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# The Modification of Poisson-Sch Integral on Cones and Its Applications

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**Abstract.** In this paper, we construct a modified Poisson-Sch integral on cones. As applications, we not only obtain the asymptotic behaviors of generalized harmonic functions but also characterize the geometrical properties of the exceptional sets with respect to the Schrödinger operator on cones.

## 1. Introduction and Main Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n (n \geq 2)$ the $n$-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance between two points $P$ and $Q$ in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin $O$ of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $\mathbf{S}$ in $\mathbf{R}^n$ are denoted by $\partial\mathbf{S}$ and $\overline{\mathbf{S}}$, respectively.

We introduce a system of spherical coordinates $(r, \Theta)$, $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set $\Omega$, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with $\Theta$ and $\Omega$, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}^{n-1}_+ = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by $\mathbf{T}_n$.

For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at $P$ and radius $r$ in $\mathbf{R}^n$. $S_r = \partial B(O, r)$. By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain $\Omega$ on $\mathbf{S}^{n-1}$. We call it a cone. Then $T_n$ is a special cone obtained by putting $\Omega = \mathbf{S}^{n-1}_+$. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$ which is $\partial C_n(\Omega) - \{O\}$.

We shall say that a set $E \subset C_n(\Omega)$ has a covering $\{r_j, R_j\}$ if there exists a sequence of balls $\{B_j\}$ with centers in $C_n(\Omega)$ such that $E \subset \cup_{j=1}^{\infty} B_j$, where $r_j$ is the radius of $B_j$ and $R_j$ is the distance between the origin and the center of $B_j$.

Let $\mathscr{A}_a$ denote the class of nonnegative radial potentials $a(P)$, i.e. $0 \leq a(P) = a(r)$, $P = (r, \Theta) \in C_n(\Omega)$, such that $a \in L^b_{loc}(C_n(\Omega))$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

This article is devoted to the stationary Schrödinger equation

$$Sch_a u(P) = -\Delta u(P) + a(P)u(P) = 0 \quad \text{for} \quad P \in C_n(\Omega),$$

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where $\Delta$ is the Laplace operator and $a \in \mathscr{A}_a$. These solutions are called generalized harmonic functions (associated with the operator $Sch_a$). Note that they are (classical) harmonic functions in the case $a = 0$. Under these assumptions the operator $Sch_a$ can be extended in the usual way from the space $C_0^\infty(C_n(\Omega))$ to an essentially self-adjoint operator on $L^2(C_n(\Omega))$ (see [13]). We will denote it $Sch_a$ as well. This last one has a Green-Sch function $G(\Omega; a)(P, Q)$. Here $G(\Omega; a)(P, Q)$ is positive on $C_n(\Omega)$ and its inner normal derivative $\partial G(\Omega; a)(P, Q)/\partial n_Q \geq 0$. We denote this derivative by $\mathbb{PI}(\Omega; a)(P, Q)$, which is called the Poisson-Sch kernel with respect to $C_n(\Omega)$. We remark that $G(\Omega; 0)(P, Q)$ and $\mathbb{PI}(\Omega; 0)(P, Q)$ are the Green function and Poisson kernel of the Laplacian in $C_n(\Omega)$ respectively.

Let $\Delta^*$ be a Laplace-Beltrami operator (spherical part of the Laplace) on $\Omega \subset \mathbf{S}^{n-1}$ and $\lambda_j$ ($j = 1, 2, 3 \ldots, 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$) be the eigenvalues of the eigenvalue problem for $\Delta^*$ on $\Omega$ (see, e.g., [14, p. 41])

$$\Delta^* \varphi(\Theta) + \lambda \varphi(\Theta) = 0 \quad \text{in} \quad \Omega,$$

$$\varphi(\Theta) = 0 \quad \text{on} \quad \partial\Omega.$$

Corresponding eigenfunctions are denoted by $\varphi_{jv}$ ($1 \leq v \leq v_j$), where $v_j$ is the multiplicity of $\lambda_j$. We set $\lambda_0 = 0$, norm the eigenfunctions in $L^2(\Omega)$ and $\varphi_1 = \varphi_{11} > 0$.

In order to ensure the existences of $\lambda_j$ ($j = 1, 2, 3 \ldots$). We put a rather strong assumption on $\Omega$: if $n \geq 3$, then $\Omega$ is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [4, p. 88-89] for the definition of $C^{2,\alpha}$-domain). Then $\varphi_{jv} \in C^2(\overline{\Omega})$ ($j = 1, 2, 3, \ldots, 1 \leq v \leq v_j$) and $\partial\varphi_1/\partial n > 0$ on $\partial\Omega$ (here and below, $\partial/\partial n$ denotes differentiation along the interior normal).

Hence well-known estimates (see, e.g., [6, p. 14]) imply the following inequality:

$$\sum_{v=1}^{v_j} \varphi_{jv}(\Theta) \frac{\partial \varphi_{jv}(\Phi)}{\partial n_\Phi} \leq M(n) j^{2n-1}, \tag{1}$$

where the symbol $M(n)$ denotes a constant depending only on $n$.

Let $V_j(r)$ ($j = 1, 2, 3, \ldots$) and $W_j(r)$ ($j = 1, 2, 3, \ldots$) stand, respectively, for the increasing and non-increasing, as $r \to +\infty$, solutions of the equation

$$-Q''(r) - \frac{n-1}{r} Q'(r) + \left( \frac{\lambda_j}{r^2} + a(r) \right) Q(r) = 0, \qquad 0 < r < \infty, \tag{2}$$

normalized under the condition $V_j(1) = W_j(1) = 1$ (see [17–19]).

We shall also consider the class $\mathscr{B}_a$, consisting of the potentials $a \in \mathscr{A}_a$ such that there exists a finite limit $\lim_{r \to \infty} r^2 a(r) = k \in [0, \infty)$, moreover, $r^{-1}|r^2 a(r) - k| \in L(1, \infty)$. If $a \in \mathscr{B}_a$, then the g.h.f.s are continuous (see [15]).

In the rest of paper, we assume that $a \in \mathscr{B}_a$ and we shall suppress this assumption for simplicity. Further, we use the standard notations $u^+ = \max(u, 0)$, $u^- = -\min(u, 0)$, $[d]$ is the integer part of $d$ and $d = [d] + \{d\}$, where $d$ is a positive real number.

Denote

$$\iota_{j,k}^{\pm} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k + \lambda_j)}}{2} \quad (j = 0, 1, 2, 3 \ldots).$$

It is known (see [5]) that in the case under consideration the solutions to the equation (2) have the asymptotics

$$V_j(r) \sim d_1 r^{\iota_{j,k}^{+}}, \quad W_j(r) \sim d_2 r^{\iota_{j,k}^{-}}, \quad \text{as} \quad r \to \infty, \tag{3}$$

where $d_1$ and $d_2$ are two positive constants.

If $a \in \mathscr{A}_a$, it is known that the following expansion for the Green function $G(\Omega;a)(P,Q)$ (see [3, Ch. 11])

$$G(\Omega;a)(P,Q) = \sum_{j=0}^{\infty} \frac{1}{\chi'(1)} V_j(\min(r,t)) W_j(\max(r,t)) \left( \sum_{v=1}^{v_j} \varphi_{jv}(\Theta)\varphi_{jv}(\Phi) \right), \tag{4}$$

where $P = (r,\Theta)$, $Q = (t,\Phi)$, $r \neq t$ and $\chi'(s) = w\left(W_1(r), V_1(r)\right)|_{r=s}$ is their Wronskian.

The series converges uniformly if either $r \leq st$ or $t \leq sr$ $(0 < s < 1)$. The expansion (4) can also be rewritten in terms of the Gegenbauer polynomials. In the case $a = 0$, this expansion $G(\Omega;0)(P,Q)$ coincides with the result by Qiao-Deng (see [8, 9]). In the case $a = 0$ and $\Omega = \mathbf{S}_+^{n-1}$, this expansion $G(\mathbf{S}_+^{n-1};0)(P,Q)$ coincides with the result by Qiao-Deng (see [7]).

For a nonnegative integer $m$ and two points $P = (r,\Theta), Q = (t,\Phi) \in C_n(\Omega)$, we put

$$K(\Omega;a,m)(P,Q) = \begin{cases} 0 & \text{if} \quad 0 < t < 1, \\ \widetilde{K}(\Omega;a,m)(P,Q) & \text{if} \quad 1 \leq t < \infty, \end{cases}$$

where

$$\widetilde{K}(\Omega;a,m)(P,Q) = \sum_{j=0}^{m} \frac{1}{\chi'(1)} V_j(r) W_j(t) \left( \sum_{v=1}^{v_j} \varphi_{jv}(\Theta)\varphi_{jv}(\Phi) \right).$$

If we modify the Green-Sch function on cones as follows

$$G(\Omega;a,m)(P,Q) = G(\Omega;a)(P,Q) - K(\Omega;a,m)(P,Q)$$

for two points $P = (r,\Theta), Q = (t,\Phi) \in C_n(\Omega)$, then the modified Poisson-Sch on cones can be defined by

$$\mathbb{PI}(\Omega;a,m)(P,Q) = \frac{\partial G(\Omega;a,m)(P,Q)}{\partial n_Q}.$$

We remark that

$$\mathbb{PI}(\Omega;a,0)(P,Q) = \mathbb{PI}(\Omega;a)(P,Q),$$

In this paper, we shall use the following modified Poisson-Sch integrals defined by

$$\mathbb{PI}_{\Omega}^{a}(m,u)(P) = \int_{S_n(\Omega)} \mathbb{PI}(\Omega;a,m)(P,Q) u(Q) d\sigma_Q,$$

where $u(Q)$ is a continuous function on $\partial C_n(\Omega)$ and $d\sigma_Q$ is the surface area element on $S_n(\Omega)$.

If $\gamma$ is a real number and $\gamma \geq 0$ (*resp.* $\gamma < 0$), we assume in addition that

$$\iota^+_{[\gamma],k} + \{\gamma\} > -\iota^+_{1,k} + 1,$$

$$\left(\textit{resp.} \; -\iota^+_{[-\gamma],k} - \{-\gamma\} > -\iota^+_{1,k} + 1,\right)$$

$$\iota^+_{[\gamma],k} + \{\gamma\} - n + 1 \leq \iota^+_{m+1,k} < \iota^+_{[\gamma],k} + \{\gamma\} - n + 2.$$

$$\left( \textit{resp.} \quad -\iota^+_{[-\gamma],k} - \{-\gamma\} - n + 1 \leq \iota^+_{m+1,k} < -\iota^+_{[-\gamma],k} - \{-\gamma\} - n + 2. \right)$$

If these conditions all hold, we write $\gamma \in \mathscr{C}(k,m,n)$ (*resp.* $\gamma \in \mathscr{D}(k,m,n)$).

Let $\gamma \in \mathscr{C}(k,m,n)$ (*resp.* $\gamma \in \mathscr{D}(k,m,n)$) and $u$ be functions on $\partial C_n(\Omega)$ satisfying

$$\int_{S_n(\Omega)} \frac{|u(t,\Phi)|}{1+t^{\iota^+_{[\gamma],k}+\{\gamma\}}} d\sigma_Q < \infty. \quad \left( resp. \int_{S_n(\Omega)} |u(t,\Phi)|(1+t^{\iota^+_{[-\gamma],k}+\{-\gamma\}}) d\sigma_Q < \infty. \right) \tag{5}$$

For $\gamma$ and $u$, we define the positive measure $\mu$ (resp. $\nu$) on $\mathbf{R}^n$ by

$$d\mu(Q) = \begin{cases} |u(t,\Phi)| t^{-\iota^+_{[\gamma],k}-\{\gamma\}} d\sigma_Q & Q=(t,\Phi) \in S_n(\Omega;(1,+\infty)), \\ 0 & Q \in \mathbf{R}^n - S_n(\Omega;(1,+\infty)). \end{cases}$$

$$\left( resp.\ d\nu(Q) = \begin{cases} |u(t,\Phi)| t^{\iota^+_{[-\gamma],k}+\{-\gamma\}} d\sigma_Q & Q=(t,\Phi) \in S_n(\Omega;(1,+\infty)), \\ 0 & Q \in \mathbf{R}^n - S_n(\Omega;(1,+\infty)). \end{cases} \right)$$

We remark that the total mass of $\mu$ and $\nu$ are finite.

Let $\epsilon > 0$, $0 \le \zeta \le n$ and $\mu$ be any positive measure on $\mathbf{R}^n$ having finite mass. For each $P=(r,\Theta) \in \mathbf{R}^n - \{O\}$, the maximal function is defined by

$$M(P;\mu,\zeta) = \sup_{0<\rho<\frac{r}{2}} \mu(B(P,\rho)) V_1(\rho) W_1(\rho) \rho^{\zeta-2}.$$

The set

$$\{P=(r,\Theta) \in \mathbf{R}^n - \{O\}; M(P;\mu,\zeta)[V_1(\rho)W_1(\rho)]^{-1}\rho^{2-\zeta} > \epsilon\}$$

is denoted by $E(\epsilon;\mu,\zeta)$.

Recently, Qiao-Deng (cf. [9, Corollary 2.1]) gave the asymptotic behavior of $\mathbb{PI}^0_\Omega(m,u)(P)$ at infinity on cones.

**Theorem A.** If $u$ is a continuous function on $\partial C_n(\Omega)$ satisfying

$$\int_{\partial C_n(\Omega)} \frac{|u(t,\Phi)|}{1+t^{\iota^+_{n,0}+m}} dQ < \infty,$$

then

$$\lim_{r\to\infty, P=(r,\Theta)\in T_n} \mathbb{PI}^0_\Omega(m,u)(P) = o(r^{\iota^+_{m+1,0}}\varphi_1^{1-n}(\Theta)).$$

Now we have

**Theorem 1.** If $\gamma \in \mathscr{C}(k,m,n)$ (*resp.* $\gamma \in \mathscr{D}(k,m,n)$) and $u$ is a measurable function on $\partial C_n(\Omega)$ satisfying (5), then there exists a covering $\{r_j, R_j\}$ $(j=0,1,2,\ldots)$ of $E(\epsilon;\mu,\zeta)$ (*resp.* $E(\epsilon;\nu,\zeta)$) $(\subset C_n(\Omega))$ satisfying

$$\sum_{j=0}^{\infty} \left(\frac{r_j}{R_j}\right)^{2-\zeta} \frac{V_j(R_j)}{V_j(r_j)} \frac{W_j(R_j)}{W_j(r_j)} < \infty \tag{6}$$

such that

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)-E(\epsilon;\mu,\zeta)} r^{-\iota^+_{[\gamma],k}-\{\gamma\}+n-1} \varphi_1^{\zeta-1}(\Theta) \mathbb{PI}^a_\Omega(m,u)(P) = 0. \tag{7}$$

$$\left( resp. \lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)-E(\epsilon;\nu,\zeta)} r^{\iota^+_{[-\gamma],k}+\{-\gamma\}+n-1} \varphi_1^{\zeta-1}(\Theta) \mathbb{PI}^a_\Omega(m,u)(P) = 0. \right) \tag{8}$$

**Remark.** In the case that $a = 0$, $\gamma = n + m$ and $\zeta = n$, then (6) is a finite sum, the set $E(\epsilon; \mu, n)$ is a bounded set and (7)-(8) hold in $C_n(\Omega)$. This is just the result of Theorem A.

As an application of modified Green-Sch kernel function and Theorem 1, we give the solutions of the Dirichlet problem for the Schrödinger operator on $C_n(\Omega)$.

**Theorem 2.** If $u$ is a continuous function on $\partial C_n(\Omega)$ satisfying

$$\int_{S_n(\Omega)} \frac{|u(t,\Phi)|}{1 + V_{m+1}(t)t^{n-1}} d\sigma_Q < \infty, \tag{9}$$

then the function $\mathbb{PI}^a_\Omega(m,u)(P)$ satisfies

$$\mathbb{PI}^a_\Omega(m,u) \in C^2(C_n(\Omega)) \cap C^0(\overline{C_n(\Omega)}),$$

$$Sch_a \mathbb{PI}^a_\Omega(m,u) = 0 \quad \text{in } C_n(\Omega),$$

$$\mathbb{PI}^a_\Omega(m,u) = u \qquad \text{on } \partial C_n(\Omega)$$

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,k}} \varphi_1^{n-1}(\Theta) \mathbb{PI}^a_\Omega(m,u)(P) = 0.$$

## 2. Lemmas

Throughout this paper, Let $M$ denote various constants independent of the variables in questions, which may be different from line to line.

**Lemma 1.**

$$\text{(i)}\mathbb{PI}(\Omega;a)(P,Q) \le M r^{\iota^-_{1,k}} t^{\iota^+_{1,k}-1} \varphi_1(\Theta)$$

$$\text{(ii)}(\textit{resp.}\ \mathbb{PI}(\Omega;a)(P,Q) \le M r^{\iota^+_{1,k}} t^{\iota^-_{1,k}-1} \varphi_1(\Theta))$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega)$ satisfying $0 < \frac{t}{r} \le \frac{4}{5}$ (*resp.* $0 < \frac{r}{t} \le \frac{4}{5}$);

$$\text{(iii)}\mathbb{PI}(\Omega;0)(P,Q) \le M\frac{\varphi_1(\Theta)}{t^{n-1}} + M\frac{r\varphi_1(\Theta)}{|P-Q|^n}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega; (\frac{4}{5}r, \frac{5}{4}r))$.

**Proof.** (i) and (ii) are obtained by B. Levin (see [3, Ch. 11]). (iii) follows from V. S. Azarin (see [2, Lemma 4 and Remark]).

**Lemma 2** (see [3, p. 356] ). For a non-negative integer $m$, we have

$$|\mathbb{PI}(\Omega;a,m)(P,Q)| \le M(n,m,s)V_{m+1}(r)\frac{W_{m+1}(t)}{t}\varphi_1(\Theta)\frac{\partial\varphi_1(\Phi)}{\partial n_\Phi} \tag{10}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and $Q = (t,\Phi) \in S_n(\Omega)$ satisfying $r \le st$ $(0 < s < 1)$, where $M(n,m,s)$ is a constant dependent of $n$, $m$ and $s$.

**Lemma 3.** Let $\mu$ be any positive measure on $\mathbf{R}^n$ having finite total mass. Then $E(\epsilon;\mu,\zeta)$ has a covering $\{r_j, R_j\}$ $(j = 1, 2, \dots)$ satisfying

$$\sum_{j=1}^{\infty}\left(\frac{r_j}{R_j}\right)^{2-\zeta}\frac{V_j(R_j)}{V_j(r_j)}\frac{W_j(R_j)}{W_j(r_j)} < \infty.$$

**Proof.** Set

$$E_j(\epsilon;\mu,\zeta) = E(\epsilon;\mu,\zeta)\cap C_n(\Omega;[2^j,2^{j+1})) \qquad (j = 2,3,4,\dots).$$

If $P = (r,\Theta) \in E_j(\epsilon;\mu,\zeta)$, then there exists a positive number $\rho(P)$ such that

$$\left(\frac{\rho(P)}{r}\right)^{2-\zeta}\frac{V_j(r)}{V_j(\rho(P))}\frac{W_j(r)}{W_j(\rho(P))} \sim \left(\frac{\rho(P)}{r}\right)^{n-\zeta} \le \frac{\mu(B(P,\rho(P)))}{\epsilon}.$$

Here $E_j(\epsilon;\mu,\zeta)$ can be covered by the union of a family of balls $(B(P_{j,i},\rho_{j,i}) : P_{j,i} \in E_j(\epsilon;\mu,n-\zeta))$ $(\rho_{j,i} = \rho(P_{j,i}))$. By the Vitali Lemma (see [16]), there exists $\Lambda_j \subset E_j(\epsilon;\mu,n-\zeta)$, which is at most countable, such that $(B(P_{j,i},\rho_{j,i}) : P_{j,i} \in \Lambda_j)$ are disjoint and $E_j(\epsilon;\mu,\zeta) \subset \cup_{P_{j,i}\in\Lambda_j} B(P_{j,i}, 5\rho_{j,i})$.

So

$$\cup_{j=2}^{\infty} E_j(\epsilon;\mu,\zeta) \subset \cup_{j=2}^{\infty}\cup_{P_{j,i}\in\Lambda_j} B(P_{j,i},5\rho_{j,i}).$$

On the other hand, note that $\cup_{P_{j,i}\in\Lambda_j} B(P_{j,i},\rho_{j,i}) \subset C_n(\Omega;[2^{j-1},2^{j+2}))$, so that

$$\begin{aligned}\sum_{P_{j,i}\in\Lambda_j}\left(\frac{5\rho_{j,i}}{|P_{j,i}|}\right)^{2-\zeta}\frac{V_j(|P_{j,i}|)}{V_j(5\rho_{j,i})}\frac{W_j(|P_{j,i}|)}{W_j(5\rho_{j,i})} &\sim \sum_{P_{j,i}\in\Lambda_j}\left(\frac{5\rho_{j,i}}{|P_{j,i}|}\right)^{n-\zeta}\\ &\le 5^{n-\zeta}\sum_{P_{j,i}\in\Lambda_j}\frac{\mu(B(P_{j,i},\rho_{j,i}))}{\epsilon}\\ &\le \frac{5^{n-\zeta}}{\epsilon}\mu(C_n(\Omega;[2^{j-1},2^{j+2}))).\end{aligned}$$

Hence we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{P_{j,i}\in\Lambda_j}\left(\frac{\rho_{j,i}}{|P_{j,i}|}\right)^{2-\zeta}\frac{V_j(|P_{j,i}|)}{V_j(\rho_{j,i})}\frac{W_j(|P_{j,i}|)}{W_j(\rho_{j,i})} &\sim \sum_{j=1}^{\infty}\sum_{P_{j,i}\in\Lambda_j}\left(\frac{\rho_{j,i}}{|P_{j,i}|}\right)^{n-\zeta}\\ &\le \sum_{j=1}^{\infty}\frac{\mu(C_n(\Omega;[2^{j-1},2^{j+2})))}{\epsilon}\\ &\le \frac{3\mu(\mathbf{R}^n)}{\epsilon}.\end{aligned}$$

Since $E(\epsilon;\mu,\zeta)\cap\{P = (r,\Theta) \in \mathbf{R}^n; r \ge 4\} = \cup_{j=2}^{\infty} E_j(\epsilon;\mu,\zeta)$. Then $E(\epsilon;\mu,\zeta)$ is finally covered by a sequence of balls $(B(P_{j,i},\rho_{j,i}), B(P_1,6))$ $(j = 2,3,\dots; i = 1,2,\dots)$ satisfying

$$\sum_{j,i}\left(\frac{\rho_{j,i}}{|P_{j,i}|}\right)^{2-\zeta}\frac{V_j(|P_{j,i}|)}{V_j(\rho_{j,i})}\frac{W_j(|P_{j,i}|)}{W_j(\rho_{j,i})} \sim \sum_{j,i}\left(\frac{\rho_{j,i}}{|P_{j,i}|}\right)^{n-\zeta} \le \frac{3\mu(\mathbf{R}^n)}{\epsilon} + 6^{n-\zeta} < +\infty,$$

where $B(P_1,6)$ $(P_1 = (1,0,\dots,0) \in \mathbf{R}^n)$ is the ball which covers $\{P = (r,\Theta) \in \mathbf{R}^n; r < 4\}$.

## 3. Proof of Theorem 1

We only prove the case $p>1$ and $\gamma\geq 0$, the remaining cases can be proved similarly.
For any $\epsilon>0$, there exists $R_\epsilon>1$ such that

$$\int_{S_n(\Omega;(R_\epsilon,\infty))}\frac{|u(Q)|}{1+t^{\iota^+_{[\gamma],k}+\{\gamma\}}}d\sigma_Q<\epsilon. \tag{11}$$

The relation $G(\Omega;a)(P,Q)\leq G(\Omega;0)(P,Q)$ implies this inequality (see [1])

$$\mathbb{PI}(\Omega;a)(P,Q)\leq\mathbb{PI}(\Omega;0)(P,Q). \tag{12}$$

For $0<s<\frac{4}{5}$ and any fixed point $P=(r,\Theta)\in C_n(\Omega)-E(\epsilon;\mu,\zeta)$ satisfying $r>\frac{5}{4}R_\epsilon$, let $I_1=S_n(\Omega;(0,1))$, $I_2=S_n(\Omega;[1,R_\epsilon])$, $I_3=S_n(\Omega;(R_\epsilon,\frac{4}{5}r])$, $I_4=S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$, $I_5=S_n(\Omega;[\frac{5}{4}r,\frac{r}{s}))$, $I_6=S_n(\Omega;[\frac{r}{s},\infty))$ and $I_7=S_n(\Omega;[1,\frac{r}{s}))$, we write

$$\begin{aligned}\mathbb{PI}^a_\Omega(m,u)(P) &= \sum_{i=1}^{6}\int_{I_i}\mathbb{PI}(\Omega;a,m)(P,Q)u(Q)d\sigma_Q\\ &= \sum_{i=1}^{5}\int_{I_i}\mathbb{PI}(\Omega;a)(P,Q)u(Q)d\sigma_Q-\int_{I_7}\frac{\partial\widetilde{K}(\Omega;a,m)(P,Q)}{\partial n_Q}u(Q)d\sigma_Q\\ &+ \int_{I_6}\mathbb{PI}(\Omega;a,m)(P,Q)u(Q)d\sigma_Q,\end{aligned}$$

which yields that

$$\mathbb{PI}^a_\Omega(m,u)(P)\leq\sum_{i=1}^{7}U_i(P),$$

where

$$U_i(P)=\int_{I_i}|\mathbb{PI}(\Omega;a)(P,Q)||u(Q)|d\sigma_Q\quad(i=1,2,3,4,5),$$

$$U_6(P)=\int_{I_6}|\mathbb{PI}(\Omega;a,m)(P,Q)||u(Q)|d\sigma_Q,$$

$$U_7(P)=\int_{I_7}|\frac{\partial\widetilde{K}(\Omega;a,m)(P,Q)}{\partial n_Q}||u(Q)|d\sigma_Q.$$

By (5), (11), Lemma 1 (i) and Hölder's inequality, we have the following growth estimates

$$\begin{aligned}U_2(P) &\leq Mr^{\iota^-_{1,k}}\varphi_1(\Theta)\int_{I_2}t^{\iota^+_{1,k}-1}|u(Q)|d\sigma_Q\\ &\leq Mr^{\iota^-_{1,k}}R_\epsilon^{\iota^+_{1,k}+\iota^+_{[\gamma],k}+\{\gamma\}-1}\varphi_1(\Theta).\end{aligned} \tag{13}$$

$$U_1(P)\leq Mr^{\iota^-_{1,k}}\varphi_1(\Theta). \tag{14}$$

$$U_3(P)\leq M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta). \tag{15}$$

We obtain by (11), Lemma 1 (ii) and Hölder's inequality

$$\begin{aligned} U_5(P) &\le Mr^{\iota_{1,k}^{+}}\varphi_1(\Theta)\int_{S_n(\Omega;[\frac{5}{4}r,\infty))} t^{\iota_{1,k}^{-}-1}|u(Q)|d\sigma_Q \\ &\le M\epsilon r^{\iota_{[\gamma],k}^{+}+\{\gamma\}-n+1}\varphi_1(\Theta). \end{aligned} \tag{16}$$

By (12) and Lemma 1 (iii), we consider the inequality

$$U_4(P) \le U_4'(P) + U_4''(P),$$

where

$$U_4'(P) = M\varphi_1(\Theta)\int_{I_4} t^{1-n}|u(Q)|d\sigma_Q, \quad U_4''(P) = Mr\varphi_1(\Theta)\int_{I_4}\frac{|u(Q)|}{|P-Q|^n}d\sigma_Q.$$

We first have

$$\begin{aligned} U_4'(P) &= M\varphi_1(\Theta)\int_{I_4} t^{\iota_{1,k}^{+}+\iota_{1,k}^{-}-1}|u(Q)|d\sigma_Q \\ &\le Mr^{\iota_{1,k}^{+}}\varphi_1(\Theta)\int_{S_n(\Omega;(\frac{4}{5}r,\infty))} t^{\iota_{1,k}^{-}-1}|u(Q)|d\sigma_Q \\ &\le M\epsilon r^{\iota_{[\gamma],k}^{+}+\{\gamma\}-n+1}\varphi_1(\Theta), \end{aligned}$$

which is similar to the estimate of $U_5(P)$.

Next, we shall estimate $U_4''(P)$.

Take a sufficiently small positive number $d_3$ such that $I_4 \subset B(P, \frac{1}{2}r)$ for any $P = (r,\Theta) \in \Pi(d_3)$, where

$$\Pi(d_3) = \{P = (r,\Theta) \in C_n(\Omega);\ \inf_{z\in\partial\Omega}|(1,\Theta)-(1,z)| < d_3,\ 0 < r < \infty\}.$$

and divide $C_n(\Omega)$ into two sets $\Pi(d_3)$ and $C_n(\Omega) - \Pi(d_3)$.

If $P = (r,\Theta) \in C_n(\Omega) - \Pi(d_3)$, then there exists a positive $d_3'$ such that $|P-Q| \ge d_3' r$ for any $Q \in S_n(\Omega)$, and hence

$$\begin{aligned} U_4''(P) &\le M\varphi_1(\Theta)\int_{I_4} t^{1-n}|u(Q)|d\sigma_Q \\ &\le M\epsilon r^{\iota_{[\gamma],k}^{+}+\{\gamma\}-n+1}\varphi_1(\Theta), \end{aligned}$$

which is similar to the estimate of $U_4'(P)$.

We shall consider the case $P = (r,\Theta) \in \Pi(d_3)$. Now put

$$H_i(P) = \{Q \in I_4;\ 2^{i-1}\delta(P) \le |P-Q| < 2^i\delta(P)\},$$

where $\delta(P) = \inf_{Q\in\partial C_n(\Omega)}|P-Q|$.

Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n : |P-Q| < \delta(P)\} = \varnothing$, we have

$$U_4''(P) = M\sum_{i=1}^{i(P)}\int_{H_i(P)} r\varphi_1(\Theta)\frac{|u(Q)|}{|P-Q|^n}d\sigma_Q,$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \le \frac{r}{2} < 2^{i(P)}\delta(P)$.

Since $r\varphi_1(\Theta) \leq M\delta(P)$ for any $P = (r, \Theta) \in C_n(\Omega)$ (see [10, 11]), similar to the estimate of $U_4'(P)$, we obtain

$$\begin{aligned} &\int_{H_i(P)} r\varphi_1(\Theta)\frac{|u(Q)|}{|P-Q|^n}d\sigma_Q \\ \leq\ & 2^{(1-i)n}\varphi_1(\Theta)\delta(P)^{\zeta-n}\int_{H_i(P)}\delta(P)^{\zeta-n}|u(Q)|d\sigma_Q \\ \leq\ & M\varphi_1^{1-\zeta}(\Theta)\delta(P)^{\zeta-n}\int_{H_i(P)} r^{1-\zeta}|u(Q)|d\sigma_Q \\ \leq\ & Mr^{n-\frac{\zeta}{p}}\varphi_1^{1-\zeta}(\Theta)\delta(P)^{\zeta-n}\int_{H_i(P)} t^{1-n}|u(Q)|d\sigma_Q \\ \leq\ & M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n-\zeta+1}{p}+n}\varphi_1^{1-\zeta}(\Theta)\frac{\mu(H_i(P))}{(2^i\delta(P))^{\zeta}} \end{aligned}$$

for $i = 0, 1, 2, \ldots, i(P)$.

Since $P = (r, \Theta) \notin E(\epsilon; \mu, \zeta)$, we have from (3)

$$\begin{aligned} \frac{\mu(H_i(P))}{\{2^i\delta(P)\}^{n-\zeta}} &\leq M\mu(B(P, 2^i\delta(P)))[V_1(2^i\delta(P))W_1(2^i\delta(P))]^p[2^i\delta(P)]^{\zeta-2} \\ &\leq MM(P; \mu, \zeta) \\ &\leq \epsilon[V_1(r)W_1(r)]^p r^{\zeta-2} \\ &\leq \epsilon r^{\zeta-n} \quad (i = 0, 1, 2, \ldots, i(P)-1). \end{aligned}$$

and

$$\frac{\mu(H_{i(P)}(P))}{\{2^i\delta(P)\}^{n-\zeta}} \leq M\mu(B(P, \frac{r}{2}))[V_1(\frac{r}{2})W_1(\frac{r}{2})]^p\left(\frac{r}{2}\right)^{\zeta-2} \leq \epsilon r^{\zeta-n}.$$

So

$$U_4''(P) \leq M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1^{1-\zeta}(\Theta). \tag{17}$$

We only consider $U_7(P)$ in the case $m \geq 1$, since $U_7(P) \equiv 0$ for $m = 0$. By the definition of $\widetilde{K}(\Omega; a, m)$, (1) and Lemma 2, we see

$$U_7(P) \leq \frac{M}{\chi'(1)}\sum_{j=0}^{m} j^{2n-1}q_j(r),$$

where

$$q_j(r) = V_j(r)\varphi_1(\Theta)\int_{I_7}\frac{W_j(t)|u(Q)|}{t}d\sigma_Q.$$

To estimate $q_j(r)$, we write

$$q_j(r) \leq q_j'(r) + q_j''(r),$$

where

$$q_j'(r) = V_j(r)\varphi_1(\Theta)\int_{I_2}\frac{W_j(t)|u(Q)|}{t}d\sigma_Q,\ q_j''(r) = V_j(r)\varphi_1(\Theta)\int_{S_n(\Omega;(R_\epsilon,\frac{r}{s}))}\frac{W_j(t)|u(Q)|}{t}d\sigma_Q.$$

If $\iota^+_{m+1,k} < \iota^+_{[\gamma],k} + \{\gamma\} - n + 1 + 1$, then $(-\iota^+_{m+1,k} - n + 2 + \frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q + n - 1 > 0$. Notice that

$$V_j(r)\frac{V_{m+1}(t)}{V_j(t)t} \leq M\frac{V_{m+1}(r)}{r} \leq Mr^{\iota^+_{m+1,k}-1} \quad (t \geq 1, R_\epsilon < \frac{r}{s}).$$

Thus, by (3), (5) and Hölder's inequality we conclude

$$\begin{aligned} q'_j(r) &= V_j(r)\varphi_1(\Theta)\int_{I_2}\frac{|u(Q)|}{V_j(t)t^{n-1}}d\sigma_Q \\ &\leq MV_j(r)\varphi_1(\Theta)\int_{I_2}\frac{V_{m+1}(t)}{t^{\iota^+_{m+1,k}}}\frac{|u(Q)|}{V_j(t)t^{n-1}}d\sigma_Q \\ &\leq Mr^{\iota^+_{m+1,k}-1}R_\epsilon^{-\iota^+_{m+1,k}+1+\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta). \end{aligned}$$

Analogous to the estimate of $q'_j(r)$, we have

$$q''_j(r) \leq M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta).$$

Thus we can conclude that

$$q_j(r) \leq M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta),$$

which yields

$$U_7(P) \leq M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta). \tag{18}$$

By (11), Lemma 2 and Hölder's inequality we have

$$\begin{aligned} U_6(P) &\leq MV_{m+1}(r)\varphi_1(\Theta)\int_{I_6}\frac{|u(Q)|}{V_{m+1}(t)t^{n-1}}d\sigma_Q \\ &\leq MV_{m+1}(r)\varphi_1(\Theta)\Big(\int_{I_6}\frac{|u(Q)|}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}}d\sigma_Q\Big)^{\frac{1}{p}}\Big(\int_{I_6}t^{(-\iota^+_{m+1,k}-n+1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q}d\sigma_Q\Big)^{\frac{1}{q}} \\ &\leq M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta). \end{aligned} \tag{19}$$

Combining (13)-(19), we obtain that if $R_\epsilon$ is sufficiently large and $\epsilon$ is sufficiently small, then $\mathbb{PI}^a_\Omega(m,u)(P) = o(r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1^{1-\zeta}(\Theta))$ as $r \to \infty$, where $P = (r,\Theta) \in C_n(\Omega;(R_\epsilon,+\infty)) - E(\epsilon;\mu,\zeta)$. Finally, there exists an additional finite ball $B_0$ covering $C_n(\Omega;(0,R_\epsilon])$, which together with Lemma 3, gives the conclusion of Theorem 1.

## 4. Proof of Theorem 2

For any fixed $P = (r,\Theta) \in C_n(\Omega)$, take a number satisfying $R > \max(1,\frac{r}{s})$ $(0 < s < \frac{4}{5})$. By (9) and Lemma 2, we have

$$\begin{aligned} &\int_{S_n(\Omega;(R,\infty))}|\mathbb{PI}(\Omega;a,m)(P,Q)||u(Q)|d\sigma_Q \\ \leq\ & V_{m+1}(r)\varphi_1(\Theta)\int_{S_n(\Omega;(R,\infty))}\frac{|u(Q)|}{V_{m+1}(t)t^{n-1}}d\sigma_Q \\ \leq\ & MV_{m+1}(r)\varphi_1(\Theta) \\ <\ & \infty. \end{aligned}$$

Then $\mathbb{PI}^a_\Omega(m,u)(P)$ is absolutely convergent and finite for any $P \in C_n(\Omega)$. Thus $\mathbb{PI}^a_\Omega(m,u)(P)$ is a generalized harmonic function on $C_n(\Omega)$.

Now we study the boundary behavior of $\mathbb{PI}^a_\Omega(m,u)(P)$. Let $Q' = (t', \Phi') \in \partial C_n(\Omega)$ be any fixed point and $l$ be any positive number satisfying $l > \max(t'+1, \frac{4}{5}R)$.

Set $\chi_{S(l)}$ is the characteristic function of $S(l) = \{Q = (t, \Phi) \in \partial C_n(\Omega), t \le l\}$ and write

$$\begin{aligned}\mathbb{PI}^a_\Omega(m,u)(P) &= \Big(\int_{S_n(\Omega;(0,1)} + \int_{S_n(\Omega;[1,\frac{5}{4}l])} + \int_{S_n(\Omega;(\frac{5}{4}l,\infty))}\Big)\mathbb{PI}(\Omega;a,m)(P,Q)u(Q)d\sigma_Q \\ &= U'(P) - U''(P) + U'''(P),\end{aligned}$$

where

$$U'(P) = \int_{S_n(\Omega;(0,\frac{5}{4}l])} \mathbb{PI}(\Omega;a)(P,Q)u(Q)d\sigma_Q,\ U''(P) = \int_{S_n(\Omega;[1,\frac{5}{4}l])} \frac{\partial K(\Omega;a,m)(P,Q)}{\partial n_Q} u(Q)d\sigma_Q,$$

$$U'''(P) = \int_{S_n(\Omega;(\frac{5}{4}l,\infty))} \mathbb{PI}(\Omega;a,m)(P,Q)u(Q)d\sigma_Q.$$

Notice that $U'(P)$ is the Poisson $a$-integral of $u(Q)\chi_{S(\frac{5}{4}l)}$, we have $\lim_{P\to Q', P\in C_n(\Omega)} U'(P) = u(Q')$. Since $\lim_{\Theta\to\Phi'} \varphi_{jv}(\Theta) = 0\ (j = 1,2,3\ldots; 1 \le v \le v_j)$ as $P = (r,\Theta) \to Q' = (t', \Phi') \in S_n(\Omega)$, we have $\lim_{P\to Q', P\in C_n(\Omega)} U''(P) = 0$ from the definition of the kernel function $K(\Omega;a,m)(P,Q)$. $U'''(P) = O(V_{m+1}(r)\varphi_1(\Theta))$ and therefore tends to zero.

So the function $\mathbb{PI}^a_\Omega(m,u)(P)$ can be continuously extended to $\overline{C_n(\Omega)}$ such that

$$\lim_{P\to Q', P\in C_n(\Omega)} \mathbb{PI}^a_\Omega(m,u)(P) = u(Q')$$

for any $Q' = (t', \Phi') \in \partial C_n(\Omega)$ from the arbitrariness of $l$, which with Theorem 1 gives the conclusion of Theorem 2.

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