Ward Continuity in 2-Normed Spaces

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Abstract. In this paper, we introduce and investigate the concept of ward continuity in 2-normed spaces. A function f defined on a 2-normed space $(X, \|., .\|)$ is ward continuous if it preserves quasi-Cauchy sequences, where a sequence (x_n) of points in X is called quasi-Cauchy if $\lim_{n \to \infty} \|\Delta x_n, z\| = 0$ for every $z \in X$. Some other kinds of continuities are also introduced, and interesting theorems are proved in 2-normed spaces.

1. Introduction

The concept of a 2-norm is needed if there is a physical situation or an abstract concept where norm topology does not work but 2-norm topology does work, especially in the cases when one needs two-inputs for a particular output but with one main input and other input is required to complete the process. So that one may expect the concept of continuity and any concept involving continuity in 2-normed spaces find very important roles not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, and biological science.

About 90 years ago, Menger ([20]) introduced a notion called a generalized metric. But many mathematicians had not paid attentions to Menger theory about generalized metrics. In 1937 Vulich ([27]) introduced a notion of higher dimensional norm in linear spaces. Unfortunately, this study had been neglected by many analysts for a long time. Then a new development began with 1962 by Gähler ([12], [13], and [14]). The notion of a 2-norm has been extensively studied in ([2, 15–17]) in which a lot for the extension of this branch of mathematics is contributed. Recently many mathematicians came out with results in 2-normed spaces, analogous with that in classical normed spaces and Banach spaces (see for example [21], [23], and [8]).

The concepts of ward continuity of a real function and ward compactness of a subset of the set of real numbers are introduced by Cakalli in [4] (see also [1], and [7]). A real function f is called ward continuous on E if the sequence $(f(x_n))$ is quasi-Cauchy whenever $x = (x_n)$ is a quasi-Cauchy sequence of points in E.

The aim of this paper is to investigate ward continuity in 2-normed spaces, and prove interesting theorems.

2. Preliminaries

Now we give some notation and definitions which will be needed in the paper. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ will denote the set of positive integers and the set of real numbers, respectively. First we recall the definition of a 2-normed space.

Definition 2.1. ([12]) Let X be a real linear space with $\dim X > 1$ and $\|.,.\| : X^2 \to \mathbb{R}$ a function. Then $(X, \|.,.\|)$ is called a *linear 2-normed space* if

1. $\|x, y\| = 0 \Leftrightarrow$ x and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|\alpha x, y\| = |\alpha| \|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for $\alpha \in \mathbb{R}$ and $x, y, z \in X$. The function $\|., .\|$ is called a —emph2-norm on X.

Observe that in any 2-normed space $(X, \|.,.\|)$ we have $\|.,.\|$ is nonnegative, $\|x - z, x - y\| = \|x - z, y - z\|$, and $\forall x, y \in X, \alpha \in \mathbb{R}$ $\|x, y + \alpha x\| = \|x, y\|$. Throughout this paper by X we will mean a 2-normed space with a 2-norm $\|., .\|$.

A classical example is the 2-normed space $X = \mathbb{R}^2$ with the 2-norm $\|., .\|$ defined by $\|a, b\| = |a_1 b_2 - a_2 b_1|$ where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$. This is the area of the parallelogram determined by the vectors a and b.

A sequence (x_n) of points in X is said to be *convergent* to an element $x \in X$ if $\lim_{n\to\infty} \|x_n - x, z\| = 0$ for every $z \in X$. This is denoted by $\lim_{n\to\infty} \|x_n, z\| = \|x, z\|$. A sequence (x_n) of points in X is called *Cauchy* if $\lim_{n,m\to\infty} \|x_n - x_m, z\| = 0$ for every $z \in X$ (see [12]).

A sequence of functions (f_n) is said to be *uniformly convergent* to a function f on a subset E of X if for each $\epsilon > 0$, an integer N can be found such that $\|f_n(x) - f(x), z\| < \epsilon$ for $n \geq N$ and for all $x, z \in X$.

3. Ward Continuity in 2-Normed Spaces

In this section we investigate the notions of ward continuity and ward compactness on a 2-normed space. Now we give some definitions that are used to describe the subject.

Definition 3.1. A sequence (x_n) of points in a 2-normed $(X, \|.,.\|)$ is called *quasi-Cauchy* if $\lim_{n\to\infty} \|\Delta x_n, z\| = 0$ for every $z \in X$ where $\Delta x_n = x_{n+1} - x_n$ for every $n \in \mathbb{N}$.

We note that any convergent sequence is quasi-Cauchy, and any Cauchy sequence is quasi-Cauchy. Any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for quasi-Cauchy sequences. A counter example is provided in the following.

Example 3.2. The subsequence $(a_{n^2}) = (n, n)$ of the quasi-Cauchy sequence $(a_n) = (\sqrt{n}, \sqrt{n})$ of points in the 2-normed space $\mathbb{R}^2$ is not quasi-Cauchy.

Definition 3.3. A subset E of X is called *ward compact* if any sequence of points in E has a quasi-Cauchy subsequence.

First, we note that any finite subset of X is ward compact, the union of two ward compact subsets of X is ward compact and the intersection of any family of ward compact subsets of X is ward compact. Furthermore any subset of a ward compact set is ward compact.

For any given 2-norm we can define norms by using 2-norm values. Consider the norm $\|.\|$ defined on a linear 2-normed space $(X, \|.,.\|)$ by the function

$$\|x\| = \|x, y\| + \|x, z\|$$

for any fixed $y, z \in X$ and $\|y, z\| \neq 0$. The function $\|.\|$ defined on X is a norm on X ([11]). On the other hand, $\|x\| = \max_{y,z}\{\|x, y\|, \|x, z\|\}$ is also a norm on X. We see that these norms can be defined for any linearly

independent pair of y, z of points in X. It is obvious that any convergent sequence on a 2-normed space $(X, \|., .\|)$ is also convergent on the normed space $(X, \|.\|)$, and any quasi-Cauchy sequence in a 2-normed space $(X, \|., .\|)$ is quasi-Cauchy on the normed space $(X, \|.\|)$. If a subset E of a 2-normed space $(X, \|., .\|)$ is ward compact, then it is also ward compact in the normed space $(X, \|.\|)$. Moreever any ward compact subset of a 2-normed space $(X, \|., .\|)$ is totally bounded in the normed space $(X, \|.\|)$ ([5, Theorem 3]).

Let $(X, \|., .\|)$ be a linear 2-normed space. For $x, z \in X$, let $p_z(x) = \|x, z\|$. Then, for each $z \in X$, p_z is a real-valued function on X such that $p_z(x) = \|x, z\| \geq 0$, $p_z(\alpha x) = |\alpha|\|x, z\| = |\alpha|p_z(x)$ and $p_z(x + y) = \|x + y, z\| = \|z, x + y\| \leq \|z, x\| + \|z, y\| = \|x, z\| + \|y, z\| = p_z(x) + p_z(y)$ for all $\alpha \in \mathbb{R}$ and all $x, y \in X$. Thus p_z is a semi-norm for each $z \in X$. For $x \in X$, if $\|x, z\| = 0$ for all $z \in X$ then $x = 0$ ([22, Lemma 1.2]). Thus $0 \neq x \in X$ implies that there is some $z \in X$ satisfying $p_z(x) = \|x, z\| \neq 0$. With this additional condition the family $\{p_z : z \in X\}$ is to be a separating family of semi-norms.

For $\epsilon > 0$ and $z \in X$, let $U_{z,\epsilon} = \{x \in X : p_z(x) < \epsilon\} = \{x \in X : \|x, z\| < \epsilon\}$. Let $\mathcal{S}_0 = \{U_{z,\epsilon} : \epsilon > 0, z \in X\}$ and $\mathcal{B}_0 = \{\bigcap \mathcal{F} : \mathcal{F}$ is a finite sub-collection of $S_0\}$. Define a topology τ on X by saying that a set U is open if and only if for every $x \in U$ there is some $N \in \mathcal{B}_0$ such that $x + N = \{x + y : y \in N\} \subset U$. That is, τ is the topology on X that has a subbase as the sets

$$\{x \in X : p_z(x - x_0) < \epsilon\}, \ z \in X, \ x_0 \in X, \ \epsilon > 0.$$

The topology τ gives X the structure of topological vector space. Since the collection $\mathcal{B}_0$ is a local base whose members are convex, X is locally convex ([22]).

Now we introduce a definition of ward continuity in a 2-normed space.

Definition 3.4. A function $f : X \to X$ is called *ward continuous* if it preserves quasi-Cauchy sequences, i.e. $\lim_{n\to\infty} \|\Delta f(x_n), w\| = 0$ for every $w \in X$ whenever $\lim_{n\to\infty} \|\Delta x_n, z\| = 0$ for every $z \in X$.

In connection with quasi-Cauchy sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on X.

1. (x_n) is quasi-Cauchy $\Rightarrow$ $(f(x_n))$ is quasi-Cauchy.
2. (x_n) is quasi-Cauchy $\Rightarrow$ $(f(x_n))$ is convergent.
3. (x_n) is convergent $\Rightarrow$ $(f(x_n))$ is convergent.
4. (x_n) is convergent $\Rightarrow$ $(f(x_n))$ is quasi-Cauchy.

It is obvious that (2) $\Rightarrow$ (1) while (1) does not imply (2), (1) $\Rightarrow$ (4) while (4) does not imply (1), (2) $\Rightarrow$ (3) while (3) does not imply (2) and lastly (3) is equivalent to (4). We see that (1) is ward continuity of f and (3) is ordinary continuity of f. We give the definition of sequential continuity in 2-normed spaces in the following before proving that (1) implies (3).

Definition 3.5. A function f on a subset E of a 2-normed space $(X, \|., .\|)$ is said to be *sequentially continuous at x_0* if for any sequence (x_n) of points in E converging to x_0, we have $(f(x_n))$ converges to $f(x_0)$ (see [19] for the definition for the special case when f is linear).

Theorem 3.6. *If $f : X \to X$ is ward continuous on a subset E of X, then it is sequentially continuous on E.*

Proof. Let (x_n) be any convergent sequence of points in E with

$$\lim_{n\to\infty} \|x_n - x_0, y\| = 0$$

for all $y \in X$. Then the sequence $\xi = (\xi_n)$ defined by

$$\xi_n = \begin{cases} x_k, & \text{if } n = 2k - 1 \text{ for a positive integer } k \\ x_0, & \text{if } n \text{ is even} \end{cases}$$

is also convergent to x_0. Therefore it is a quasi-Cauchy sequence. As f is ward continuous on E, the sequence $f(\xi) = (f(\xi_n))$ obtained by

$$f(\xi_n) = \begin{cases} f(x_k), & \text{if } n = 2k - 1 \text{ for a positive integer } k \\ f(x_0), & \text{if } n \text{ is even} \end{cases}$$

is also quasi-Cauchy. Now it follows that $\lim_{n\to\infty} \|f(x_n) - f(x_0), z\| = 0$ for every $z \in X$. It implies that the sequence $(f(x_n))$ converges to $f(x_0)$. This completes the proof of the theorem. $\square$

The converse of this theorem is not always true. We give the following example.

Example 3.7. Let $f(x, y) = (x^2, y^2)$ be the function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R} \times \mathbb{R}$. We define a 2-norm by $\|(a_1, a_2), (b_1, b_2)\| = |a_1 b_2 - a_2 b_1|$. $\forall z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}$

$$\lim_{n\to\infty} \|\Delta x_n, z\| = \lim_{n\to\infty} \|x_{n+1} - x_n, z\|$$

$$= \lim_{n\to\infty} \|(\sqrt{n+1}, \sqrt{n+1}) - (\sqrt{n}, \sqrt{n}), (z_1, z_2)\|$$

$$= \lim_{n\to\infty} \|(\sqrt{n+1} - \sqrt{n}, \sqrt{n+1} - \sqrt{n}), (z_1, z_2)\|$$

$$= 0.$$

Thus the sequence $x_n = (\sqrt{n}, \sqrt{n})$ is quasi-Cauchy. On the other hand $(f(x_n)) = (f(\sqrt{n}, \sqrt{n})) = (n, n)$ is not quasi-Cauchy. Since we have for $w = (1, 2)$

$$\lim_{n\to\infty} \|\Delta f(x_n), w\| = \lim_{n\to\infty} \|f(x_{n+1}) - f(x_n), w\|$$

$$= \lim_{n\to\infty} \|(n+1, n+1) - (n, n), w\|$$

$$= \lim_{n\to\infty} \|(1, 1), (1, 2)\| = 1 \neq 0.$$

Observing that any normed space (single normed) is a first countable topological Hausdorff group, we have the following result related to G-sequential continuity (see [3] for the definition of G-sequential continuity).

Corollary 3.8. *Let X be a finite dimensional 2-normed space with dimension m, and $\{e_1, e_2, ..., e_m\}$ be a base of X. Consider the norm defined by $\|x\|_\infty = \max_{i=1,2,...,m} \|x, e_i\|$. If $f : X \to X$ is ward continuous on a subset E of the 2-normed space X, then it is G-sequentially continuous on E in the normed space X for every regular subsequential method G.*

Corollary 3.9. *Let X be a finite dimensional 2-normed space with dimension m, and $\{e_1, e_2, ..., e_m\}$ be a base of X. Consider the norm defined by $\|x\|_\infty = \max_{i=1,2,...,m} \|x, e_i\|$. If $f : X \to X$ is ward continuous on a subset E of the 2-normed space X, then it is sequentially continuous on E in the normed space $(X, \|.\|_\infty)$.*

The following theorem guarantees that the ward continuous image of a ward compact subset of X is ward compact.

Theorem 3.10. *Let E be a ward compact subset of X. If $f : X \to X$ is ward continuous on E, then $f(E)$ is ward compact.*

Proof. Ward compactness of E implies that there is a subsequence $z = (z_k)$ of $\mathbf{x} = (x_n)$ with $\lim_{k\to\infty} \|\Delta z_k, y\| = 0$, $\forall y \in X$. Let $(t_k) = (f(z_k))$. (t_k) is the subsequence of the sequence $f(x)$ with $\lim_{k\to\infty} \|\Delta t_k, f(y)\| = 0$. This completes the proof of the theorem. $\square$

Definition 3.11. A function $f : X \to X$ is called *uniformly continuous on a subset E* of X if for any $\varepsilon > 0$ and for any $w \in X$ there exist positive real numbers $\delta_1, \delta_2, ..., \delta_p$ and $z_1, z_2, ..., z_p \in X$ such that $\|f(x) - f(y), w\| < \epsilon$ whenever $\|x - y, z_k\| < \delta_k$, $k = 1, 2, ..., p$.

We note that a function $f : X \to X$ is uniformly continuous on a subset E of X if for any finite choose of $\epsilon_1, \epsilon_2, ..., \epsilon_m$ and $w_1, w_2, ..., w_m$ there exist $\delta_1, \delta_2, ..., \delta_p$ and $z_1, z_2, ..., z_p$ such that $\|f(x) - f(y), w_j\| < \epsilon_j$ for $j = 1, 2, ..., m$ whenever $\|x - y, z_k\| < \delta_k$, $k = 1, 2, ..., p$. On the other hand, taking one ϵ and one δ in the consideration we give the following definition.

Definition 3.12. A function $f : X \to X$ is called *u-continuous on a subset E of X* if for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|f(x) - f(y), w\| < \epsilon$ for any $w \in X$ whenever $\|x - y, z\| < \delta$ for any $x, y \in E$, and $z \in X$.

Theorem 3.13. *If a function $f : X \to X$ is u-continuous on a subset E of X, then it is ward continuous on E.*

Proof. Let f be u-continuous on E, and (x_n) be any quasi-Cauchy sequence of points in E. Then for given any $\epsilon > 0$ there exists a $\delta > 0$ such that $\|f(x) - f(y), w\| < \epsilon$ for any $w \in X$ whenever $\|x - y, z\| < \delta$ for any $x, y, z \in X$. For the choice of $\delta > 0$, there exists an $N = N(\delta) = N_1(\epsilon) \in N$ such that $\|\Delta x_n, z\| < \delta$ for every $n > N$. So $\|\Delta f(x_n), f(z)\| < \epsilon$ for every $n > N$. This completes the proof of the theorem. □

Corollary 3.14. *Any u-continuous function is sequentially continuous.*

Proof. Proof follows easily from Theorem 3.6, so is omitted. □

Corollary 3.15. *u-continuous image of any ward compact subset of X is ward compact.*

Any u-continuous function is uniformly-continuous. But the converse is not always true. It is not difficult to construct an example.

Theorem 3.16. *If a function $f : X \to X$ is uniformly continuous on a subset E of X, then it is ward continuous on E.*

Proof. Although the proof can be obtained by Theorem 3.13 we give a proof for completeness. Let f be uniformly continuous on E, and (x_n) be any quasi-Cauchy sequence of points in E. Choose any $\epsilon > 0$ and any $w \in X$. As f is uniformly continuous on E for this ϵ and w there exist $\delta_1, \delta_2, ..., \delta_p$ and $z_1, z_2, ..., z_p$ such that $\|f(x) - f(y), w\| < \epsilon$ whenever $\|x - y, z_k\| < \delta_k, k = 1, 2, ..., p$. Since (x_n) is a quasi-Cauchy sequence, for δ_1 there exists a positive integer n_1 such that $\|x_{n+1} - x_n, z_1\| < \delta_1$. Similarly for δ_2 there exists a positive integer n_2 such that $\|x_{n+1} - x_n, z_2\| < \delta_2$. Having found positive integers $n_3, n_4, ..$, and so on we finally can find a positive integer n_p such that $\|x_{n+1} - x_n, z_p\| < \delta_p$. Now write $n_0 = \max\{n_1, n_2, n_3, ..., n_p\}$. Then $\|f(x_{n+1}) - f(x_n), w\| < \epsilon$ for $n \geq n_0$. Hence $(f(x_n))$ is a quasi-Cauchy sequence, therefore f preserves quasi-Cauchy sequences. This completes the proof of the theorem. □

It is a well known result that the uniform limit of a sequence of continuous functions is continuous. Some related results in 2-normed spaces are obtained in [25]. For the case of u-continuity, we have the following.

Theorem 3.17. *If (f_n) is a sequence of u-continuous functions defined on a subset E of X, and (f_n) is uniformly convergent to a function f, then f is u-continuous on E.*

Proof. Since (f_n) is uniformly convergent to f, for given any $\epsilon > 0$ there exists a positive integer N such that $\|f_n(x) - f(x), w\| < \frac{\epsilon}{3}$ whenever $n \geq N$ for all $x, w \in E$. Since f_N is u-continuous on E, for $\frac{\epsilon}{3}$, there exists a positive real number δ such that $\|f_N(x) - f_N(y), w\| < \frac{\epsilon}{3}$ for any $w \in X$ whenever $\|x - y, z\| < \delta$ for any $x, y, z \in X$. Then whenever $\|x - y, z\| < \delta$, we have

$$\|f(x) - f(y), w\| \leq \|f(x) - f_N(x), w\| + \|f_N(x) - f_N(y), w\| + \|f_N(y) - f(y), w\|$$
$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for every $w \in X$. So f is u-continuous on E, and the proof is completed. □

In the real case it was proved that uniform limit of a sequence of ward continuous functions is ward continuous ([4]). It is also true in 2-normed spaces.

Theorem 3.18. *If* (f_n) *is a sequence of ward continuous functions defined on a subset E of X, and* (f_n) *is uniformly convergent to a function* f*, then* f *is ward continuous on E.*

Proof. To prove that f is ward continuous on E, take any quasi-Cauchy sequence (x_n) of points in E. Let ε be any positive real number. Since (f_n) is uniformly convergent to f, for $\frac{\epsilon}{3} > 0$ there exists a positive integer N such that $\|f_n(x) - f(x), z\| < \frac{\epsilon}{3}$ whenever $n \geq N$ for all $x, z \in E$. Since f_N is ward continuous on E, there exists a positive integer $N_1 \geq N$ such that $\|f_N(x_{n+1}) - f_N(x_n), z\| < \frac{\epsilon}{3}$ for $n \geq N_1$. Then we have for $n \geq N_1$

$$\|f(x_{n+1}) - f(x_n), z\|$$
$$\leq \|f(x_{n+1}) - f_N(x_{n+1}), z\| + \|f_N(x_{n+1}) - f_N(x_n), z\| + \|f_N(x_n) - f(x_n), z\|$$
$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for every $z \in X$. So f is ward continuous on E, and the proof is completed. $\square$

4. Conclusion

The concept of a 2-normed space was extensively studied by Gähler who asked what the real motivation for studying 2-norm structure is, and if there is a physical situation or an abstract concept where norm topology does not work but 2-norm topology does work. We observe that if a term in the definition of 2-norm represents the change of a shape, and the 2-norm stands for the associated area, we can think of some plausible application of the notion of a 2-norm, and then the generalized convergence make sense. This can also be viewed as: suppose for a particular output we need two-inputs but with one main input and other input is required to complete the process. So that one may expect to be a more useful tool in the field of a 2-normed space in modelling various problems, occuring in many areas of science, especially in computer science and information theory. Such possible applications attract researchers to be involved in investigation on 2-normed spaces. We note that the present work contains not only an investigation of quasi-Cauchy sequences as it has been presented in a very different setting, i.e. in a 2-normed space which is quite different from the real case, and metric case, but also an investigation of some other kinds of sequential continuities.

We note that the study in this paper can be carried to n-normed spaces without much effort (see for example [24] for the definition of an n-normed space). For a further study, we suggest to investigate quasi-Cauchy sequences of points and fuzzy functions in a 2-normed fuzzy space. However due to the change in the setting, the definitions and methods of proofs will not always be analogous to those of the present work (for example see [6], [26], and [18]). For another further study we suggest to investigate quasi-Cauchy sequences of double sequences in 2-normed spaces (see for example [8], [9], [21], and [10] for the definitions and related concepts in the double case).

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