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# Hermite-Hadamard-Fejer Type Inequalities for $s-$Convex Function in the Second Sense via Fractional Integrals

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**Abstract.** In this paper, we established Hermite-Hadamard-Fejer type inequalities for $s-$convex functions in the second sense via fractional integrals. The some results presented here would provide extansions of those given in earlier works.

## 1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality[10]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a.b \in I$ with $a < b$. A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if whenever $x, y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

In [9], Fejér gave a generalization of the inequalities (1) as the following:

If $f : [a, b] \to \mathbb{R}$ is a convex function, and $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{2}$$

If $g \equiv 1$, then we are talking about the Hermite–Hadamard inequalities.More about those inequalities can be found in a number of papers and monographies (for example, see [7]-[19]).

In [11], Hudzik and Maligrada considered among others the class of functions which are s-convex in the second sense.

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**Definition 1.1.** *A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y).$$

*for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.*

It can be easily seen that $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.
In [7], Dragomir and Fitzpatrick proved Hadamard's inequality which holds for s-convex functions in the second sense.

**Theorem 1.2.** *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex functions in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L[a, b]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throught this paper.

**Definition 1.3.** *Let $f \in L[a, b]$. The Riemann − Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by integrals hold:*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \quad x > a$$

*and*

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \quad x < b$$

*respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du$. Here is $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$*

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. The recent results and the properties concerning this operator can be found ([1]-[6])

In [15], Sarıkaya *et.al.* represented Hermite-Hadamard's inequalities in fractional integral forms as follows.

**Theorem 1.4.** *Let $f : [a, b] \to \mathbb{R}$ be positive function with $0 \leq a < b$ and $f \in L[a, b]$. If $f$ is a convex function on $[a, b]$, than the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a) + f(b)}{2} \tag{3}$$

*with $\alpha > 0$.*

In [12], İşcan gave the following Hermite-Hadamard-Fejer integral inequalities via fractional integrals:

**Theorem 1.5.** *Let $f : [a, b] \to \mathbb{R}$ be convex function with $a < b$ and $f \in L[a, b]$. If $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] \leq \left[J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a)\right] \leq \frac{f(a) + f(b)}{2} \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right]$$

*with $\alpha > 0$.*

Set *et al*. established some inequalities connected with the left-hand side of the inequality (2) used the following lemma.

**Lemma 1.6.** *[19] Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$ and let $g : [a,b] \to \mathbb{R}$. If $f', g \in L[a,b]$, then the following identity for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right] = \frac{1}{\Gamma(\alpha)}\int_a^b k(t) f'(t)\, dt \tag{4}$$

*where*

$$k(t) = \begin{cases} \int_a^t (s-a)^{\alpha-1} g(s)\, ds, & t \in \left[a, \frac{a+b}{2}\right] \\ -\int_t^b (b-s)^{\alpha-1} g(s)\, ds, & t \in \left[\frac{a+b}{2}, b\right] \end{cases}.$$

Set *et al. proved the following three theorems.*

**Theorem 1.7.** *[19] Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$and $f' \in L[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\ \le\ & \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha+1}\Gamma(\alpha+1)(\alpha+1)}\left[|f'(a)| + |f'(b)|\right] \end{aligned} \tag{5}$$

*with $\alpha > 0$.*

**Theorem 1.8.** *[19] Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$and $f' \in L[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\ \le\ & \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha+1+1/q}(\alpha+1)(\alpha+2)^{1/q}\Gamma(\alpha+1)} \\ &\times\left\{\left[(\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q\right]^{1/q} + \left[(\alpha+1)|f'(a)|^q + (\alpha+3)|f'(b)|^q\right]^{1/q}\right\} \end{aligned} \tag{6}$$

*with $\alpha > 0$.*

**Theorem 1.9.** *[19] Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$and $f' \in L[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\ \le\ & \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{2^{\alpha+1+2/q}(\alpha p+1)^{1/q}\Gamma(\alpha+1)}\left[3|f'(a)|^q + |f'(b)|^{q1/q} + \left(|f'(a)|^q + 3|f'(b)|^q\right)^{1/q}\right] \end{aligned} \tag{7}$$

*where $1/p + 1/q = 1$.*

We recall the following function:
The incomplete Beta function defined by

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt.$$

In this paper, motivated by the recent results given in [11],[19], we establish Hermite-Hadamard-Fejer type inequalities for s-convex functions in the second sense via fractional integral. An interesting feature of our results is that they provide new estimates on these types of inequalities for fractional integrals.

## 2. Main Results

Now, by using the Lemma 1.6 we prove our main theorems.

**Theorem 2.1.** *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on $I^o$ and $f' \in L[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continious. If $|f'|$ is $s-$convex on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\
\leq\; & \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{\Gamma(\alpha+1)}\left\{B_{1/2}(\alpha+1,s+1) + \frac{1}{2^{\alpha+s+1}(\alpha+s+1)}\right\}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].
\end{aligned} \tag{8}$$

*Proof.* Since $|f'|$ is $s-convex$ on $[a,b]$ for some fixed $s \in (0,1]$, we know that for $t \in [a,b]$

$$\left|f'(t)\right| = \left|f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right| \leq \left(\frac{b-t}{b-a}\right)^s \left|f'(a)\right| + \left(\frac{t-a}{b-a}\right)^s \left|f'(b)\right|$$

From Lemma 1.6 we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\
\leq\; & \frac{1}{\Gamma(\alpha)}\left\{\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}g(s)\,ds\right|\left|f'(t)\right|dt + \int_{\frac{a+b}{2}}^b\left|\int_t^b (b-s)^{\alpha-1}g(s)\,ds\right|\left|f'(t)\right|dt\right\} \\
\leq\; & \frac{\|g\|_{[a,\frac{a+b}{2}],\infty}}{(b-a)^s\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\left(\int_a^t (s-a)^{\alpha-1}ds\right)\left((b-t)^s\left|f'(a)\right| + (t-a)^s\left|f'(b)\right|\right)dt \\
&+\frac{\|g\|_{[\frac{a+b}{2},b],\infty}}{(b-a)^s\Gamma(\alpha)}\int_{\frac{a+b}{2}}^b\left(\int_t^b (b-s)^{\alpha-1}ds\right)\left((b-t)^s\left|f'(a)\right| + (t-a)^s\left|f'(b)\right|\right)dt \\
=\; & \frac{\|g\|_{[a,\frac{a+b}{2}],\infty}}{(b-a)^s\Gamma(\alpha+1)}\int_a^{\frac{a+b}{2}}(t-a)^{\alpha}\left[(b-t)^s\left|f'(a)\right| + (t-a)^s\left|f'(b)\right|\right]dt \\
&+\frac{\|g\|_{[\frac{a+b}{2},b],\infty}}{(b-a)^s\Gamma(\alpha+1)}\int_{\frac{a+b}{2}}^b(b-t)^{\alpha}\left[(b-t)^s\left|f'(a)\right| + (t-a)^s\left|f'(b)\right|\right]dt \\
=\; & \frac{\|g\|_{[a,\frac{a+b}{2}],\infty}}{(b-a)^s\Gamma(\alpha+1)}\left[\left|f'(a)\right|\int_a^{\frac{a+b}{2}}(t-a)^{\alpha}(b-t)^s\,dt + \left|f'(b)\right|\int_a^{\frac{a+b}{2}}(t-a)^{\alpha+s}\,dt\right] \\
&+\frac{\|g\|_{[\frac{a+b}{2},b],\infty}}{(b-a)^s\Gamma(\alpha+1)}\left[\left|f'(a)\right|\int_{\frac{a+b}{2}}^b(b-t)^{\alpha+s}\,dt + \left|f'(b)\right|\int_{\frac{a+b}{2}}^b(b-t)^{\alpha}(t-a)^s\,dt\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\|g\|_{[a,\frac{a+b}{2}],\infty}}{(b-a)^s\Gamma(\alpha+1)}\left[\left|f'(a)\right|(b-a)^{\alpha+s+1}B_{1/2}(\alpha+1,s+1)+\left|f'(b)\right|\frac{(b-a)^{\alpha+s+1}}{2^{\alpha+s+1}(\alpha+s+1)}\right]\\
&\quad+\frac{\|g\|_{[\frac{a+b}{2},b],\infty}}{(b-a)^s\Gamma(\alpha+1)}\left[\left|f'(a)\right|\frac{(b-a)^{\alpha+s+1}}{2^{\alpha+s+1}(\alpha+s+1)}+\left|f'(b)\right|(b-a)^{\alpha+s+1}B_{1/2}(\alpha+1,s+1)\right]\\
&= \frac{(b-a)^{\alpha+s+1}}{(b-a)^s\Gamma(\alpha+1)}\left\{\|g\|_{[a,\frac{a+b}{2}],\infty}\left(\left|f'(a)\right|B_{1/2}(\alpha+1,s+1)+\left|f'(b)\right|\frac{1}{2^{\alpha+s+1}(\alpha+s+1)}\right)\right.\\
&\quad\left.+\|g\|_{[\frac{a+b}{2},b],\infty}\left(\left|f'(a)\right|\frac{1}{2^{\alpha+s+1}(\alpha+s+1)}+\left|f'(b)\right|B_{1/2}(\alpha+1,s+1)\right)\right\}\\
&\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{\Gamma(\alpha+1)}\left\{B_{1/2}(\alpha+1,s+1)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]+\frac{\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{2^{\alpha+s+1}(\alpha+s+1)}\right\}\\
&= \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{\Gamma(\alpha+1)}\left\{B_{1/2}(\alpha+1,s+1)+\frac{1}{2^{\alpha+s+1}(\alpha+s+1)}\right\}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]
\end{aligned}$$

where

$$\int_a^{\frac{a+b}{2}}(t-a)^{\alpha+s}\,dt=\int_{\frac{a+b}{2}}^b(b-t)^{\alpha+s}\,dt=\frac{(b-a)^{\alpha+s+1}}{2^{\alpha+s+1}(\alpha+s+1)}$$

and

$$\int_a^{\frac{a+b}{2}}(t-a)^{\alpha}(b-t)^s\,dt=\int_{\frac{a+b}{2}}^b(b-t)^{\alpha}(t-a)^s\,dt=(b-a)^{\alpha+s+1}B_{1/2}(\alpha+1,s+1).$$

□

**Remark 2.2.** *In Theorem 7, if we choose $s = 1$ , then* (8) *reduces inequality* (5) *of Theorem 4.*

**Theorem 2.3.** *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on $I^o$ and $f' \in L[a,b]$ with $a < b$ and let $g : [a,b] \to \mathbb{R}$ is continious. If $\left|f'\right|^q$ is $s-$convex on $[a,b]$ for some fixed $s \in (0,1]$, $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{(\frac{a+b}{2})-}g(a)+J^{\alpha}_{(\frac{a+b}{2})+}g(b)\right]-\left[J^{\alpha}_{(\frac{a+b}{2})-}(fg)(a)+J^{\alpha}_{(\frac{a+b}{2})+}(fg)(b)\right]\right|\\
&\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha+1+\frac{1}{q}}(\alpha+1)(\alpha+2)^{1/q}(\alpha+s+q)^{1/q}\Gamma(\alpha+1)} \qquad (9)\\
&\quad\times\left\{\left((\alpha+s+1)(\alpha+3)\left|f'(a)\right|^q+2^{1-s}(\alpha+1)(\alpha+2)\left|f'(b)\right|^q\right)^{1/q}\right.\\
&\quad\left.+\left(2^{1-s}(\alpha+1)(\alpha+2)\left|f'(a)\right|^q+(\alpha+s+1)(\alpha+3)\left|f'(b)\right|^q\right)^{1/q}\right\}.
\end{aligned}$$

*Proof.* Since $\left|f'\right|$ is $s-$ *convex* on $[a,b]$ for some fixed $s \in (0,1]$, we know that for $t \in [a,b]$

$$\left|f'(t)\right|^q=\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\leq\left(\frac{b-t}{b-a}\right)^s\left|f'(a)\right|^q+\left(\frac{t-a}{b-a}\right)^s\left|f'(b)\right|^q$$

Using Lemma 1.6, power mean inequality and convexity of $\left|f'\right|^q$, it follows that

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right|\\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-1/q}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}g(s)\,ds\right|\left|f'(t)\right|^q dt\right)^{1/q}\\
&+\frac{1}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b (b-s)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-1/q}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b (b-s)^{\alpha-1}g(s)\,ds\right|\left|f'(t)\right|^q dt\right)^{1/q}\\
\leq\ & \frac{\left\|g\right\|_{[a,\frac{a+b}{2}],\infty}}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}ds\right|dt\right)^{1-1/q}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}ds\right|\left|f'(t)\right|^q dt\right)^{1/q}\\
&+\frac{\left\|g\right\|_{[\frac{a+b}{2},b],\infty}}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b (b-s)^{\alpha-1}ds\right|dt\right)^{1-1/q}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b (b-s)^{\alpha-1}ds\right|\left|f'(t)\right|^q dt\right)^{1/q}\\
\leq\ & \frac{1}{\alpha\Gamma(\alpha)}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{1-1/q}\left\{\frac{\left\|g\right\|_{[a,\frac{a+b}{2}],\infty}}{(b-a)^{s/q}}\left[\int_a^{\frac{a+b}{2}}\left((t-a)^{\alpha}(b-t)^s\left|f'(a)\right|^q+(t-a)^{\alpha+s}\left|f'(b)\right|^q\right)dt\right]^{1/q}\right.\\
&\left.+\frac{\left\|g\right\|_{[\frac{a+b}{2},b],\infty}}{(b-a)^{s/q}}\left[\int_{\frac{a+b}{2}}^b\left((b-t)^{\alpha+s}\left|f'(a)\right|^q+(b-t)^{\alpha}(t-a)^s\left|f'(b)\right|^q\right)dt\right]^{1/q}\right\}\\
=\ & \frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{1-1/q}\\
&\times\left\{\frac{\left\|g\right\|_{[a,\frac{a+b}{2}],\infty}}{(b-a)^{s/q}}\left[(b-a)^{\alpha+s+1}B_{1/2}(\alpha+1,s+1)\left|f'(a)\right|^q+\frac{(b-a)^{\alpha+s+1}}{2^{\alpha+s+1}(\alpha+s+1)}\left|f'(b)\right|^q\right]^{1/q}\right.\\
&\left.+\frac{\left\|g\right\|_{[\frac{a+b}{2},b],\infty}}{(b-a)^{s/q}}\left[\frac{(b-a)^{\alpha+s+1}}{2^{\alpha+s+1}(\alpha+s+1)}\left|f'(a)\right|^q+(b-a)^{\alpha+s+1}B_{1/2}(\alpha+1,s+1)\left|f'(b)\right|^q\right]^{1/q}\right\}\\
\leq\ & \frac{(b-a)^{\alpha+1}\left\|g\right\|_{[a,b],\infty}}{2^{\alpha+1+\frac{s}{q}}(\alpha+1)^{1-1/q}(\alpha+s+q)^{1/q}\Gamma(\alpha+1)}\left\{\left(2^{\alpha+s+1}(\alpha+s+1)B_{1/2}(\alpha+1,s+1)\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\right.\\
&\left.\left(\left|f'(a)\right|^q+2^{\alpha+s+1}(\alpha+s+1)B_{1/2}(\alpha+1,s+1)\left|f'(b)\right|^q\right)^{1/q}\right\}
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{(b-a)^{\alpha+1}\left\|g\right\|_{[a,b],\infty}}{2^{\alpha+1+\frac{1}{q}}(\alpha+1)(\alpha+2)^{1/q}(\alpha+s+q)^{1/q}\Gamma(\alpha+1)}\left\{\left((\alpha+s+1)(\alpha+3)\left|f'(a)\right|^q+2^{1-s}(\alpha+1)(\alpha+2)\left|f'(b)\right|^q\right)^{1/q}\right.\\
&\left.+\left(2^{1-s}(\alpha+1)(\alpha+2)\left|f'(a)\right|^q+(\alpha+s+1)(\alpha+3)\left|f'(b)\right|^q\right)^{1/q}\right\}.
\end{aligned}$$

□

**Remark 2.4.** *In Theorem 8, if we choose* $s=1$ *, then* (10) *reduces inequality* (6) *of Theorem 5.*

**Theorem 2.5.** *Let* $f: I\subseteq[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* $I^o$ *and* $f'\in L[a,b]$ *with* $a<b$ *and let* $g:[a,b]\to\mathbb{R}$ *is continious. If* $\left|f'\right|^q$ *is* $s-$*convex on* $[a,b]$ *for some fixed* $s\in(0,1]$, $q>1$, *then the following inequality* *for fractional integrals holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\ \leq\ & \frac{(b-a)^{\alpha+1}\left\|g\right\|_{[a,b],\infty}}{2^{\alpha+1+\frac{s}{q}}(\alpha p+1)(\alpha+2)^{1/p}(s+1)^{1/q}\Gamma(\alpha+1)} \\ &\times\left[\left(\left|f'(a)\right|^q\left(2^{s+1}-1\right)+\left|f'(b)\right|^q\right)^{1/q}+\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\left(2^{s+1}-1\right)\right)^{1/q}\right] \end{aligned} \tag{10}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* Using Lemma 1.6, Hölder's inequality and the s-convex of $\left|f'\right|^q$ it follows that

$$\begin{aligned} &\left| f\left(\tfrac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\ \leq\ & \tfrac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}g(s)\,ds\right|dt+\int_{\frac{a+b}{2}}^b\left|\int_t^b (s-a)^{\alpha-1}g(s)\,ds\right|\left|f'(t)\right|^q dt\right] \\ \leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}g(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^{\frac{a+b}{2}}\left|f'(t)\right|^q dt\right)^{1/q} \\ &+\frac{1}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b (s-a)^{\alpha-1}g(s)\,ds\right|^p dt\right)^{1/p}\left(\int_{\frac{a+b}{2}}^b\left|f'(t)\right|^q dt\right)^{1/q} \\ =\ & \frac{1}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}g(s)\,ds\right|^p dt\right)^{1/p}\left[\left(\int_a^{\frac{a+b}{2}}\left|f'(t)\right|^q dt\right)^{1/q}+\left(\int_{\frac{a+b}{2}}^b\left|f'(t)\right|^q dt\right)^{1/q}\right] \\ \leq\ & \frac{(b-a)^{\alpha+1}\left\|g\right\|_{[a,b],\infty}}{2^{\alpha+1+\frac{s}{q}}(\alpha p+1)(\alpha+2)^{1/p}(s+1)^{1/q}\Gamma(\alpha+1)} \\ &\times\left[\left(\left|f'(a)\right|^q\left(2^{s+1}-1\right)+\left|f'(b)\right|^q\right)^{1/q}+\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\left(2^{s+1}-1\right)\right)^{1/q}\right]. \end{aligned}$$

Here we use

$$\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}g(s)\,ds\right|^p dt=\int_{\frac{a+b}{2}}^b\left|\int_t^b (s-a)^{\alpha-1}g(s)\,ds\right|^p dt=\frac{(b-a)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)\alpha^p}$$

$$\int_a^{\frac{a+b}{2}}\left|f'(t)\right|^q dt\leq\frac{b-a}{2^{s+1}(s+1)}\left[\left|f'(a)\right|^q\left(2^{s+1}-1\right)+\left|f'(b)\right|^q\right]$$

$$\int_{\frac{a+b}{2}}^b\left|f'(t)\right|^q dt\leq\frac{b-a}{2^{s+1}(s+1)}\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\left(2^{s+1}-1\right)\right].$$

□

**Remark 2.6.** *In Theorem 9, if we choose* $s=1$ *, then* (10) *reduces inequality* (7) *of Theorem 6.*

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