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Holomorphically Projective Mappings of (Pseudo-) Kähler Manifolds Preserve the Class of Differentiability

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Abstract. In this paper we study fundamental equations of holomorphically projective mappings of (pseudo-) Kähler manifolds with respect to the smoothness class of metrics C^r, $r \geq 1$. We show that holomorphically projective mappings preserve the smoothness class of metrics.

1. Introduction

First we study the general dependence of holomorphically projective mappings of classical and pseudo-Kähler manifolds (shortly *Kähler*) on the smoothness class of the metric. We present well known facts, which were proved by Otsuki, Tashiro [31], Tashiro, Ishihara [44], Domashev, Mikeš [8], Mikeš [19, 20], A.V. Aminova, D. Kalinin [2–5], etc., see [6, 9, 25, 27, 28, 35, 36, 45]. To the theory of holomorphically projective mappings and their generalization are devoted many publications, eg. [1, 7, 10, 14–18, 21–23, 26, 29, 30, 32, 33, 38–41]. In these results no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects.

The following results are connected to the paper [11] where it was proved that holomorphically projective mappings preserve the smoothness class C^r of the metrics in the case $r \geq 2$. In the following paper we generalize this result to the case $r \geq 1$.

2. Main Results

Let $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ be (pseudo-) Kähler manifolds, where M and $\bar{M}$ are n-dimensional manifolds with dimension $n \geq 4$, g and $\bar{g}$ are metrics, F and $\bar{F}$ are structures. All the manifolds are assumed to be connected.

Definition 2.1. A diffeomorphism $f\colon K_n \to \bar{K}_n$ is called a *holomorphically projective mapping* of K_n onto $\bar{K}_n$ if f maps any holomorphically planar curve in K_n onto a holomorphically planar curve in $\bar{K}_n$.

We obtain the following theorem.

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Theorem 2.2. *If the (pseudo-) Kähler manifold K_n ($K_n \in C^r$, $r \geq 1$) admits a holomorphically projective mapping onto $\bar{K}_n \in C^1$, then $\bar{K}_n$ belongs to C^r.*

Briefly, this means that:

holomorphically projective mappings preserve the class of smoothness of the metric.

The analogous property for geodesic mappings of (pseudo-) Riemannian manifolds is proved in [12].

Here and later $K_n = (M, g, F) \in C^r$ denotes that $g \in C^r$, i.e. in a coordinate neighborhood (U, x) for the components of the metric g holds $g_{ij}(x) \in C^r$. If $K_n \in C^r$, then $M \in C^{r+1}$. This means that the atlas on the manifold M has the differentiability class C^{r+1}, i.e. for non disjoint charts (U, x) and (U', x') on $U \cap U'$ it is true that the transformation $x' = x'(x) \in C^{r+1}$.

The differentiability class r is equal to $0, 1, 2, \ldots, \infty, \omega$, where $0, \infty$ and ω denotes continuous, infinitely differentiable, and real analytic functions respectively.

Remark 2.3. It's easy to prove that the Theorem 2.2 is valid also for $r = \infty$ and for $r = \omega$. This follows from the theory of solvability of differential equations. Of course we can apply this theorem only locally, because differentiability is a local property.

Remark 2.4. A minimal requirement for holomorphically projective mappings is K_n, $\bar{K}_n \in C^1$.

Mikeš, see [19, 21, 22, 24, 25], [28, p. 82] found equidistant Kähler metrics g in canonical coordinates x:

$$g_{ab} = g_{a+m\,b+m} = \partial_{ab} f + \partial_{a+m\,b+m} f \ \text{ and } \ g_{a\,b+m} = \partial_{a\,b+m} f - \partial_{a+m\,b} f,$$

where $a = 1, 2, \ldots, m$, $m = n/2$, $f = \exp(2x^1) \cdot G(x^2, x^3, \ldots, x^m, x^{2+m}, x^{3+m}, \ldots, x^{2m})$, $G \in C^3$, which admit holomorphically projective mappings. Evidently, if $G \in C^{r+2}$ ($r \in \mathbb{N}$), $G \in C^\infty$ and C^ω, then $K_n \in C^r$, $K_n \in C^\infty$ and $K_n \in C^\omega$, respectively. From these metrics we can easily see examples of non trivial holomorphically projective mappings $K_n \to \bar{K}_n$, where

$$K_n, \bar{K}_n \in C^r \text{ and } \notin C^{r+1} \text{ for } r \in \mathbb{N}; \qquad K_n, \bar{K}_n \in C^\infty \text{ and } \notin C^\omega; \qquad K_n, \bar{K}_n \in C^\omega.$$

3. (Pseudo-) Kähler Manifolds

In the following definition we introduce generalizations of Kähler manifolds.

Definition 3.1. An n-dimensional (pseudo-) Riemannian manifold ($n \geq 4$) is called a *(pseudo-) Kähler manifold* K_n=(M, g, F), if beside the metric tensor g, a tensor field F of type $(1, 1)$ is given on the manifold M, called a *structure* F, such that the following conditions hold:

$$F^2 = -Id; \quad g(X, FX) = 0; \quad \nabla F = 0, \tag{1}$$

where X is an arbitrary vector of TM, and ∇ denotes the covariant derivative in K_n.

These spaces were first considered as *A-spaces* by P.A. Shirokov, see [34]. Independently such spaces with positive definite metric were studied by E. Kähler [13]. The tensor field F is called a *complex structure* [45].

The following lemma specifies the properties of the differentiability of geometrical objects on (pseudo-) Kähler manifolds.

Lemma 3.2. *If $K_n = (M, g, F) \in C^r$, i.e. $g \in C^r$, then $F \in C^r$, for $r \in \mathbb{N}$ and $r = \infty, \omega$.*

Proof. Let $K_n \in C^r$, i.e. the components of the metric $g_{ij}(x) \in C^r$ in a coordinate chart x. It is a priori valid that $F^h_i \in C^1$. The formula $\nabla F = 0$ can be written $\partial_k F^h_i = F^h_a \Gamma^a_{ik} - F^a_i \Gamma^h_{ak}$, where $\Gamma_{ijk} = {}^1\!/_2\, (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$, $\partial_k = \partial/\partial x^k$, and $\Gamma^h_{ij} = g^{hk}\Gamma_{ijk}$ are Christoffel symbols of the first and second kind, respectively. It holds that Γ_{ijk} and $\Gamma^h_{ij} \in C^{r-1}$. From this equation follows immediately $F^h_i(x) \in C^r$, i.e. $F \in C^r$. □

Moreover, due to the differentiability of $g \in C^r$ according to (1), each point has a coordinate neighborhood $(U, x) \in C^{r+1}$ in which the structure F has the following canonical form:

$$F_b^{a+m} = -F_{b+m}^a = \delta_b^a, \quad F_b^a = F_{b+m}^{a+m} = 0, \quad a, b = 1, \cdots, m; \ m = \frac{n}{2}. \tag{2}$$

We get, as an immediate consequence, that the dimension is even, $n = 2m$. Such a coordinate system will be called *canonical*.

Due to the conditions (1) and (2), the components of the metric tensor and Christoffel symbols of the second kind in a canonical coordinate system satisfy

$$g_{a+m,b+m} = g_{ab}, \quad g_{ab+m} = -g_{a+mb}, \quad \text{and} \quad \Gamma_{bc}^a = \Gamma_{b+mc+m}^{a+m} = -\Gamma_{b+mc+m}^a, \qquad \Gamma_{b+mc+m}^{a+m} = \Gamma_{b+mc}^a = -\Gamma_{bc}^{a+m}. \tag{3}$$

Obviously, the coordinate transformation $x'^h = x'^h(x)$ preserves a canonical coordinate system if and only if the Jacobi matrix $J = (\partial x'^h / \partial x^i)$ satisfies

$$\frac{\partial x'^{a+m}}{\partial x^{b+m}} = \frac{\partial x'^a}{\partial x^b} \quad \text{and} \quad \frac{\partial x'^{a+m}}{\partial x^b} = -\frac{\partial x'^a}{\partial x^{b+m}}. \tag{4}$$

Let us set $z^a = x^a + ix^{a+m}$, $z'^a = x'^a + ix'^{a+m}$ (where i is the imaginary unit). Then (4) can be interpreted as Cauchy-Riemann conditions for the complex functions $z'^a = z'^a(z^1, \cdots, z^m)$, and we will call this transformation *analytic*.

4. Holomorphically Projective Mappings $K_n \to \bar{K}_n$ of Class C^1

Assume the (pseudo-) Kähler manifolds $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here K_n, $\bar{K}_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$.

Likewise, as in [31], see [6], [35, p. 205], [36], [25], [28, p. 240], we introduce the following notations.

Definition 4.1. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$ $(\neq 0), t \in I$, where t is a parameter is called *holomorphically planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = Span\,\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(t)F\lambda,$$

where $a(t)$ and $b(t)$ are some functions of the parameter t.

Particularly, in the case $b(t) = 0$, a holomorphically planar curve is a geodesic.

We recall the Definition 2.1: A diffeomorphism f: $K_n \to \bar{K}_n$ is called a *holomorphically projective mapping* of K_n onto $\bar{K}_n$ if f maps any holomorphically planar curve in K_n onto a holomorphically planar curve in $\bar{K}_n$.

Assume a holomorphically projective mapping f: $K_n \to \bar{K}_n$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally f: $K_n \to \bar{K}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$.

A manifold K_n admits a holomorphically projective mapping onto $\bar{K}_n$ if and only if the following equations [28, 36]:

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X - \psi(FX)FY - \psi(FY)FX \tag{5}$$

hold for any tangent fields X, Y and where ψ is a differential form. In local form:

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \psi_i \delta_j^h + \psi_j \delta_i^h - \psi_{\bar{i}} \delta_{\bar{j}}^h - \psi_{\bar{j}} \delta_{\bar{i}}^h,$$

where Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are the Christoffel symbols of K_n and $\bar{K}_n$, ψ_i, F_i^h are components of ψ, F and δ_i^h is the Kronecker delta, $\psi_{\bar{i}} = \psi_a F_i^a$, $\delta_{\bar{i}}^h = F_i^h$. Here and in the following we will use the conjugation operation of indices in the way

$$A_{\cdots\bar{i}\cdots}^{\cdots} = A_{\cdots k \cdots}^{\cdots} F_i^k, \qquad A_{\cdots}^{\cdots\bar{i}\cdots} = A_{\cdots}^{\cdots k \cdots} F_k^i.$$

If $\psi \equiv 0$, then f is *affine* or *trivially holomorphically projective*. Beside these facts it was proved [28, 36] that $\bar{F} = \pm F$; for this reason we can suppose that $\bar{F} = F$.

It is known that

$$\psi_i = \nabla_i \Psi, \quad \Psi = \frac{1}{2(n+2)} \ln \left| \frac{\det \bar{g}}{\det g} \right|.$$

Equations (5) are equivalent to the following equations

$$\nabla_Z \bar{g}(X,Y) = 2\psi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z) + \psi(FX)\bar{g}(FY,Z) + \psi(FY)\bar{g}(FX,Z). \tag{6}$$

In local form:

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik} + \psi_{\bar{i}} \bar{g}_{\bar{j}k} + \psi_{\bar{j}} \bar{g}_{\bar{i}k},$$

where $\bar{g}_{ij}$ are components of the metric $\bar{g}$ on $\bar{K}_n$.

The above formulas are well known for $\bar{F} = F$, see [31], [6], [35, p. 206], [36], [25], [28, p. 240-242].

Domashev and Mikeš ([8], see [35, p. 212], [36], [25], [28, p. 246]) proved that equations (5) and (6) are equivalent to

$$\nabla_Z a(X,Y) = \lambda(X)g(Y,Z) + \lambda(Y)g(X,Z) + \lambda(FX)g(FY,Z) + \lambda(FY)g(FX,Z); \tag{7}$$

in local form:

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_{\bar{i}} g_{\bar{j}k} + \lambda_{\bar{j}} g_{\bar{i}k},$$

where

$$\text{(a)}\ \ a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{ab} g_{ai} g_{bj}; \quad \text{(b)}\ \ \lambda_i = -\,\mathrm{e}^{2\Psi} \bar{g}^{ab} g_{bi} \psi_a. \tag{8}$$

From (7) follows $\lambda_i = \nabla_i \Lambda$ and $\Lambda = \frac{1}{4} a_{bc} g^{bc}$. On the other hand [28]:

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi} \tilde{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \tilde{g}}{\det g} \right|, \quad \|\tilde{g}_{ij}\| = \|g^{ib} g^{jc} a_{bc}\|^{-1}. \tag{9}$$

The above formulas are the criterion for holomorphically projective mappings $K_n \to \bar{K}_n$, globally as well as locally.

5. Holomorphically Projective Mapping for $K_n \in C^2 \to \bar{K}_n \in C^1$

I. Hinterleitner [11] proved the theorem:

Theorem 5.1. *If a (pseudo-) Kähler manifold $K_n \in C^r$, $r \geq 2$, admits a holomorphically projective mapping onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^r$.*

It is easy to see that Theorem 2.2 follows from Theorem 5.1 and the following theorem.

Theorem 5.2. *If $K_n \in C^2$ admits a holomorphically projective mapping onto $\bar{K}_n \in C^1$, then $\bar{K}_n \in C^2$.*

Proof. We will suppose that the (pseudo-) Kähler manifold $K_n = (M, g, F) \in C^2$ admits a holomorphically projective mapping f onto the (pseudo-) Kähler manifold $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F}) \in C^1$. Furthermore, we can assume that $\bar{M} = M$ and $\bar{F} = F$. The corresponding points $x \in M$ and $\bar{x} = f(x) \in \bar{M}$ have common coordinates $(x^1, x^2, \ldots, x^n)$, shortly x, in the coordinate chart (U, x), $U \subset M$, .

We study the coordinate neighborhood (U, x) of any point p at M. Moreover, we suppose that the coordinate system x is canonical (2). On (U, x) formulae (5)–(9) hold, and formula (7) may be written in the following form

$$\partial_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k + \bar{\lambda}^i F^j_k + \bar{\lambda}^j F^i_k - f^{ij}_k, \tag{10}$$

where $a^{ij} = a_{bc}g^{bi}g^{cj}$, $\lambda^i = \lambda_a g^{ia}$, $\bar{\lambda}^i = \lambda^a F^i_a$, and $f^{ij}_k = a^{ib}\Gamma^j_{bk} + a^{jb}\Gamma^i_{bk}$.

The components $g_{ij}(x) \in C^2$ and $\bar{g}_{ij}(x) \in C^1$ on $U \subset M$ and from that facts follows that the functions $g^{ij}(x) \in C^2$, $\bar{g}^{ij}(x) \in C^1$, $\Psi(x) \in C^1$, $\psi_i(x) \in C^0$, $a^{ij}(x) \in C^1$, $\lambda^i(x) \in C^0$, and $\Gamma^h_{ij}(x) \in C^1$. It is easy to see, that $f^{ij}_k \in C^1$.

In the canonical coordinate system x we can calculate the following derivatives for fixed different indices $a, b = 1, \ldots, m$, $m = n/2$:

$$\begin{aligned} \partial_b a^{ab} &= \lambda^a - f^{ab}_b, & \partial_{b+m} a^{ab} &= -\lambda^{a+m} - f^{ab}_{b+m}, \\ \partial_b a^{ab+m} &= \lambda^{a+m} - f^{ab+m}_b, & \partial_{b+m} a^{ab+m} &= -\lambda^a - f^{ab+m}_{b+m}. \end{aligned} \tag{11}$$

Eliminating λ^a and λ^{a+m} we obtain the equations

$$\begin{aligned} \partial_b a^{ab} - \partial_{b+m} a^{ab+m} &= -f^{ab}_b + f^{ab+m}_b \\ \partial_{b+m} a^{ab} + \partial_b a^{ab+m} &= -f^{ab}_{b+m} - f^{ab+m}_b. \end{aligned} \tag{12}$$

We denote $w = a^{ab} + i \cdot a^{ab+m}$, $z = x^b + i \cdot x^{b+m}$, where i is the imaginary unit. Then (12) can be rewritten

$$\partial_z w = F \equiv (-f^{ab}_b + f^{ab+m}_b) + i \cdot (-f^{ab}_{b+m} - f^{ab+m}_b),$$

and because $F \in C^1$, then exists $\partial^2_{z\bar{z}} w$.

So there are the second partial derivatives of the functions a^{ab} and a^{ab+m} of the variables x^b and x^{b+m}; and, clearly, also of x^a and x^{a+m}. After this from formula (11) follows that $\lambda^h \in C^1$; and equations (10) imply that $a^{ij}, a_{ij} \in C^2$. Finally, formula (9) shows that $\bar{g}_{ij} \in C^2$. □

6. Holomorphically Projective Mapping $K_n \to \bar{K}_n$ of Class C^2

Let K_n and $\bar{K}_n \in C^2$ be (pseudo-) Kähler manifolds, then for holomorphically projective mappings $K_n \to \bar{K}_n$ the Riemann and the Ricci tensors transform in the following way

$$\begin{aligned} &\text{(a)} \quad \bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} + \delta^h_{\bar{k}} \psi_{i\bar{j}} - \delta^h_{\bar{j}} \psi_{i\bar{k}} - 2\delta^h_{\bar{i}} \psi_{j\bar{k}}; \\ &\text{(b)} \quad \bar{R}_{ij} = R_{ij} - (n+2)\psi_{ij}, \end{aligned} \tag{13}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j + \psi_{\bar{i}} \psi_{\bar{j}}$ $(\psi_{ij} = \psi_{ji} = \psi_{\bar{i}\bar{j}})$. Here the Ricci tensor is defined by $R_{ik} = R^a_{iak}$. In many papers it is defined with the opposite sign [19, 25, 35, 46], etc.

The tensor of the holomorphically projective curvature, which is defined in the following form

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik} + \delta^h_{\bar{k}} R_{i\bar{j}} - \delta^h_{\bar{j}} R_{i\bar{k}} - 2\delta^h_{\bar{i}} R_{j\bar{k}}\right), \tag{14}$$

is invariant with respect to holomorphically projective mappings, i.e. $\bar{P}^h_{ijk} = P^h_{ijk}$.

The above mentioned formulae can be found in the papers [6, 28, 35].

The integrability conditions of equations (7) have the following form

$$a_{ia} R^a_{jkl} + a_{ja} R^a_{ikl} \quad = \quad g_{ik} \nabla_l \lambda_j + g_{jk} \nabla_l \lambda_i - g_{il} \nabla_k \lambda_j - g_{jl} \nabla_k \lambda_i + g_{\bar{i}k} \nabla_l \lambda_{\bar{j}} + g_{\bar{j}k} \nabla_l \lambda_{\bar{i}} - g_{\bar{i}l} \nabla_k \lambda_{\bar{j}} - g_{\bar{j}l} \nabla_k \lambda_{\bar{i}}. \tag{15}$$

After contraction with g^{jl} we get:

$$a_{ib} R^b_k + a_{bc} R^b{}_{ik}{}^c = -\nabla_{\bar{k}} \lambda_{\bar{i}} - (n-1) \nabla_k \lambda_i,$$

where $R^b{}_{il}{}^c = g^{ck} R^b{}_{ilk}$; $R^b_l = g^{bj} R_{jl}$ and $\mu = \nabla_c \lambda_b g^{bc}$.

We contract this formula with $F^i_{i'} F^k_{k'}$ and from the properties of the Riemann and the Ricci tensors of K_n we obtain

$$\nabla_{\bar{k}} \lambda_{\bar{i}} = \nabla_k \lambda_i, \tag{16}$$

and ([8, 25, 28, 35])

$$n \nabla_k \lambda_i = \mu g_{ik} - a_{ib} R^b_k - a_{bc} R^b{}_{ik}{}^c \,. \tag{17}$$

Because λ_i is a gradient-like covector, from equation (17) follows $a_{ib} R^b_j = a_{jb} R^b_i$.

From (16) follows that the vector field $\lambda_{\bar{i}}$ ($\equiv \lambda_a F^a_i$) is a Killing vector field, i.e. $\nabla_j \lambda_{\bar{i}} + \nabla_i \lambda_{\bar{j}} = 0$. But the other side of the equations (16) can be written in the form $\nabla_a \lambda^h F^a_i = \nabla_i \lambda^a F^h_a$. In the canonical coordinate system x they are given by

$$\partial_b \lambda^a - \partial_{b+m} \lambda^{a+m} = 0 \;\text{ and }\; \partial_{b+m} \lambda^a + \partial_b \lambda^{a+m} = 0, \quad a, b = 1, \dots, m, \; m = n/2.$$

These are Cauchy-Riemann equations, which implies that the functions $\lambda^h(x)$ are real analytic. After this differentiation of the Killing equations we obtain $\nabla_j(\nabla_i \bar{\lambda}^h) = \bar{\lambda}^a R^h_{ija}$, and by contraction with F^i_h, we finally obtain

$$\nabla_j \mu = -2 \lambda^a R_{ai}.$$

These equations were found earlier under the assumption $K_n \in C^3$ and $\bar{K}_n \in C^3$, [20], see [35, p. 212], [28, pp. 247–248].

From that we proof the following theorem

Theorem 6.1. *A Kähler manifold $K_n \in C^2$ admits holomorphically projective mappings onto $\bar{K}_n \in C^1$ if and only if the system of differential equations*

$$\begin{aligned} \nabla_k a_{ij} &= \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_{\bar{i}} g_{\bar{j}k} + \lambda_{\bar{j}} g_{\bar{i}k}, \\ n \nabla_k \lambda_i &= \mu g_{ik} - a_{ib} R^b_k - a_{bc} R^b{}_{ik}{}^c, \\ \nabla_j \mu &= -2 \lambda^b R_{bj}, \end{aligned} \tag{18}$$

has a solution a_{ij}, λ_i and μ satisfying the following conditions

$$a_{ij} = a_{ij} = a_{\bar{i}\bar{j}}, \quad \det(a_{ij}) \neq 0. \tag{19}$$

Remark 6.2. *Moreover if $K_n \in C^r$, it follows that $\bar{K}_n \in C^r$, the function $\lambda_i \in C^r$ and $\mu \in C^{r-1}$.*

Remark 6.3. *If $K_n \in C^\infty$, then $\bar{K}_n \in C^\infty$, and if $K_n \in C^\omega$, then $\bar{K}_n \in C^\omega$.*

Theorem 6.1 was proved in the case $K_n, \bar{K}_n \in C^3$, see [20].

The family of differential equations (18) is linear with coefficients of intrinsic character in K_n and independent of the choice of coordinates. If the metric tensor g and the structure tensor F of the Kähler manifold K_n are real then for the initial data

$$a_{ij}(x_0) = \overset{o}{a}_{ij}, \qquad \lambda_i(x_0) = \overset{o}{\lambda}_i, \qquad \mu(x_0) = \overset{o}{\mu},$$

the system (18) has at most one solution. Accounting that the initial data must satisfy (19), it follows that the general solution of (18) depends on r_{hpm} significant parameters, where $r_{hpm} \leq (n/2 + 1)^2$.

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