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On Hermite-Hadamard Type Integral Inequalities for n-times Differentiable Log-Preinvex Functions

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Abstract. In this paper, new Hermite-Hadamard type inequalities for n-times differentiable log-preinvex functions are established. The established results generalize some of those results proved in recent papers for differentiable log-preinvex functions and differentiable log-convex functions.

1. Introduction

It is well known in mathematics literature that if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. The inequalities (1.1) are known as Hermite-Hadamard inequalities, a result first noticed by Ch. Hermite in 1883 and rediscovered ten years later by J. Hadamard. Since the discovery of (1.1) in 1883, Hermite-Hadamard inequality (see [10]) has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements, counterparts and new Hermite-Hadamard-type inequalities and numerous applications, see [4]-[7], [9], [11]-[15], [25], [27]-[30], [32, 33] and the references therein.

In recent years, many mathematicians generalized the classical convexity in many ways and some of those are given as follows.

Definition 1. *[36] A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to $\eta : K \times K \to \mathbb{R}^n$ if*

$$u + t\eta(v,u) \in K, \forall u, v \in K, t \in [0,1].$$

The invex set K is also called an η-connected set.

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Definition 2. *[36] Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq (1 - t) f(u) + t f(v)$$

for all $u, v \in K$ and $t \in [0, 1]$. The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every preinvex function is convex with respect to the map $\eta(u, v) = u - v$ but the converse is not true see for instance [36].

Definition 3. *[36] Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be prequasi-invex with respect to η, if*

$$f(u + t\eta(v, u)) \leq \max\{f(u), f(v)\}, \forall u, v \in K, t \in [0, 1].$$

Definition 4. *[21] Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to (0, \infty)$ is said to be logarithmic preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq (f(u))^{1-t} (f(v))^t, \forall u, v \in K, t \in [0, 1].$$

It is clear from the arithmetic-geometric mean inequality that if $f : K \to (0, \infty)$ is logarithmic preinvex function, we have

$$\begin{aligned} f(u + t\eta(v, u)) &\leq (f(u))^{1-t} (f(v))^t \\ &\leq (1 - t) f(u) + t f(v) \\ &\leq \max\{f(u), f(v)\}, \end{aligned}$$

$\forall u, v \in K, t \in [0, 1]$.

Most recently, Noor [20] has obtained the following Hermite-Hadamard inequalities for the preinvex and log-preinvex functions.

Theorem 1. *[20] Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a < a + \eta(b, a)$. Then the following inequality holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a) + f(b)}{2}. \tag{1.2}$$

Theorem 2. *[20] Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be a log-preinvex function. Then*

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a) - f(b)}{\log f(a) - \log f(b)}.$$

The other results connected with (1.2) in which two log-preinvex functions are involved can be found in [24].

For log-preinvex functions, following Hermite-Hadamard type inequalities were also proved in [31].

Theorem 3. *[31] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is log-preinvex on K, for every $a, b \in K$ with $\eta(b, a) > 0$, we have the inequality*

$$\left|\frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a + \frac{1}{2}\eta(b, a)\right)\right| \leq \eta(b, a) \left[\frac{\sqrt{|f'(b)|} - \sqrt{|f'(a)|}}{\log\left(|f'(b)|\right) - \log\left(|f'(a)|\right)}\right]^2. \tag{1.3}$$

Theorem 4. *[31] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|^q$, $q > 1$, $q \in \mathbb{R}$, is a log-preinvex on K, for every $a, b \in K$ with $\eta(b,a) > 0$, we have the inequality*

$$\left| f\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \le \frac{\eta(b,a)\sqrt{\left|f'(a)\right|}}{2^{1/p}(p+1)^{1/p} q^{1/q}} \left[\frac{\left(\left|f'(b)\right|\right)^{q/2} - \left(\left|f'(a)\right|\right)^{q/2}}{\log\left(\left|f'(b)\right|\right) - \log\left(\left|f'(a)\right|\right)} \right]^{1/q}, \quad (1.4)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For more results on Hermite-Hadamard type inequalities for preinvex functions and n-times differentiable preinvex functions, we refer the readers to the recent works of Sarikaya et. al , [31] and Latif [16].

The main purpose of the present paper is to establish new Hermite-Hadamard type inequalities in Section 2 that are connected with the right-side and left-side of Hermite-Hadamard inequality for n-times differentiable log-preinvex functions which generalize those results established for differentiable log-preinvex functions given in [31].

2. Main Results

In order to prove our main results, we need the following two lemmas:

Lemma 1. *[16] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \ge 1$. If $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$, where $a, b \in K$ with $\eta(b,a) > 0$, the following equality holds*

$$-\frac{f(a) + f(a + \eta(b,a))}{2} + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx + \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a + \eta(b,a))$$
$$= \frac{(-1)^{n-1}(\eta(b,a))^n}{2n!} \int_0^1 t^{n-1}(n-2t) f^{(n)}(a + t\eta(b,a))dt, \quad (2.1)$$

where the sum above takes 0 when $n = 1$ and $n = 2$.

Lemma 2. *[16] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \ge 1$. If $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$, where $a, b \in K$ with $\eta(b,a) > 0$, the following equality holds*

$$\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx$$
$$= \frac{(-1)^{n+1}(\eta(b,a))^n}{n!} \int_0^1 K_n(t) f^{(n)}(a + t\eta(b,a))dt, \quad (2.2)$$

where

$$K_n(t) := \begin{cases} t^n, & t \in \left[0, \frac{1}{2}\right] \\ (t-1)^n, & t \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

The following useful results will also help us establishing our results.

Lemma 3. *If* $\mu > 0$ *and* $\mu \neq 1$, *then*

$$\int_0^1 t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}}. \tag{2.3}$$

Proof. For $n = 0$, we have

$$\int_0^1 \mu^t dt = \frac{\mu - 1}{\ln \mu},$$

which coincides with the right hand side of (2.3) for $n = 0$.

For $n = 1$, we have

$$\int_0^1 t\mu^t dt = \frac{\mu}{\ln \mu} - \frac{\mu}{(\ln \mu)^2} + \frac{1}{(\ln \mu)^2},$$

and it coincides with the right hand side of (2.3) for $n = 1$.

Suppose (2.3) is true for $n - 1$, i.e.

$$\int_0^1 t^{n-1} \mu^t dt = \frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)!\mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)! (\ln \mu)^{k+1}}. \tag{2.4}$$

Now by integration by parts and using (2.4), we have

$$\begin{aligned}
\int_0^1 t^n \mu^t dt &= \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \int_0^1 t^{n-1} \mu^t dt \\
&= \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \left[\frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)!\mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)! (\ln \mu)^{k+1}} \right] \\
&= \frac{\mu}{\ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(n-1-k)! (\ln \mu)^{k+2}} \\
&= \frac{n!\mu}{n! \ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=1}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}} \\
&= \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4. *If* $\mu > 0$ *and* $\mu \neq 1$, *then*

$$\int_0^{\frac{1}{2}} t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu^{1/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k} (n-k)! (\ln \mu)^{k+1}}. \tag{2.5}$$

Proof. It follows from Lemma 3 after making use of the substitution $t = \frac{u}{2}$. □

Lemma 5. *If* $\mu > 0$ *and* $\mu \neq 1$, *then*

$$\int_{\frac{1}{2}}^1 (1-t)^n \mu^t dt = \frac{n!\mu}{(\ln \mu)^{n+1}} - n!\mu^{1/2} \sum_{k=0}^{n} \frac{1}{2^{n-k} (n-k)! (\ln \mu)^{k+1}}. \tag{2.6}$$

Proof. It follows from Lemma 4 afer making the substitution $1-t=u$. □

Lemma 6. *[35] For $\alpha>0$ and $\mu>0$, we have*

$$I(\alpha,\mu):=\int_0^1 t^{\alpha-1}\mu^t dt=\mu\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(\alpha)_k}<\infty,$$

where

$$(\alpha)_k=\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1).$$

Moreover, it holds

$$\left|I(\alpha,\mu)-\mu\sum_{k=1}^{m}(-1)^{k-1}\frac{(\ln\mu)^{k-1}}{(\alpha)_k}\right|\le\frac{|\ln\mu|}{\alpha\sqrt{2\pi(m-1)}}\left(\frac{|\ln\mu|e}{m-1}\right)^{m-1}.$$

We are now ready to give our first result.

Theorem 5. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\ge 2$, where $a,b\in K$ with $\eta(b,a)>0$. If $\left|f^{(n)}\right|^q$ is log-preinvex on K for $q\ge 1$, we have the inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|$$
$$\le\frac{(\eta(b,a))^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\left[E_1(n,q)\right]^{1/q},\quad(2.7)$$

where

$$E_1(n,q)=\frac{(-1)^n n!\left\{q\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]+2\right\}\left|f^{(n)}(a)\right|^q}{q^{n+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}}-\frac{2\left|f^{(n)}(b)\right|^q}{q\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]}$$
$$-n!\left|f^{(n)}(b)\right|^q\sum_{k=1}^{n}\frac{(-1)^k\left\{q\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]+2\right\}}{(n-k)!q^{k+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}.$$

Proof. Suppose $n\ge 2$. Since K is an invex set with respect to η, for every $a,b\in K$ and $t\in[0,1]$, we have $a+t\eta(b,a)\in K$. By the log-preinvexity of $\left|f^{(n)}\right|^q$ on K, Lemma 1 and Hölder inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.$$
$$\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\le\frac{(\eta(b,a))^n}{2n!}$$
$$\times\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-1/q}\left(\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{1/q}$$
$$\le\frac{(\eta(b,a))^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\left(\int_0^1 t^{n-1}(n-2t)\left(\left(\left|f^{(n)}(a)\right|\right)^{q(1-t)}\left(\left|f^{(n)}(b)\right|\right)^{qt}\right)dt\right)^{1/q}$$
$$=\frac{(\eta(b,a))^n\left|f^{(n)}(a)\right|}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\left(n\int_0^1 t^{n-1}\mu^t dt-2\int_0^1 t^n\mu^t dt\right)^{1/q},\quad(2.8)$$

where $\mu = \frac{|f^{(n)}(b)|^q}{|f^{(n)}(a)|^q} \neq 1$.
By Lemma 3, we have

$$\begin{aligned} & n\int_0^1 t^{n-1}\mu^t dt - 2\int_0^1 t^n \mu^t dt \\ & = \frac{(-1)^n n!}{(\ln\mu)^n} - n!\mu\sum_{k=1}^{n}\frac{(-1)^k}{(n-k)!(\ln\mu)^k} - \frac{2(-1)^{n+1} n!}{(\ln\mu)^{n+1}} - 2n!\mu\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!(\ln\mu)^{k+1}} \\ & = \frac{(-1)^n n! [\ln\mu + 2] - 2\mu(\ln\mu)^n}{(\ln\mu)^{n+1}} - n!\mu\sum_{k=1}^{n}\frac{(-1)^k[\ln\mu+2]}{(n-k)!(\ln\mu)^{k+1}}. \end{aligned} \tag{2.9}$$

Applying (2.9) in (2.8) and replacing $\mu = \frac{|f^{(n)}(b)|^q}{|f^{(n)}(a)|^q} \neq 1$, we get the desired inequality (2.7). This completes the proof of the theorem □

Corollary 1. *Suppose the assumptions of Theorem 5 are satisfied and if $q = 1$, we have the inequality*

$$\begin{aligned} & \left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a))\right| \\ & \leq \frac{(\eta(b,a))^n}{2n!} E_1(n,1), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} E_1(n,1) = & \frac{(-1)^n n! \left\{\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right] + 2\right\}\left|f^{(n)}(a)\right|}{\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}} - \frac{2\left|f^{(n)}(b)\right|}{\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]} \\ & - n!\left|f^{(n)}(b)\right|\sum_{k=1}^{n}\frac{(-1)^k\left\{\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right] + 2\right\}}{(n-k)!\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}. \end{aligned}$$

Corollary 2. *Under the assumptions of Theorem 5, if $n = 2$, we have the inequality*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^2}{4}\left(\frac{1}{3}\right)^{1-1/q}[E_1(2,q)]^{1/q}, \tag{2.11}$$

where

$$E_1(2,q) = \frac{2\left\{q\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] + 2\right\}\left|f''(a)\right|^q}{q^3\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3} + \frac{2\left\{q\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] - 2\right\}\left|f''(b)\right|^q}{q^3\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3}.$$

Corollary 3. *Under the assumptions of Theorem 5, if $n = 2$ and $q = 1$, we have the inequality*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^2}{4}[E_1(2,1)], \tag{2.12}$$

where

$$E_1(2,1) = \frac{2\left\{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] + 2\right\}\left|f''(a)\right|}{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3} + \frac{2\left\{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] - 2\right\}\left|f''(b)\right|}{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3}.$$

Remark 1. *If $\eta(b,a) = b - a$ in the inequalities (2.11) and (2.12), one can get inequalities for the bounds of the difference between middle and the right most terms in the Hermite-Hadamard inequalities (1.1) in terms of second order derivatives for log-convex functions.*

Theorem 6. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 2$, where $a, b \in K$ with $\eta(b,a) > 0$. If $\left|f^{(n)}\right|^q$ is log-preinvex on K for $q > 1$, we have the inequality*

$$\left|\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=2}^{n-1}\frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a + \eta(b,a))\right|$$
$$\leq \frac{(\eta(b,a))^n\left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}\left|f^{(n)}(b)\right|}{2^{2-1/q}n!}$$
$$\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\sum_{k=1}^{\infty}(-q)^{k-1}\frac{\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k-1}}{(q(n-1)+1)_k}\right)^{1/q}. \tag{2.13}$$

Proof. By the log-preinvexity of $\left|f^{(n)}\right|^q$ on K, Lemma 1 and Hölder inequality, we have

$$\left|\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=2}^{n-1}\frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a + \eta(b,a))\right|$$
$$\leq \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 (n-2t)^{q/(q-1)}\,dt\right)^{1-1/q}\left(\int_0^1 t^{q(n-1)}\left|f^{(n)}(a + t\eta(b,a))\right|^q dt\right)^{1/q}$$
$$\leq \frac{(\eta(b,a))^n\left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\int_0^1 t^{q(n-1)}\left(\left(\left|f^{(n)}(a)\right|\right)^{q(1-t)}\left(\left|f^{(n)}(b)\right|\right)^{qt}\right)dt\right)^{1/q}$$
$$= \frac{(\eta(b,a))^n\left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}\left|f^{(n)}(a)\right|}{2^{2-1/q}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\int_0^1 t^{q(n-1)}\mu^t dt\right)^{1/q}, \tag{2.14}$$

where $\mu = \frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q} \neq 1$. Applying Lemma 6 to the last integral in the inequality (2.14) and simplifying, we get the required inequality (2.13). □

Corollary 4. *Suppose the assumptions of Theorem 6 are satisfied and $n = 2$. Then*

$$\left|\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^2\left|f''(b)\right|}{2}$$
$$\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\sum_{k=1}^{\infty}\frac{(-q)^{k-1}\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^{k-1}}{(q+1)_k}\right)^{1/q}. \tag{2.15}$$

***Corollary** 5.* *If $\eta(b,a) = b - a$ in Corollary 4, we have*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2\left|f''(b)\right|}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left(\sum_{k=1}^{\infty}\frac{(-q)^{k-1}\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^{k-1}}{(q+1)_k}\right)^{1/q}. \tag{2.16}$$

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable log-preinvex functions.

Theorem 7. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 1$, where $a, b \in K$ with $\eta(b,a) > 0$. If $\left|f^{(n)}\right|^q$ is log-preinvex on K for $q \geq 1$, we have the following inequality*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n \left|f^{(n)}(a)\right|}{2^{(n+1)(q-1)/q}(n+1)^{1-1/q}(n!)^{1-1/q}} \left\{\left[E_2(n,q)\right]^{1/q} + \left[E_3(n,q)\right]^{1/q}\right\}, \quad (2.17)$$

where

$$E_2(n,q) = \frac{(-1)^{n+1}}{q^{n+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}} + \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2} \sum_{k=0}^{n} \frac{(-1)^k}{q^{k+1}2^{n-k}(n-k)!\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}$$

and

$$E_3(n,q) = \frac{\left|f^{(n)}(b)\right|^q}{q^{n+1}\left[\ln\left[\left|f^{(n)}(b)\right|\right] - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}\left|f^{(n)}(a)\right|^q} - \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2} \sum_{k=0}^{n} \frac{1}{q^{k+1}2^{n-k}(n-k)!\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}.$$

Proof. Suppose $n \geq 1$. By using Lemma 2 and the log-preinvexity of $\left|f^{(n)}\right|^q$ on K for $n \in \mathbb{N}$, $n \geq 1$, we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n}{n!}\left[\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(a + t\eta(b,a))\right| dt + \int_{\frac{1}{2}}^1 (1-t)^n \left|f^{(n)}(a + t\eta(b,a))\right| dt\right]$$
$$\leq \frac{(\eta(b,a))^n \left|f^{(n)}(a)\right|}{n!}\left[\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-1/q}\left(\int_0^{\frac{1}{2}} t^n \mu^t dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-1/q}\left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^t dt\right)^{1/q}\right], \quad (2.18)$$

where $\mu = \frac{\left|f^{(n)}(b)\right|^{q\,q}}{\left|f^{(n)}(a)\right|} \neq 1$. Applying Lemma 4 and Lemma 5 to the integrals in the inequality (2.18) and replacing $\mu = \frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q} \neq 1$, we get the desired inequality (2.17). This completes the proof of the theorem. □

Corollary 6. *Suppose the assumptions of Theorem 7 are fulfilled and if $q = 1$, we have*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq (\eta(b,a))^n \left|f^{(n)}(a)\right| \left\{\left[E_2(n,1)\right] + \left[E_3(n,1)\right]\right\}, \quad (2.19)$$

where

$$E_2(n,1)=\frac{(-1)^{n+1}}{\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}}+\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{1/2}\sum_{k=0}^{n}\frac{(-1)^k}{2^{n-k}(n-k)!\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}$$

and

$$E_3(n,1)=\frac{\left|f^{(n)}(b)\right|}{\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}\left|f^{(n)}(a)\right|}-\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{1/2}\sum_{k=0}^{n}\frac{1}{2^{n-k}(n-k)!\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}.$$

Corollary 7. *[31] If we take $n=1$ in Corollary 6, we have*

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\le\eta(b,a)\left[\frac{\sqrt{\left|f'(b)\right|}-\sqrt{\left|f'(a)\right|}}{\ln\left(\left|f'(b)\right|\right)-\ln\left(\left|f'(a)\right|\right)}\right]^2. \tag{2.20}$$

Corollary 8. *[31] If $\eta(b,a)=b-a$ in Corollary 7, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^{b}f(x)\,dx\right|\le(b-a)\left[\frac{\sqrt{\left|f'(b)\right|}-\sqrt{\left|f'(a)\right|}}{\ln\left(\left|f'(b)\right|\right)-\ln\left(\left|f'(a)\right|\right)}\right]^2. \tag{2.21}$$

Theorem 8. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\ge1$, where $a,b\in K$ with $\eta(b,a)>0$. If $\left|f^{(n)}\right|^q$ is log-preinvex on K for $q>1$, we have the inequality*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\le\frac{(\eta(b,a))^n\left[\sqrt{\left|f^{(n)}(a)\right|}+\sqrt{\left|f^{(n)}(b)\right|}\right]}{2^{n+1/p}(np+1)^{1/p}q^{1/q}n!}\left[\frac{\left(\left|f^{(n)}(b)\right|\right)^{q/2}-\left(\left|f^{(n)}(a)\right|\right)^{q/2}}{\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)}\right]^{1/q}, \tag{2.22}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, the Hölder integral inequality and the log-preinvexity of $\left|f^{(n)}\right|^q$ on K, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\le\frac{(\eta(b,a))^n\left|f^{(n)}(a)\right|}{n!}\left[\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{qt}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{qt}dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{(\eta(b,a))^n\left[\sqrt{\left|f^{(n)}(a)\right|}+\sqrt{\left|f^{(n)}(b)\right|}\right]}{2^{n+1/p}(np+1)^{1/p}q^{1/q}n!}\left[\frac{\left(\left|f^{(n)}(b)\right|\right)^{q/2}-\left(\left|f^{(n)}(a)\right|\right)^{q/2}}{\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)}\right]^{1/q}. \tag{2.23}$$

Which is the required inequality. This completes the proof of the theorem. □

Corollary 9. *Under the assumptions of Theorem 8, if $n = 1$, we have the inequality*

$$\left| f\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \leq \frac{\eta(b,a)\left[\sqrt{|f'(a)|} + \sqrt{|f'(b)|}\right]}{2^{1+1/p}(p+1)^{1/p} q^{1/q}} \left[\frac{\left(|f'(b)|\right)^{q/2} - \left(|f'(a)|\right)^{q/2}}{\ln\left(|f'(b)|\right) - \ln\left(|f'(a)|\right)}\right]^{1/q}, \quad (2.24)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 10. *If we take $\eta(b,a) = b - a$ in (2.24), we get the inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{(b-a)\left[\sqrt{|f'(a)|} + \sqrt{|f'(b)|}\right]}{2^{1+1/p}(p+1)^{1/p} q^{1/q}} \left[\frac{\left(|f'(b)|\right)^{q/2} - \left(|f'(a)|\right)^{q/2}}{\log\left(|f'(b)|\right) - \log\left(|f'(a)|\right)}\right]^{1/q}. \quad (2.25)$$

Remark 2. *Inequalities (2.24) and (2.25) are the corrected inequalities that are given in Theorem 4 and its related corollary from [31].*

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