcal{F}$ is an essential z-filter if and only if $\mathcal{Z}_A^{-1}[\mathcal{F}]$ is an essential ideal in $A(X)$.*

Proof. (1) It is enough to show that $\bigcap \mathcal{Z}_A[\mathcal{Z}_A^{-1}[\mathcal{F}]] = \bigcap \mathcal{F}$. Always we have $\mathcal{Z}_A[\mathcal{Z}_A^{-1}[\mathcal{F}]] \subseteq \mathcal{F}$. Thus $\bigcap \mathcal{F} \subseteq \bigcap \mathcal{Z}_A[\mathcal{Z}_A^{-1}[\mathcal{F}]]$. Suppose, in order to obtain a contradiction, that there is an $x \in \bigcap \mathcal{Z}_A[\mathcal{Z}_A^{-1}[\mathcal{F}]]$ and $E \in \mathcal{F}$ such that $x \notin E$. By complete regularity of X, there is a $g \in C^*(X)$ such that $x \notin Z(g)$ and $E \subseteq Z(g) = \bigcap \mathcal{Z}_A(g)$. Therefore $\mathcal{Z}_A(g) \subseteq \mathcal{F}$. This implies that $g \in \mathcal{Z}_A^{-1}[\mathcal{F}]]$, and so $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A[\mathcal{Z}_A^{-1}[\mathcal{F}]]$. But $x \notin Z(g)$. So there is $K \in \mathcal{Z}_A(g)$ such that $x \notin K$. This shows that $x \notin \bigcap \mathcal{Z}_A[\mathcal{Z}_A^{-1}[\mathcal{F}]]$, a contradiction.

(2) Assume that $\mathcal{F}$ is an essential z-filter, $0 \neq f \in A(X)$ and $\mathcal{Z}_A^{-1}[\mathcal{F}] \cap (f) = 0$. We show that it follows that $\mathcal{F} \cap \mathcal{Z}_A(f) = \{X\}$. To see this, consider $Z(g) \in \mathcal{F} \cap \mathcal{Z}_A(f)$. Then we have $\mathcal{Z}_A(g) \subseteq \mathcal{F}$ and $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A(f)$. On the other hand, $fg \in \mathcal{Z}_A^{-1}[\mathcal{F}] \cap (f) = 0$. Thus by Lemma 2.2(2), $\mathcal{Z}_A(g) = \mathcal{Z}_A(g) \cap \mathcal{Z}_A(f) = \mathcal{Z}_A(fg) = \{X\}$, i.e., $E = X$. But $\mathcal{Z}_A(f)$ is a non-trivial z-filter, hence $\mathcal{F} \cap \mathcal{Z}_A(f) = \{X\}$ contradicts the essentiality of $\mathcal{F}$, so we are done. Conversely, suppose that $\mathcal{Z}_A^{-1}[\mathcal{F}]$ is an essential ideal. By Proposition 3.2, $\mathcal{Z}_A[\mathcal{Z}_A^{-1}[\mathcal{F}]]$ is an essential z-filter. On the other hand $\mathcal{Z}_A[\mathcal{Z}_A^{-1}[\mathcal{F}]] \subseteq \mathcal{F}$. Thus for each non-trivial z-filter $\mathcal{F}'$, $\mathcal{Z}_A[\mathcal{Z}_A^{-1}[\mathcal{F}]] \cap \mathcal{F}' \subseteq \mathcal{F} \cap \mathcal{F}'$. So $\mathcal{F} \cap \mathcal{F}' \neq \{X\}$, i.e., $\mathcal{F}$ is an essential z-filter. □

It is well known that any maximal ideal of $A(X)$ is of the form M_A^p, where $p \in \beta X$ and $M_A^p = \{f \in A(X) :\ p$ is cluster point of $\mathcal{Z}_A(f)$ in $\beta X\}$, see [24, Theorem 5]. We also note that a maximal ideal in $A(X)$ is free if and only if it is of the form M_A^p for some $p \in \beta X \setminus X$. H.L. Byun and S. Watson in [6] defined the ideal $O_A^p = \{f \in A(X) : p \in int_{\beta X} cl_{\beta X} \mathcal{Z}_A(f)\}$, where $cl_{\beta X} \mathcal{Z}_A(f) = \bigcap_{E \in \mathcal{Z}_A(f)} cl_{\beta X} E$. It is an analogue of the ideal O^p in $C(X)$ for each $p \in \beta X$. Afterwards, J.M. Dominguez and J. Gomez showed that $int_{\beta X} cl_{\beta X} \mathcal{Z}_A(f) = int_{\beta X} cl_{\beta X} Z(f)$ for each $f \in A(X)$, and therefore $O_A^p = O^p \cap A(X)$, see also [9, Proposition 3.4]. Now, by [6, Theorem 4.1], which states that $\bigcap \mathcal{Z}_A[O_A^p] = \bigcap \mathcal{Z}_A[M_A^p]$, Proposition 3.2 and Corollary 3.5, we have the following result.

Corollary 3.8. *The following statements are equivalent.*

1. *The ideal M^p_A is essential.*
2. *The ideal O^p_A is essential*
3. *$p \in \beta X \setminus I(X)$, where $I(X)$ is the set of isolated points of X.*

If $x \in X \setminus I(X)$, then by Corollary 3.8, M^x_A is an essential ideal which is not a free ideal.

Remark 3.9. *Every prime ideal of $A(X)$ is either an essential ideal or a maximal ideal which is at the same time a minimal prime ideal. For, every prime ideal P of $A(X)$ is contained in a unique maximal ideal M^p_A for some $p \in \beta X$ (see [6, Corollary 1.9]), and a prime ideal P in $A(X)$ is contained in M^p_A if and only if P contains O^p_A (see [23, p. 48] and the proof of [6, Theorem 4.2]). Now, if $p \in I(X)$, then $O^p_A = P = M^p_A$ and so it is maximal which is at the same time a minimal prime ideal. Otherwise, by Corollary 3.8, O^p_A is essential and so P is essential.*

Corollary 3.10. *X is finite if and only if $A(X)$ has no proper essential ideals.*

Proof. If X is finite, then we are done. Conversely, by hypothesis, the ideal M^p is a non-essential ideal for each $p \in \beta X$, and hence by Corollary 3.8, $\beta X \setminus I(X) = \emptyset$, i.e., $\beta X = I(X)$. Thus βX is finite. □

It was proved in [29, Theorem 3.6] that the intersection of all essential minimal prime ideals of $C(X)$ and $\mathrm{Soc}(C(X))$ are equal. Now, we see that this result is true for any subring $A(X)$. To achieve this goal, we need the following lemma. Before we recall that for $S \subseteq \beta X$, $M^S_A = \bigcap_{p\in S} M^p_A$ and $O^S_A = \bigcap_{p \in S} O^p_A$ (see [6]).

Lemma 3.11. *The intersection of all essential prime ideals in $A(X)$ is equal to the intersection of all essential minimal prime ideals of $A(X)$.*

Proof. Always we have the intersection of all essential prime ideals is contained in the intersection of all essential minimal prime ideals. Now let f be an element of the intersection of all essential minimal prime ideals of $A(X)$, P be an essential prime ideal and Q be a minimal prime ideal contained in P (see [28, 3.53]). Then by reasoning given in Remark 3.9, there is a unique $p \in \beta X$ such that $O^p_A \subseteq Q \subseteq P \subseteq M^p_A$. The essentiality of P implies that M^p_A is an essential ideal. By Corollary 3.8, O^p_A is essential and therefore Q is essential. Thus $f \in Q \subseteq P$, so we are done. □

Proposition 3.12. *The intersection of all essential minimal prime ideals of $A(X)$ coincides with the $Soc(A(X))$ (i.e., $Soc(C(X))$).*

Proof. First we show that the intersection of all essential minimal prime ideals of $A(X)$ is equal to the ideal $O^{\beta X\setminus I(X)}_A$. Let P be an essential minimal prime ideal of $A(X)$. Then by reasoning given in Remark 3.9, there is a unique $p \in \beta X$ such that $O^p_A \subseteq P \subseteq M^p_A$. The essentiality of P implies that M^p_A is essential and so by Corollary 3.8, O^p_A is an essential ideal. This shows that $O^{\beta X\setminus I(X)}_A$ is contained in the intersection of all essential minimal prime ideals of $A(X)$. Now let f be an element of the intersection of all essential minimal prime ideals of $A(X)$ and $p \in \beta X \setminus I(X)$. By Corollary 3.8, O^p_A is an essential ideal. By [9, Remark 3.14], O^p_A is an intersection of prime ideals. Since any ideal containing an essential ideal is essential, each of these prime ideals containing O^p_A is essential. Now Lemma 3.11 implies that $f \in O^p_A$. Hence the intersection of all essential minimal prime ideals of $A(X)$ is contained in $O^{\beta X\setminus I(X)}_A$. Finally, by [9, Proposition 3.4] (i.e., $O^p_A = O^p \cap C(X)$ for each $p \in \beta X$), [29, Theorem 3.6] which states that $O^{\beta X\setminus I(X)} = \mathrm{Soc}(C(X))$ and Lemma 3.1, we have $O^{\beta X\setminus I(X)}_A = O^{\beta X\setminus I(X)} \cap A(X) = \mathrm{Soc}(C(X)) \cap A(X) = \mathrm{Soc}(A(X))$. □

Corollary 3.13. *The intersection of all essential minimal prime ideals of $A(X)$ coincides with the intersection of all essential minimal prime ideals of $C(X)$ (i.e., $O^{\beta X\setminus I(X)}_A = O^{\beta X\setminus I(X)}$).*

Proof. This follows from [29, Theorem 3.6], Lemma 3.1 and Proposition 3.12. □

4. Intersection of Essential Maximal Ideals in $A(X)$

Let $A_n(fg) = \{x \in X : |f(x)||g(x)| \geq \frac{1}{n}\}$, where $f, g \in A(X)$ and $n \in \mathbb{N}$.

Theorem 4.1. *[2, Theorem 2.2] The intersection of all free maximal ideals in $A(X)$ is equal to $K_A = \{f \in A(X) : A_n(fg)$ is compact $\forall g \in A(X)$ and $\forall n \in \mathbb{N}\}$.*

We are ready to present our characterization of the intersection of all essential maximal ideals of $A(X)$. Note that the set of those points of a space X with compact neighborhoods is denoted by X_L.

Theorem 4.2. *The following statements hold.*

1. *The intersection of all essential maximal ideals of $A(X)$ is equal to $J_A = \{f \in A(X) : A_n(fg) \subseteq I(X)$ is finite $\forall n \in \mathbb{N}$ and $\forall g \in A(X)\}$*
2. *The intersection of all free maximal ideals and the intersection of all essential maximal ideals in $A(X)$ are equal if and only if $I(X) = X_L$.*

Proof. (1) Let f be an element of the intersection of all essential maximal ideals of $A(X)$. Then by Corollary 3.8, $f \in M_A^{\beta X \setminus I(X)}$. Thus $\beta X \setminus I(X) \subseteq cl_{\beta X}\mathcal{Z}_A(f)$. So $A_n(fg) \subseteq X \setminus Z(f) \subseteq I(X)$ for each $g \in A(X)$. By Corollary 3.5, every free ideal is essential. Therefore $f \in M_A^{\beta X \setminus X}$. By Theorem 4.1, $A_n(fg)$ is compact. Hence $A_n(fg)$ is a finite subset of $I(X)$, i.e., $f \in J_A$. Now let $f \in J_A$ and M be an essential maximal ideal such that $f \notin M$. Then there is a $g \in A(X)$ such that $1 - fg \in M$. By hypothesis, $G = \{x \in X : f(x)g(x) > \frac{1}{2}\} \subseteq \{x \in X : |f(x)g(x)| \geq \frac{1}{2}\}$ is a finite subset of $I(X)$. Put $h = |1 - fg| + \chi_G$, where χ_G is the characteristic function of the subset G. Then $\chi_G \in C(X)$ and $X \setminus Z(\chi_G)$ is finite, so by Lemma 3.1, $\chi_G \in \mathrm{Soc}(C(X)) \subseteq M$. Also, we have $|1 - fg| \in M$. Thus $h \in M$. We claim that h is a unit in $A(X)$. To see this, define $k(x) = \frac{1}{1+|1-f(x)g(x)|}$ for $x \in G$ and $k(x) = \frac{1}{|1-f(x)g(x)|}$ for $x \notin G$. Then $k \in C^*(X) \subseteq A(X)$ (since $x \notin G$ implies that $\frac{1}{|1-f(x)g(x)|} \leq 2$ and G is finite) and $hk = 1$; this is a contradiction, so f is an element of any essential maximal ideal of $A(X)$.

(2) Suppose that $M_A^{\beta X \setminus X} = M_A^{\beta X \setminus I(X)}$. Always $I(X) \subseteq X_L$ holds. Now let $x \in X_L$. Then there is a compact subset U in X such that $x \in intU \subseteq U$. By complete regularity of X there exists a function $f \in C^*(X)$ such that $x \in X \setminus Z(f) \subseteq U$. Therefore $A_n(fg) \subseteq X \setminus Z(f)$ is compact for each $g \in A(X)$, $n \in \mathbb{N}$, and so $f \in K_A$. By hypothesis and Theorem 4.1, $f \in M_A^{\beta X \setminus I(X)}$. So $x \in X \setminus Z(f) \subseteq I(X)$.

Conversely, by hypothesis and [1, Theorem 2.2] which states that $X_L = int_{\beta X}X$, we have $int_{\beta X}X = I(X)$. This implies that $cl_{\beta X}(\beta X \setminus X) = \beta X \setminus I(X)$. On the other hand for each $S \subseteq \beta X$, $S \subseteq cl_{\beta X}\mathcal{Z}_A(f)$ implies $cl_{\beta X}S \subseteq cl_{\beta X}\mathcal{Z}_A(f)$, so $M_A^S \subseteq M_A^{cl_{\beta X}S}$ and of $S \subseteq cl_{\beta X}S$ we obtain $M_A^{cl_{\beta X}S} \subseteq M_A^S$. Hence $M_A^{cl_{\beta X}S} = M_A^S$. Therefore $M_A^{\beta X \setminus X} = M_A^{cl_{\beta X}(\beta X \setminus X)} = M_A^{\beta X \setminus I(X)}$. Thus the proof is complete. □

M. Ghirati and A. Taherifar proved in [12, Theorem 2.2(3)] that the intersection of all essential maximal ideals of $C(X)$ coincides with the intersection of all free maximal ideals if and only if $I(X) = X_L$. So by the above theorem, the equality of the intersection of all free maximal ideals and the intersection of all essential maximal ideals in $A(X)$ implies the equality of them in $C(X)$ and vice versa.

Corollary 4.3. *The intersection of all essential maximal ideals in $C^*(X)$ is equal to $\{f\colon \{x : |f(x)| \geq \frac{1}{n}\} \subseteq I(X)$ for each $n \in \mathbb{N}$ and is finite$\}$.*

Proof. It is enough to take $A = C^*(X)$ in Theorem 4.2. □

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