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Polynomials of Unitary Cayley Graphs

Milan Bašić[a], Aleksandar Ilić[a]

[a]*Faculty of Sciences and Mathematics, University of Niš, Serbia*

Abstract. The unitary Cayley graph X_n has the vertex set $Z_n = \{0, 1, 2, \ldots, n-1\}$ and vertices a and b are adjacent, if and only if $\gcd(a-b, n) = 1$. In this paper, we present some properties of the clique, independence and distance polynomials of the unitary Cayley graphs and generalize some of the results from [W. Klotz, T. Sander, *Some properties of unitary Cayley graphs*, Electr. J. Comb. 14 (2007), #R45]. In addition, using some properties of Laplacian polynomial we determine the number of minimal spanning trees of any unitary Cayley graph.

1. Introduction

Let $G = (V, E)$ be a connected simple graph with $n = |V(G)| = |V|$ vertices and $m = |E(G)| = |E|$ edges. For vertices $u, v \in V$, the distance $d(u, v)$ is defined as the length of the shortest path between u and v in G. The maximum distance in the graph G is its diameter, denoted by d.

The unitary Cayley graph X_n has the vertex set $Z_n = \{0, 1, 2, \ldots, n-1\}$ and vertices a and b are adjacent, if and only if $\gcd(a-b, n) = 1$. Integral circulant graphs are a generalization of unitary Cayley graphs, recently studied by Klotz and Sander in [11]. The integral circulant graph $X_n(D)$ has the vertex set $Z_n = \{0, 1, 2, \ldots, n-1\}$ and vertices a and b are adjacent, if and only if $\gcd(a-b, n) \in D$, where $D = \{d_1, d_2, \ldots, d_k\}$ is a set of divisors of n. These graphs play an important role in modeling quantum spin networks supporting the perfect state transfer [1, 2] and also have applications in chemical graph theory [8].

In this paper we compute the clique polynomial of X_n in Section 2 and present some properties of independence polynomial in Section 3- thus generalizing results from [11] on the clique and independence number of the unitary Cayley graphs. In Section 4, using some properties of Laplacian polynomial we determine the number minimal spanning tress of any unitary Cayley graph. In section 5 we determine the distance polynomial of X_n.

2. The Clique Polynomial

Recall that the clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G.

Let p be the smallest prime divisor of n. In [11], the authors proved that $\omega(X_n) = p$ and here we will calculate the number of cliques of size l, where $l \geq 1$. This will further generalize the results from [11] on the number of triangles in the graph X_n.

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Email addresses: basic_milan@yahoo.com (Milan Bašić), aleksandari@gmail.com (Aleksandar Ilić)

The clique polynomial $C(G;x)$ for the graph G is defined as the polynomial

$$C(G;x) = \sum_{l=1}^{\omega(G)} c_l x^l,$$

where the coefficient of x^l for $l > 0$ is the number of cliques c_l in a graph with l vertices, and the constant term is 1.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ and $m = p_1 p_2 \cdot \ldots \cdot p_k$. Let $S = \{x_1, x_2, \ldots, x_l\}$ be a clique of size l. Then the residues of the numbers from S must be all different for modules $p_1, p_2, \ldots, p_k$. In other words, for any $1 \leq i \leq k$, $1 \leq s, t \leq l$ and $s \neq t$ holds that $x_s \not\equiv_{p_i} x_t$. This further means that for each fixed prime number p_i there exists $l-$tuple of mutually different number $(r_{1i}, r_{2i}, \ldots, r_{li})$, $0 \leq r_{ji} < p_i$, such that

$$x_1 \equiv_{p_i} r_{1i}, x_2 \equiv_{p_i} r_{2i}, \ldots, x_l \equiv_{p_i} r_{li}.$$

So the total number of the $l-$tuples is equal to the number of $l-$element variations of p_i elements with repetition not allowed, which is $\binom{p_i}{l} l!$.

Using the Chinese Remainder Theorem we can determine uniquely any x_s modulo m, for $1 \leq s \leq l$. So, each element from S can be replaced with arbitrary number with the same residue modulo m and there are exactly $\frac{n}{m}$ such possibilities. Thus, for any fixed choice of k $l-$ tuples of residues we have $(\frac{n}{m})^l$ $l-$tuples $(x_1, \ldots, x_l)$ and the total number of $l-$tuples is equal to

$$\left(\frac{n}{m}\right)^l \cdot \prod_{i=1}^{k} \binom{p_i}{l} l!.$$

Since S is a set (not $l-$tuple), for counting we should divide the overall number of combinations by $l!$. Therefore, the total number of cliques of size l equals

$$\begin{aligned} c_l &= \left(\frac{n}{m}\right)^l \cdot \frac{1}{l!} \cdot \prod_{i=1}^{k} \binom{p_i}{l} l! = \frac{\left(p_1^{\alpha_1-1} \cdot p_2^{\alpha_2-1} \cdot \ldots \cdot p_k^{\alpha_k-1}\right)^l}{l!} \prod_{p|n} \frac{p!}{(p-l)!} \\ &= \frac{n^l}{l!} \prod_{p|n} \frac{p!}{(p-l)!p^l} \end{aligned}$$

For $l = 1$, we have the number of vertices $c_1 = n$.

For $l = 2$, we have the number of edges

$$c_2 = \frac{n^2}{2p_1 p_2 \cdot \ldots \cdot p_k} \cdot p_1(p_1 - 1) p_2(p_2 - 1) \cdot \ldots \cdot p_k(p_k - 1) = \frac{n\varphi(n)}{2}.$$

For $l = 3$, we have the number of triangles

$$c_3 = \frac{n^3}{6} \prod_{p|n} (1 - \frac{1}{p})(1 - \frac{2}{p}). \tag{1}$$

Finally for $l > p_1$, we have $c_l = 0$ as $\binom{n}{s} = 0$ for $s > n$. Therefore, it follows that $\omega(X_n) = p_1$.

The authors in [3] completely solved the problem of finding the clique number for integral circulant graphs with exactly one and two divisors–and these results can be used to characterize the clique polynomial of $X_n(d_1, d_2)$.

3. Independence Polynomial

Recall that an independent set or stable set is a set of vertices in a graph, no two of which are adjacent and independence number is maximal size independent set.

Let s_k be the number of independent sets of cardinality k in a graph G. The polynomial

$$\alpha(G,x) = \sum_{k=0}^{\alpha(G)} s_k \cdot x^k$$

where $\alpha(G)$ is the independence number, is called the independence polynomial of G.

By definition we have $s_0 = 1$, $s_1 = n$ and s_2 is the number of edges in complement of G, that is $\frac{n(n-1-\varphi(n))}{2}$.

The authors in [11] proved that the independence number of X_n is equal to $\frac{n}{p}$, where p is the smallest prime number dividing n. Here we extend the results by counting the number of different independent sets with size $\frac{n}{p}$ and $\frac{n}{p} - 1$.

First, we prove that $s_{\alpha(X_n)} = p$. All p independent sets with size $\frac{n}{p}$ are given by $I_k = \{k, k+p, \ldots, k+n-p\}$, for $k = 0, 1, \ldots, p-1$. It easy to see that the sets I_k are independent, since for every $a, b \in I_k$ we have $p \mid a-b$ and thus $\gcd(a-b, n) \neq 1$. Now, let $I = \{a_1, a_2, \ldots, a_{n/p}\}$ be an independent set, with $a_1 < a_2 < \ldots < a_{n/p}$. It can be easily seen that $a_{i+1} - a_i \geq p$, or otherwise $\gcd(a_{i+1} - a_i, n) = 1$. Using this inequality, we similarly obtain $a_{n/p} - a_1 = \sum_{i=1}^{n/p-1} (a_{i+1} - a_i) \geq (n/p - 1)p$ implying that $a_1 - a_{n/p} + n \leq p$. On the other hand, it holds that $a_{n/p} - a_1 \leq n/p - 1 = (n-p)/p \leq n - p$ implying that $a_1 - a_{n/p} + n \geq p$. It follows that the differences between consecutive elements of I must be equal p. Therefore, $s_{\alpha(X_n)} = p$.

For n being a prime number, X_n is a complete graph and only independent sets are vertices of X_n, so it holds that $\alpha(X_n, x) = n \cdot x$. We will further prove that $s_{\alpha(X_n)-1} = n$, for $n > 6$ and not a prime number. Obviously, by removing one element from each independent set of the size $\alpha(X_n)$ we get an independent set of size $\alpha(X_n) - 1$. Namely, if we want to remove the element $m = ps + r$, for some $0 \leq r \leq p-1$, then we will remove it from the independent set $I_r = \{r, r+p, \ldots, r+n-p\}$. As all such newly obtained sets $I_r \setminus \{m\}$ are different, it follows that $s_{\alpha(X_n)-1} \geq n$. Let $I = \{a_1, a_2, \ldots, a_{n/p-1}\}$ be an arbitrary independent set of size $\frac{n}{p} - 1$, with $a_1 < a_2 < \ldots < a_{n/p-1}$. Let the consecutive differences be $b_i = a_{i+1} - a_i$, $1 \leq i < \frac{n}{p} - 1$ and $b_{n/p-1} = n - a_{n/p-1} + a_1$. As in the previous case we have $b_i \geq p$ and $\sum_{i=1}^{\frac{n}{p}-1} b_i = n$. From these relations we conclude that at least one difference b_i is greater than p, i.e. $b_i \geq p^2$.

Assume first that $n = p^\alpha$, $\alpha > 1$. Let $b_k = a_{k+1} - a_k \geq p^2$ for some $1 \leq k < \frac{n}{p}$. Using the same reasoning about the sum of consecutive differences, we get

$$p^\alpha = \sum_{i=1}^{\frac{n}{p}-1} b_i \geq p(p^{\alpha-1} - 2) + p^2 = p^\alpha + p(p-2).$$

We see that the last inequality is satisfied for $p = 2$ and there is exactly one index $1 \leq k \leq p^{\alpha-1} - 1$ such that $b_k = p^2$. Since in that case holds $p^2 = 2p$, we conclude that every independent set is of the form $I_r \setminus \{m\}$, for $0 \leq m \leq p^\alpha - 1$, and thus $s_{\alpha(X_{p^\alpha})-1} = p^\alpha$.

Assume now that $n > 6$ has at least two prime divisors, p and q $(p < q)$. Also suppose that $b_i > p$ implying that $b_i \geq \min\{p^2, q\}$, for $1 \leq i < \frac{n}{p}$. Now it is easy to see that $\sum_{i=1}^{\frac{n}{p}-1} \geq (\frac{n}{p} - 1) \min\{p^2, q\} > n$, for $n > 6$. Therefore, we conclude that there exists some k such that $b_k = p$ and $b_{k+1} = q$ for some prime number q dividing n. Furthermore, using the sum of the differences, it holds that $n = \sum_{i=1}^{\frac{n}{p}-1} b_i \geq p(\frac{n}{p} - 2) + q$ implying that $p < q < 2p$. Next we have $\gcd(a_{k+1} - a_{k-1}, n) = \gcd(p+q, n) > 1$. We will now consider two cases based on the parity of n. Let β be the largest power of q such that $q^\beta \mid n$.

Case 1. If n is even, we have that $p = 2$ and $q = 3$ (since $q < 2p$). Furthermore, there are exactly two differences among $b_1, b_2, \ldots, b_{n/p-1}$ that are equal q and all others are equal to p. Indeed, suppose that there is an index i such that $b_i = r$ for some prime $r > 3$ dividing n. This means that $\sum_{i=1}^{\frac{n}{p}-1} b_i \geq 2(\frac{n}{2}-2)+3+r > n$, which is a contradiction. Now, suppose that $b_i \in \{p, q\}$ for $1 \leq i < \frac{n}{p}$ and $s = |\{1 \leq i < \frac{n}{p} \mid b_i = q\}|$. From the inequality $n = \sum_{i=1}^{\frac{n}{p}-1} b_i \geq 2(\frac{n}{2}-1-s)+3s$ we obtain that $s \leq 2$. For $s = 1$ we have that $(\frac{n}{p}-1)p + q = n$, which is impossible since $p \nmid q$.

Let $b_i = b_j = q$, for some $1 \leq i < j \leq \frac{n}{p} - 1$. These two numbers divide the difference array b into two parts, and we can consider the larger group. So, we have that $t = \max\{j-i-1, i-j+n/p-2\}$ is the cardinality of that subsequence. Without loss of generality suppose that $t = j-i-1$. Therefore, all numbers $a_{i+1} - a_i = q, a_{i+2} - a_i = q+2, a_{i+3} - a_i = q+4, \ldots, a_{i+t-1} - a_i = q + 2t$ have a common divisor with n, where

$$t \geq \frac{n-2q}{2p} = \frac{n-6}{4} > \frac{n}{4} - 2 = \frac{n}{2p} - 2.$$

In particular, this means that $q + 2 \cdot \frac{n}{2q^\beta} = q + \frac{n}{q^\beta}$ has a common factor with n, since it holds that $\frac{n}{2q^\beta} \leq \frac{n}{2p} - 2 < t$. But, as $\gcd(q + \frac{n}{q^\beta}, n) = 1$ we obtain a contradiction.

Case 2. Let n be odd and $q_1, q_2, \ldots, q_s$ primes dividing n that are different than p and appear as values of the sequence b_i. In other words, we have that $s = |\{1 \leq i \leq \frac{n}{p} \mid b_i \neq p\}|$ and $n = \left(\frac{n}{p} - 1 - s\right)p + \sum_{i=1}^{s} q_i$ implying that

$$p(s+1) = \sum_{i=1}^{s} q_i. \tag{2}$$

If $q = \min\{q_1, \ldots, q_s\}$ then we obtain $sq \leq p(s+1)$ and $s \leq \frac{p}{q-p} \leq \frac{p}{2}$, since $q - p \geq 2$. Therefore, there can be at most $\lfloor \frac{p}{2} \rfloor$ numbers not equal to p among $b_1, b_2, \ldots, b_{n/p-1}$. Let t be the maximal number of consecutive elements of b which are equal to p (the elements are taken in a cyclic order). Thus, it follows that $tsp + \sum_{i=1}^{s} q_s \geq n$. Furthermore, according to (2) and $s \leq \lfloor \frac{p}{2} \rfloor$ it holds that $t\lfloor \frac{p}{2} \rfloor p + p(\lfloor \frac{p}{2} \rfloor + 1) \geq n$. This means that there is again a chain of elements $q_i, q_i + p, q_i + 2p, \ldots, q_i + tp$ that have a common divisor with n, for some $1 \leq i \leq s$. For the length t holds the following estimation

$$t \geq \frac{n - (\lfloor p/2 \rfloor + 1)p}{\lfloor p/2 \rfloor \cdot p} = \frac{2n - (p+1)p}{p(p-1)} > \frac{n}{pq}.$$

In particular this means that $q + p \cdot \frac{n}{pq^\beta}$ has a common factor with n– which is impossible. So, in both cases we have that only independent sets od the size $\alpha(X_n) - 1$ are given in the form $I_r \setminus \{m\}$, which ends the proof.

We implemented backtrack algorithm for computing the independence polynomials for small values of n and presented the results in Table 1. Following the results from [10], these results can be generalized to characterize the properties of the independence polynomial of $X_n(d_1, d_2)$.

4. Laplacian Polynomial

The Laplacian polynomial is the characteristic polynomial of the Laplacian matrix. The Laplacian matrix is defined by $L = D - A$, where $D = diag(d_1, ..., d_n)$ is the degree matrix, which is the diagonal matrix formed from the vertex degrees and A *is* the adjacency matrix. As X_n is regular graph with the regularity $\varphi(n)$, the Laplacian matrix of X_n is given by $L(X_n) = \varphi(n)I_n - A(X_n)$, where I_n represents a unit matrix of order n.

In this section we assume that n has the following prime factor factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ and $m = p_1 p_2 \cdot \ldots \cdot p_k$.

Denote by $c(j, n)$ the following expression

$$c(j,n) = \mu(t_{n,j}) \frac{\varphi(n)}{\varphi(t_{n,j})}, \quad t_{n,j} = \frac{n}{\gcd(n, j)},$$

n	The independence polynomial
4	$4x + 2x^2$
5	$5x$
6	$6x + 9x^2 + 2x^3$
7	$7x$
8	$8x + 12x^2 + 8x^3 + 2x^4$
9	$9x + 9x^2 + 3x^3$
10	$10x + 25x^2 + 20x^3 + 10x^4 + 2x^5$
11	$11x$
12	$12x + 42x^2 + 52x^3 + 33x^4 + 12x^5 + 2x^6$
13	$13x$
14	$14x + 49x^2 + 70x^3 + 70x^4 + 42x^5 + 14x^6 + 2x^7$
15	$15x + 45x^2 + 35x^3 + 15x^4 + 3x^5$
16	$16x + 56x^2 + 112x^3 + 140x^4 + 112x^5 + 56x^6 + 16x^7 + 2x$
17	$17x$
18	$18x + 99x^2 + 222x^3 + 297x^4 + 270x^5 + 171x^6 + 72x^7 + 18x^8 + 2x^9$
19	$19x$
20	$20x + 110x^2 + 260x^3 + 425x^4 + 504x^5 + 420x^6 + 240x^7 + 90x^8 + 20x^9 + 2x^{10}$
21	$21x + 84x^2 + 112x^3 + 105x^4 + 63x^5 + 21x^6 + 3x^7$
22	$22x + 121x^2 + 330x^3 + 660x^4 + 924x^5 + 924x^6 + 660x^7 + 330x^8 + 110x^9 + 22x^{10} + 2x^{11}$
23	$23x$
24	$24x + 180x^2 + 584x^3 + 1194x^4 + 1752x^5 + 1932x^6 + 1608x^7 + 993x^8 + 440x^9 + 132x^{10} + 24x^{11} + 2x^{12}$
25	$25x + 50x^2 + 50x^3 + 25x^4 + 5x^5$

Table 1: The independence polynomials for small n.

where φ is Euler's totient function and μ is the Möbius function defined as

$$\mu(n) \quad = \quad \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n \text{ is not square–free} \\ (-1)^k, & \text{if } n \text{ is product of } k \text{ distinct prime numbers.} \end{cases} \tag{3}$$

The expression $c(j, n)$ is known as the *Ramanujan function* ([6, p. 55]).

As noted in [11], the eigenvalues μ_i of the adjacency matrix of X_n is given by $\mu_i = c(i, n)$, for $0 \leq i \leq n-1$. Since X_n is regular graph with the regularity equal to $\varphi(n)$, then we see that μ_0 is the greatest eigenvalue equal to $\varphi(n)$. Thus all the Laplacian eigenvalues $\lambda_i = \varphi(n) - \mu_i$ are positive, for $1 \leq i \leq n-1$.

In this section we determine the number of spanning tress in unitary Cayley graphs using the famous Kirchhoff's theorem ([12, p. 138]). Then the number of spanning trees of X_n is

$$t(X_n) = \frac{1}{n}\lambda_1\lambda_2 \cdot \ldots \cdot \lambda_{n-1}.$$

Lemma 4.1. *If n and $1 \leq j \leq n-1$ are two arbitrary positive integer number then the Ramanujan function $c(j, n) \neq 0$ if and only if $p_i^{\alpha_i-1} \mid n$ for $1 \leq i \leq k$.*

Proof. From the definition of the function $c(j, n)$, we have that $c(j, n) \neq 0$ if and only if $\mu(t_{n,j}) \neq 0$. The last inequality is true if and only if $t_{n,j}$ is a square–free number. According to the definition of $t_{n,j}$, we have that $t_{n,j}$ is square–free if and only if $p_i^{\alpha_i-1} \mid \gcd(n, j)$ for $1 \leq i \leq k$ implying that $p_i^{\alpha_i-1} \mid j$ for $1 \leq i \leq k$. $\square$

Theorem 4.2. *The number of spanning trees in the unitary Cayley graph X_n is equal to*

$$\frac{1}{n}\varphi(n)^{n-1} \prod_{\substack{d \mid m \\ d>1}} \left(1 - \frac{\mu(d)}{\varphi(d)}\right)^{\varphi(d)}.$$

Proof. If there exists some $1 \le i \le k$ such that $p_i^{\alpha_i-1} \nmid j$ for some $1 \le j \le n-1$ then according to Lemma 4.1 it holds that $\lambda_j = \varphi(n)$.

Now, assume that $p_i^{\alpha_i-1} \mid j$ for each $1 \le i \le k$. Without loss of generality we can also assume that $p_1^{\alpha_1-1} \| j, p_2^{\alpha_2-1} \| j, \ldots, p_s^{\alpha_s-1} \| j$ (where $p^\beta \| j$ means that $p^\beta \mid j$ and $p^{\beta+1} \nmid j$) and $p_i^{\alpha_i} \mid j$ for some $1 \le s \le k$ and every $s < i \le k$. Notice that the number of previously described elements j is equal to

$$|\{1 \le j \le n-1 \mid \gcd(j,n) = p_1^{\alpha_1-1} \cdot \ldots \cdot p_s^{\alpha_s-1} p_{s+1}^{\alpha_{s+1}} \cdot \ldots \cdot p_k^{\alpha_k}\}| = \varphi(p_1 p_2 \cdot \ldots \cdot p_s).$$

Now, we calculate the Laplacian eigenvalue λ_j. Since $\gcd(j,n) = p_1^{\alpha_1-1} \cdot \ldots \cdot p_s^{\alpha_s-1} p_{s+1}^{\alpha_{s+1}} \cdot \ldots \cdot p_k^{\alpha_k}$ we have that $t(n,j) = p_1 p_2 \cdot \ldots \cdot p_s$ and finally

$$\lambda_j = \varphi(n) - (-1)^s \frac{\varphi(n)}{\varphi(p_1 p_2 \cdot \ldots \cdot p_s)}.$$

According to Kirchhoff's theorem we conclude that

$$t(X_n) = \frac{1}{n} \varphi(n)^{n-1} \prod_{\{i_1,\ldots,i_s\} \subseteq \{1,\ldots,k\}} \left(1 - \frac{(-1)^s}{\varphi(p_{i_1} \cdot \ldots \cdot p_{i_s})}\right)^{\varphi(p_{i_1} \cdot \ldots \cdot p_{i_s})}.$$

Since there is a bijection between sets of indices $\{i_1, \ldots, i_s\}$ and divisors $d = p_{i_1} \cdot \ldots \cdot p_{i_s} \mid m$ we obtain the formula from the statement of the theorem. $\square$

For n being prime we obtain that $t(X_n) = \frac{1}{n}(n-1)^{n-1}(1 + \frac{1}{n-1})^{n-1} = n^{n-2}$. This result is excepted since in this case X_n is a complete graph and the number of spanning trees is given by Cayley's formula.

If n is a prime power number p^α for some prime p and integer $\alpha \ge 1$, then

$$t(X_n) = \frac{p^{(\alpha-1)(p^\alpha-1)}(p-1)^{p^\alpha-1}}{p^\alpha} \cdot \left(1 + \frac{1}{p-1}\right)^{p-1} = p^{(\alpha-1)p^\alpha+p-2\alpha}(p-1)^{p^\alpha-p}.$$

We see that the formula is much more complicated in comparison with the previous case.

If n has two prime factors p and q, in particular if $n = pq$, then the product is

$$\begin{aligned} t(X_n) &= \frac{(p-1)^{pq-1}(q-1)^{pq-1}}{pq} \cdot \frac{p^{p-1}}{(p-1)^{p-1}} \cdot \frac{q^{q-1}}{(q-1)^{q-1}} \cdot \frac{((p-1)(q-1)-1)^{(p-1)(q-1)}}{((p-1)(q-1))^{(p-1)(q-1)}} \\ &= (p-1)^{q-1} \cdot (q-1)^{p-1} \cdot p^{p-2} \cdot q^{q-2} \cdot ((p-1)(q-1)-1)^{(p-1)(q-1)} \end{aligned}$$

In general, there is no evident further cancellation or factorization (especially when n is not square–free), and only a small amount of such terms can be collected. Since it doesn't seem like there is much else to be done here, we will simplify the formula $t(X_n)$, for arbitrary n, by giving its asymptotic form.

Writing $(1 - \mu(d)/\varphi(d))^{\varphi(d)}$ as $\exp(\varphi(d)\ln(1 - \mu(d)/\varphi(d)))$ and using Taylor series for $\ln(1-x)$ around 0 ($|\mu(d)/\varphi(d)| \le 1$), we obtain that

$$\begin{aligned} t(X_n) &= \frac{1}{n}\varphi(n)^{n-1} \prod_{\substack{d \mid m \\ d>1}} \exp\left(\varphi(d)\left(-\frac{\mu(d)}{\varphi(d)} - \frac{\mu(d)^2}{2\varphi(d)^2} + O\left(\frac{\mu(d)^3}{\varphi(d)^3}\right)\right)\right) \\ &= \frac{1}{n}\varphi(n)^{n-1} \exp(-\sum_{\substack{d \mid m \\ d>1}} \mu(d) - \sum_{\substack{d \mid m \\ d>1}} \frac{\mu(d)^2}{2\varphi(d)} + \sum_{\substack{d \mid m \\ d>1}} O(\frac{1}{\varphi(d)^2})). \end{aligned}$$

According to well-known equalities $\sum_{d|n} \mu(d) = 0$, for $n \geq 2$, and $\sum_{d|n} \frac{\mu(d)^2}{\varphi(d)} = \frac{n}{\varphi(n)}$ ([7, p. 24, 36]), it holds that $\sum_{d|m} \mu(d) = -1$ and $\sum_{d|m} \frac{\mu(d)^2}{\varphi(d)} = \frac{m}{\varphi(m)} - 1$. This finally yields that

$$\begin{aligned} t(X_n) &= \frac{1}{n}\varphi(n)^{n-1} \exp(\frac{1}{2}(3 - \frac{m}{\varphi(m)}) + \sum_{\substack{d|m \\ d>1}} O(\frac{1}{\varphi(d)^2})) \\ &= \frac{1}{n}\varphi(n)^{n-1} \exp(\frac{1}{2}(3 - \prod_{i=1}^{k} \frac{p_i}{p_i - 1})) \cdot \exp(\sum_{\substack{d|m \\ d>1}} O(\frac{1}{\varphi(d)^2})). \end{aligned}$$

Using Taylor series for e^x around 0 and since m is square–free we further have that

$$\begin{aligned} t(X_n) &= \frac{1}{n}\varphi(n)^{n-1} \exp\left(\frac{1}{2}(3 - \prod_{i=1}^{k}(1 + \frac{1}{p_i - 1}))\right) \cdot (1 + O(\sum_{\substack{d|m \\ d>1}} \frac{1}{\varphi(d)^2})) \\ &= \frac{1}{n}\varphi(n)^{n-1} \exp\left(\frac{1}{2}(3 - \prod_{i=1}^{k}(1 + \frac{1}{p_i - 1}))\right) \cdot (1 + O(\sum_{i=1}^{k} \frac{1}{p_i^2})) \\ &= \frac{1}{n}\varphi(n)^{n-1} \exp\left(\frac{1}{2}(3 - \frac{m}{\varphi(m)})\right) \cdot (1 + O(\sum_{i=1}^{k} \frac{1}{p_i^2})). \end{aligned}$$

Asymptotic formulas only makes sense when prime factors (and all divisors d) of n are sufficiently large. In that case $\varphi(d)$ is always large. In the following table we give a several examples showing the accuracy of these formulas by calculating the relative error.

n	Relative error
$5 \cdot 7$	-0,024899
$7 \cdot 11$	-0,0111825
$11 \cdot 13$	-0,00524387
$13 \cdot 17$	-0,00340821
$29 \cdot 31$	-0,000774676

We see that the relative error is smaller for the greater values of the prime factors p_i, which makes this approximation satisfactory.

5. Distance Polynomial

For a parameter x, the distance polynomial (or Hosoya polynomial) of G is defined as [5]

$$W(G, x) = \sum_{u,v \in V(G)} x^{d(u,v)} = \sum_{k=1}^{diam(G)} W_k(G) \cdot x^k.$$

Obviously, the sum of all coefficients of the distance polynomial is $\frac{n(n-1)}{2}$ and the coefficient with x equals the number of edges of a graph.

According to the following theorem we will distinguish three cases to determine the forms of distance polynomial.

Theorem 5.1. *[11] For a given X_n and $n \geqslant 2$, we have that*

$$diam\ (X_n) = \begin{cases} 1, & n \text{ is a prime} \\ 2, & n \text{ is an odd composite integer or a power of } 2 \\ 3, & \text{otherwise.} \end{cases} \quad (4)$$

Case 1. n is a prime number.
The graph X_n is a complete graph, therefore $diam(G) = 1$ implying that $W(X_n, x) = \frac{n(n-1)}{2}x$.

Case 2. n is odd composite number or $n = 2^k$ for $k > 1$.
The graph X_n has diameter two, and the adjacent pairs of vertices are on the distance equal to one and nonadjacent on distance equal to two. Thus we have $W(X_n, x) = \frac{n\varphi(n)}{2}x + \frac{n(n-1-\varphi(n))}{2}x^2$.

Case 3. n is even and has an odd prime divisor.
We count the number of vertices in X_n which are on the distance 3 from the vertex 0. Using 1, as $2 \mid n$, we obtain that the number of triangles are equal to 0. This means that only the vertices which are on the distance 2 from the vertex 0 have the common neighbors with the vertex 0. By Theorem 6 from [11] and its generalization Theorem 4.1 from [4] we have that the number of common neighbors between the vertices 0 and l is given by $F_n(l) = n\prod_{p|l}(1-\frac{1}{p})\prod_{p\nmid l}(1-\frac{2}{p})$. Therefore, according to above discussion we have

$$\{1 \le l < n \mid d(0,l) \in \{1,3\}\} = \{1 \le l < n \mid F_n(l) = 0\}.$$

As $F_n(l) = 0$ only for l even we see the cardinality of the righthand side of the equation is equal to $\frac{n}{2}$. Furthermore, from $|\{1 \le l < n \mid d(0,l) = 1\}| = \varphi(n)$ we see that $|\{1 \le l < n \mid d(0,l) = 3\}| = \frac{n}{2} - \varphi(n)$. Finally, $|\{1 \le l < n \mid d(0,l) = 2\}| = n-1-|\{1 \le l < n \mid d(0,l) \in \{1,3\}\}| = n-1-\frac{n}{2}=\frac{n}{2}-1$. According to the above discussion finally we get the distance polynomial

$$W(X_n, x) = \frac{n\varphi(n)}{2}x + \frac{n(n-2)}{4}x^2 + \frac{n(n-2\varphi(n))}{4}x^3.$$

This result can be generalized for $X_n(d_1, d_2)$ using the results from [4].

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