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Spectral Radius Inequalities for Functions of Operators Defined by Power Series

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Abstract. By the help of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series that has as coefficients the absolute values of the coefficients of f, namely $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$. Utilising these functions we show among others that

$$r[f(T)] \le f_a[r(T)]$$

where $r(T)$ denotes the spectral radius of the bounded linear operator T on a complex Hilbert space while $\|T\|$ is its norm. When we have A and B two commuting operators, then

$$r^2[f(AB)] \le f_a\left(r^2(A)\right) f_a\left(r^2(B)\right)$$

and

$$r[f(AB)] \le \frac{1}{2}\left[f_a(\|AB\|) + f_a\left(\left\|A^2\right\|^{1/2}\left\|B^2\right\|^{1/2}\right)\right].$$

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on a complex Hilbert space H. For an operator $T \in B(H)$, let $r(T)$ and $\|T\|$ denote the *spectral radius* and the usual *operator norm* of A, respectively. It is well known that for every $T \in B(H)$, we have the fundamental inequality

$$r(T) \le \|T\| \tag{1}$$

and that equality holds in the inequality (1) if T is normal.

In addition to the inequality (1), the most important properties of the spectral radius are the *spectral radius formula*

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}, \tag{2}$$

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a special case of the *spectral mapping theorem*, which asserts that

$$r(T^m) = r^m(T) \tag{3}$$

for every natural number m, and the *commutativity property*, which asserts that

$$r(AB) = r(BA) \text{ for every } A, B \in B(H). \tag{4}$$

It follows from the spectral radius formula (2) that if $A, B \in B(H)$ are *commutative* then the following *subadditivity*

$$r(A + B) \le r(A) + r(B) \tag{5}$$

and *submultiplicativity*

$$r(AB) \le r(A)r(B) \tag{6}$$

properties hold.

For additional properties of the spectral radius, the reader is referred to the classical book [3] and the papers [5]-[14].

There are simple examples, see for instance [4], showing that the properties (5) and (6) are not true for non-commutative operators A and B.

In [4] the author has proved the following inequality

$$\begin{aligned} & r\left(AB \pm BA\right) \\ & \le \frac{1}{2}\left(\|AB\| + \|BA\| + \sqrt{\left(\|AB\| - \|BA\|\right)^2 + 4\left\|A^2\right\|\left\|B^2\right\|}\right) \end{aligned} \tag{7}$$

for any $A, B \in B(H)$.

If A and B are commutative, then from (7) we get

$$r\left(AB\right) \le \frac{1}{2}\left(\|AB\| + \left\|A^2\right\|^{1/2}\left\|B^2\right\|^{1/2}\right), \tag{8}$$

which is of interest in itself and also has some nice applications for functions of operators as follows.

In the same paper [4], the author also provided the inequality below

$$\begin{aligned} r\left(AB \pm BA\right) & \le \|AB\| + \min\left\{\|A\|^{1/2}\left\|AB^2\right\|^{1/2}, \left\|A^2B\right\|^{1/2}\|B\|^{1/2}\right\} \\ & \le \|AB\| + \begin{cases} \|A\|^{1/2}\|B\|^{1/2}\|AB\|^{1/2}, \\ \\ \min\left\{\|A\|\left\|B^2\right\|^{1/2}, \left\|A^2\right\|^{1/2}\|B\|\right\}, \end{cases} \end{aligned} \tag{9}$$

which produces in the case of commutative A and B the string of inequalities that are also useful in what follows:

$$\begin{aligned} r\left(AB\right) & \le \frac{1}{2}\left[\|AB\| + \min\left\{\|A\|^{1/2}\left\|AB^2\right\|^{1/2}, \left\|A^2B\right\|^{1/2}\|B\|^{1/2}\right\}\right] \\ & \le \frac{1}{2}\|AB\| + \frac{1}{2} \times \begin{cases} \|A\|^{1/2}\|B\|^{1/2}\|AB\|^{1/2}, \\ \\ \min\left\{\|A\|\left\|B^2\right\|^{1/2}, \left\|A^2\right\|^{1/2}\|B\|\right\}. \end{cases} \end{aligned} \tag{10}$$

Motivated by the above results we establish in this paper some inequalities for the spectral radius of functions of operators defined by power series, which incorporate many fundamental functions of interest such as the exponential function, some trigonometric functions, the functions $f(z) = (1 \pm z)^{-1}$, $g(z) = \log(1 \pm z)^{-1}$ and others. Some examples of interest are also provided.

2. Inequalities for One Operator

We start with the following lemmas.

Lemma 2.1. *Let $\left(V_j\right)_{j\in\mathbb{N}}$ be a sequence of bounded linear operators such that $V_jV_k = V_kV_j$ for any $j,k \in \mathbb{N}$. The for any $m \in \mathbb{N}$, $m \geq 1$ we have*

$$r\left(\sum_{j=0}^{m} V_j\right) \leq \sum_{j=0}^{m} r\left(V_j\right). \tag{11}$$

Proof. By induction over m.

If $m = 1$, the inequality follows by (5).

Assume that (11) is true for $m > 1$. Since the operators $\sum_{j=0}^{m} V_j$ and V_{m+1} are commutative, then by (5) we also have

$$r\left(\sum_{j=0}^{m+1} V_j\right) = r\left(\sum_{j=0}^{m} V_j + V_{m+1}\right) \leq r\left(\sum_{j=0}^{m} V_j\right) + r\left(V_{m+1}\right)$$
$$\leq \sum_{j=0}^{m} r\left(V_j\right) + r\left(V_{m+1}\right) = \sum_{j=0}^{m+1} r\left(V_j\right),$$

where for the last inequality we used the induction hypothesis.

This proves the inequality (11) for any $m \geq 1$. □

Lemma 2.2. *If $V, S \in B(H)$ are commutative then the following continuity property holds*

$$\left|r(V) - r(S)\right| \leq r(V - S). \tag{12}$$

Proof. Since $V - S$ and S are commutative, then by (5) we have

$$r(V) = r(V - S + S) \leq r(V - S) + r(S)$$

giving that

$$r(V) - r(S) \leq r(V - S). \tag{13}$$

From (2) we have that $r(-T) = r(T)$ for any operator T.

Since the operators $S - V$ and V also commute, then

$$r(S) \leq r(S - V) + r(V)$$

showing that

$$r(S) - r(V) \leq r(S - V) = r(V - S)$$

or, equivalently

$$-r(V - S) \leq r(V) - r(S). \tag{14}$$

Utilising (13) and (14) we obtain (12). □

Lemma 2.3. *Let $\left(V_j\right)_{j\in\mathbb{N}} \subset B(H)$ and $V \in B(H)$. If $V_n \to V$ in $B(H)$ and $V_jV = VV_j$ for any $j \in \mathbb{N}$, then* $\lim_{n\to\infty} r(V_n) = r(V)$.

Proof. Utilising (12) and (2) we have

$$|r(V_n) - r(V)| \le r(V_n - V) \le \|V_n - V\|$$

for any $n \in \mathbb{N}$ which produces the desired result. □

For power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients we can naturally construct another power series which have as coefficients the absolute values of the coefficient of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series have the same radius of convergence as the original series, and that if all coefficients $a_n \ge 0$, then $f_a = f$.

We can state and prove now our first result:

Theorem 2.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $T \in B(H)$ with $\|T\| < R$, then*

$$r[f(T)] \le f_a(r(T)). \tag{15}$$

Proof. Let $m \ge 1$ and define $V_j := a_j T^j$ for $j \in \{0, ..., m\}$. We observe that $V_j V_k = V_k V_j$ for any $j, k \in \{0, ..., m\}$ and by Lemma 2.1 we then have

$$\begin{aligned} r\left(\sum_{j=0}^{m} a_j T^j\right) &\le \sum_{j=0}^{m} r\left(a_j T^j\right) = \sum_{j=0}^{m} \left|a_j\right| r\left(T^j\right) \\ &= \sum_{j=0}^{m} \left|a_j\right| r^j(T), \end{aligned} \tag{16}$$

where for the last equality we used the property (3).

Now, consider the sequence $V_m := \sum_{j=0}^{m} a_j T^j$. Since $\|T\| < R$ it follows that $V_m \to f(T)$ in $B(H)$. Also, since $f(T)$ commutes with each of the $a_j T^j$ it follows that $f(T)$ also commutes with V_m and by Lemma 2.3 we have that

$$\lim_{m\to\infty} r\left(\sum_{j=0}^{m} a_j T^j\right) = r\left(\sum_{j=0}^{\infty} a_j T^j\right) = r[f(T)].$$

Therefore, by taking the limit over $m \to \infty$ and taking into account the fact that $\sum_{j=0}^{\infty} \left|a_j\right| r^j(T)$ is convergent since $r(T) \le \|T\| < R$, we deduce the desired result (15). □

Corollary 2.5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $T \in B(H)$ with $\|T\| < R$, then*

$$r[f(T)] \le f(r(T)). \tag{17}$$

3. Inequalities for Two Commuting Operators

We can consider now the case of two operators.

Theorem 3.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $A, B \in B(H)$ are commutative and for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$*

$$\|A\|^p, \|B\|^q < R, \tag{18}$$

then we have

$$r[f(AB)] \le \min\{L_1, L_2\} \tag{19}$$

where

$$L_1 := f_a^{1/p}\left(r^p(A)\right) f_a^{1/q}\left(r^q(B)\right)$$

and

$$L_2 := \frac{f_a\left(r^p(A)\right) f_a\left(r^q(B)\right)}{f_a\left(r^{p-1}(A)\,r^{q-1}(B)\right)}.$$

In particular, if $\|A\|^2, \|B\|^2 < R$, *then*

$$r^2\left[f(AB)\right] \le f_a\left(r^2(A)\right) f_a\left(r^2(B)\right). \tag{20}$$

Proof. Let $m \ge 1$ and write the inequality (16) for $T = AB$ to get

$$r\left(\sum_{j=0}^{m} a_j (AB)^j\right) \le \sum_{j=0}^{m} \left|a_j\right| r^j(AB). \tag{21}$$

Since A and B are commutative, then by (6) we also have

$$\sum_{j=0}^{m} \left|a_j\right| r^j(AB) \le \sum_{j=0}^{m} \left|a_j\right| r^j(A)\, r^j(B) \tag{22}$$

for any $m \ge 1$.

Now, by Hölder's weighted inequality we have

$$\sum_{j=0}^{m} \left|a_j\right| r^j(A)\, r^j(B) \le \left(\sum_{j=0}^{m} \left|a_j\right| r^{jp}(A)\right)^{1/p} \left(\sum_{j=0}^{m} \left|a_j\right| r^{jq}(B)\right)^{1/q} \tag{23}$$

for any $m \ge 1$.

Utilising (21)-(23) we have

$$r\left(\sum_{j=0}^{m} a_j (AB)^j\right) \le \left(\sum_{j=0}^{m} \left|a_j\right| r^{jp}(A)\right)^{1/p} \left(\sum_{j=0}^{m} \left|a_j\right| r^{jq}(B)\right)^{1/q} \tag{24}$$

for any $m \ge 1$.

From the condition (18) we observe that the series whose partial sums are involved in the inequality (24) are convergent and by letting $m \to \infty$ in (24) we obtain the first inequality in (19).

Further, by utilizing the following Hölder's type inequality obtained by Dragomir and Sándor in 1990 [2] (see also [1, Corollary 2.34]):

$$\sum_{k=0}^{n} m_k |x_k|^p \sum_{k=0}^{n} m_k \left|y_k\right|^q \ge \sum_{k=0}^{n} m_k \left|x_k y_k\right| \sum_{k=0}^{n} m_k |x_k|^{p-1} \left|y_k\right|^{q-1}, \tag{25}$$

that holds for nonnegative numbers m_k and complex numbers x_k, y_k where $k \in \{0, ..., n\}$, we observe that the convergence of the series $\sum_{k=0}^{\infty} m_k |x_k|^p$ and $\sum_{k=0}^{\infty} m_k \left|y_k\right|^q$ imply the convergence of the series $\sum_{k=0}^{\infty} m_k |x_k|^{p-1} \left|y_k\right|^{q-1}$.

On applying the inequality (25) we also have

$$\sum_{j=0}^{m} \left|a_j\right| r^j(A)\, r^j(B) \le \frac{\sum_{j=0}^{m} \left|a_j\right| r^{jp}(A) \sum_{j=0}^{m} \left|a_j\right| r^{jq}(B)}{\sum_{j=0}^{m} \left|a_j\right| r^{j(p-1)}(A)\, r^{j(q-1)}(B)} \tag{26}$$

for any $m \geq 1$.

Utilising (21)-(22) and (26) we get

$$r\left(\sum_{j=0}^{m} a_j (AB)^j\right) \leq \frac{\sum_{j=0}^{m} |a_j| r^{jp}(A) \sum_{j=0}^{m} |a_j| r^{jq}(B)}{\sum_{j=0}^{m} |a_j| r^{j(p-1)}(A) r^{j(q-1)}(B)} \tag{27}$$

for any $m \geq 1$.

Since all the series whose partial sums are involved in the inequality (27) are convergent, then by letting $m \to \infty$ in (27) we obtain the second part of (19). □

Remark 3.2. *If the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has nonnegative coefficients, then f_a in the inequalities (19) and (20) may be replaced with f.*

From a different perspective, we also have

Theorem 3.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $A, B \in B(H)$ are commutative and $\|A\|^2, \|B\|^2 < R$, then*

$$\begin{aligned} r[f(AB)] &\leq \frac{1}{2}\left[f_a(\|AB\|) + f_a\left(\left\|A^2\right\|^{1/2}\left\|B^2\right\|^{1/2}\right)\right] \\ &\leq \frac{1}{2}\left[f_a(\|AB\|) + f_a^{1/2}\left(\left\|A^2\right\|\right) f_a^{1/2}\left(\left\|B^2\right\|\right)\right] \end{aligned} \tag{28}$$

and

$$\begin{aligned} &r[f(AB)] \\ &\leq \frac{1}{2} f_a(\|AB\|) + \frac{1}{2}\min\left\{f_a\left(\|A\|^{1/2}\left\|AB^2\right\|^{1/2}\right), f_a\left(\left\|A^2B\right\|^{1/2}\|B\|^{1/2}\right)\right\} \\ &\leq \frac{1}{2} f_a(\|AB\|) + \frac{1}{2}\min\left\{f_a^{1/2}(\|A\|) f_a^{1/2}\left(\left\|AB^2\right\|\right), f_a^{1/2}\left(\left\|A^2B\right\|\right) f_a^{1/2}(\|B\|)\right\} \end{aligned} \tag{29}$$

provided also that $\|A\|, \|B\| < R$.

Proof. Since A and B commute, then A^j and B^j commute for each $j \in \mathbb{N}$ and by the inequality (8) and the properties of norms we have

$$\begin{aligned} r\left((AB)^j\right) = r\left(A^jB^j\right) &\leq \frac{1}{2}\left(\left\|A^jB^j\right\| + \left\|A^{2j}\right\|^{1/2}\left\|B^{2j}\right\|^{1/2}\right) \\ &\leq \frac{1}{2}\left(\|AB\|^j + \left\|A^2\right\|^{j/2}\left\|B^2\right\|^{j/2}\right) \end{aligned} \tag{30}$$

for each $j \in \mathbb{N}$.

If we multiply (30) by $|a_j|$, sum over j from 0 to $m \geq 1$ and use the weighted Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\begin{aligned} &\sum_{j=0}^{m} |a_j| r\left((AB)^j\right) \\ &\leq \frac{1}{2}\left(\sum_{j=0}^{m} |a_j| \|AB\|^j + \sum_{j=0}^{m} |a_j| \left\|A^2\right\|^{j/2}\left\|B^2\right\|^{j/2}\right) \\ &\leq \frac{1}{2}\left(\sum_{j=0}^{m} |a_j| \|AB\|^j + \left(\sum_{j=0}^{m} |a_j| \left\|A^2\right\|^j\right)^{1/2}\left(\sum_{j=0}^{m} |a_j| \left\|B^2\right\|^j\right)^{1/2}\right) \end{aligned} \tag{31}$$

for any $m \geq 1$.

Now, utilising (21) we can state the following string of inequalities

$$\begin{aligned} & r\left(\sum_{j=0}^{m} a_j (AB)^j\right) \\ & \leq \frac{1}{2}\left(\sum_{j=0}^{m} \left|a_j\right| \|AB\|^j + \sum_{j=0}^{m} \left|a_j\right| \left\|A^2\right\|^{j/2} \left\|B^2\right\|^{j/2}\right) \\ & \leq \frac{1}{2}\left(\sum_{j=0}^{m} \left|a_j\right| \|AB\|^j + \left(\sum_{j=0}^{m} \left|a_j\right| \left\|A^2\right\|^j\right)^{1/2} \left(\sum_{j=0}^{m} \left|a_j\right| \left\|B^2\right\|^j\right)^{1/2}\right) \end{aligned} \tag{32}$$

for any $m \geq 1$.

Since all the series whose partial sums are involved in the inequality (32) are convergent, then by letting $m \to \infty$ in (32) we obtain the desired inequality (28).

Now, on making use of the first inequality in (10) we have

$$\begin{aligned} & r\left(A^j B^j\right) \\ & \leq \frac{1}{2}\left[\left\|A^j B^j\right\| + \min\left\{\left\|A^j\right\|^{1/2} \left\|A^j B^{2j}\right\|^{1/2}, \left\|A^{2j} B^j\right\|^{1/2} \left\|B^j\right\|^{1/2}\right\}\right] \\ & \leq \frac{1}{2}\left[\|AB\|^j + \min\left\{\|A\|^{j/2} \left\|AB^2\right\|^{j/2}, \left\|A^2 B\right\|^{j/2} \|B\|^{j/2}\right\}\right] \end{aligned} \tag{33}$$

for any $j \in \mathbb{N}$.

Utilising the elementary inequality for nonnegative numbers p_j, c_j, d_j with $j \in \{0, ..., m\}$, $m \geq 1$

$$\sum_{j=0}^{m} p_j \min\left\{c_j, d_j\right\} \leq \min\left\{\sum_{j=0}^{m} p_j c_j, \sum_{j=0}^{m} p_j d_j\right\},$$

we obtain from (32) by multiplying with $\left|a_j\right|$ and summing over $j \in \{0, ..., m\}$ that

$$\sum_{j=0}^{m} \left|a_j\right| r\left((AB)^j\right) \leq \frac{1}{2} \sum_{j=0}^{m} \left|a_j\right| \|AB\|^j + \frac{1}{2} \min\left\{\sum_{j=0}^{m} \left|a_j\right| \|A\|^{j/2} \left\|AB^2\right\|^{j/2}, \sum_{j=0}^{m} \left|a_j\right| \left\|A^2 B\right\|^{j/2} \|B\|^{j/2}\right\} \tag{34}$$

for any $m \geq 1$.

Following a similar argument to the one outlined above we get the first inequality in (29).

The second inequality follows by the Cauchy-Bunyakovsky-Schwarz inequality and the details are omitted. □

Remark 3.4. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with nonnegative coefficients then f_a from the inequalities (28) and (29) may be replaced with f.*

Finally, on making use of the inequality

$$r(AB) \leq \frac{1}{2}\|AB\| + \frac{1}{2} \times \begin{cases} \|A\|^{1/2} \|B\|^{1/2} \|AB\|^{1/2} \\ \\ \min\left\{\|A\| \left\|B^2\right\|^{1/2}, \left\|A^2\right\|^{1/2} \|B\|\right\} \end{cases} \tag{35}$$

we can also state the result:

Proposition 3.5. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If $A,B\in B(H)$ are commutative and $\|A\|^2,\|B\|^2<R$, then*

$$
\begin{aligned}
& r\left[f(AB)\right] \\
& \leq \frac{1}{2}f_a(\|AB\|) \\
& +\frac{1}{2}\times\begin{cases} f_a\left(\|A\|^{1/2}\|B\|^{1/2}\|AB\|^{1/2}\right) \\ \min\left\{f_a\left(\|A\|\left\|B^2\right\|^{1/2}\right),f_a\left(\left\|A^2\right\|^{1/2}\|B\|\right)\right\}\end{cases} \\
& \leq \frac{1}{2}f_a(\|AB\|) \\
& +\frac{1}{2}\times\begin{cases} f_a^{1/2}(\|A\|\|B\|)f_a^{1/2}(\|AB\|) \\ \min\left\{f_a^{1/2}\left(\|A\|^2\right)f_a^{1/2}\left(\left\|B^2\right\|\right),f_a^{1/2}\left(\left\|A^2\right\|\right)f_a^{1/2}\left(\|B\|^2\right)\right\}.\end{cases}
\end{aligned} \tag{36}
$$

4. Some Examples

As some natural examples that are useful for applications, we can point out that, if

$$
\begin{aligned}
f(z) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln\frac{1}{1+z},\ z\in D(0,1); \\
g(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\ z\in\mathbb{C}; \\
h(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\ z\in\mathbb{C}; \\
l(z) &= \sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\ z\in D(0,1);
\end{aligned} \tag{37}
$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$
\begin{aligned}
f_a(z) &= \sum_{n=1}^{\infty}\frac{1}{n}z^n=\ln\frac{1}{1-z},\ z\in D(0,1); \\
g_a(z) &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C}; \\
h_A(z) &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C}; \\
l_A(z) &= \sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1).
\end{aligned} \tag{38}
$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned}
\exp(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \qquad z \in \mathbb{C}, \\
\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1); \\
\sin^{-1}(z) &= \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1}, \qquad z \in D(0,1); \\
\tanh^{-1}(z) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1) \\
{}_2F_1(\alpha,\beta,\gamma,z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0, \\
& z \in D(0,1);
\end{aligned} \tag{39}$$

where Γ is *Gamma function*.

If $T \in B(H)$ with $\|T\| < 1$, then by (15) we have the inequalities

$$r\left[(I \pm T)^{-1}\right] \leq [1 - r(T)]^{-1},$$

$$r\left[\ln(I \pm T)^{-1}\right] \leq \ln[1 - r(T)]^{-1},$$

$$r\left[\sin^{-1}(T)\right] \leq \sin^{-1}[r(T)]$$

and

$$r[{}_2F_1(\alpha,\beta,\gamma,T)] \leq {}_2F_1(\alpha,\beta,\gamma,r(T)).$$

If $T \in B(H)$, then by the same inequality we have

$$r[\exp(T)] \leq \exp[r(T)],$$

$$r[\sin(T)], r[\sinh(T)] \leq \sinh(r(T))$$

and

$$r[\cos(T)], r[\cosh(T)] \leq \cosh(r(T)).$$

If $A, B \in B(H)$ are commutative and $\|A\|, \|B\| < 1$, then by the inequality (19) we have

$$r\left[(I \pm AB)^{-1}\right] \leq \begin{cases} (1 - r^p(A))^{-1/p} (1 - r^q(B))^{-1/q}, \\ \frac{(1-r^p(A))^{-1}(1-r^q(B))^{-1}}{\left(1-r^{p-1}(A)r^{q-1}(B)\right)^{-1}}, \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and by the inequalities (28), (29) and (36) we have

$$r\left[(I \pm AB)^{-1}\right] \le \frac{1}{2}(1 - \|AB\|)^{-1} + \frac{1}{2} \times \begin{cases} \left(1 - \left\|A^2\right\|^{1/2}\left\|B^2\right\|^{1/2}\right)^{-1}, \\ \min\left\{\left(1 - \|A\|^{1/2}\left\|AB^2\right\|^{1/2}\right)^{-1}, \left(1 - \left\|A^2B\right\|^{1/2}\|B\|^{1/2}\right)^{-1}\right\}, \\ \left(1 - \|A\|^{1/2}\|B\|^{1/2}\|AB\|^{1/2}\right)^{-1}, \\ \min\left\{\left(1 - \|A\|\left\|B^2\right\|^{1/2}\right)^{-1}, \left(1 - \left\|A^2\right\|^{1/2}\|B\|\right)^{-1}\right\}. \end{cases}$$

By the same inequalities, if $A, B \in B(H)$ are commutative, then

$$r[\exp(AB)] \le \begin{cases} \exp\left[\frac{1}{p}r^p(A) + \frac{1}{q}r^q(B)\right], \\ \exp\left[r^p(A) + r^q(B) - r^{p-1}(A)\,r^{q-1}(B)\right], \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and

$$r[\exp(AB)] \le \frac{1}{2}\exp(\|AB\|) + \frac{1}{2} \times \begin{cases} \exp\left(\left\|A^2\right\|^{1/2}\left\|B^2\right\|^{1/2}\right), \\ \exp\left[\min\left\{\|A\|^{1/2}\left\|AB^2\right\|^{1/2}, \left\|A^2B\right\|^{1/2}\|B\|^{1/2}\right\}\right], \\ \exp\left(\|A\|^{1/2}\|B\|^{1/2}\|AB\|^{1/2}\right), \\ \exp\left[\min\left\{\|A\|\left\|B^2\right\|^{1/2}, \left\|A^2\right\|^{1/2}\|B\|\right\}\right]. \end{cases}$$

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