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A Pair of Fractional Powers of Hankel-Clifford Transformations of Arbitrary Order

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Abstract. The main objective of this paper is to extend a pair of fractional powers of Hankel-Clifford transformations to arbitrary values of ν. Moreover, we obtain some interesting results for these extension. To illustrate some problems are given.

1. Introduction

Prasad *et al.* [9] introduced a pair of fractional powers of α $(0 < \alpha < \pi)$ of Hankel-Clifford transformations of order $\nu \geq 0$ depending on an arbitrary real parameter μ, which is a generalization of a pair of Hankel-Clifford transformations [1, 6, 7]. In this work the fractional powers of first Hankel-Clifford transformation is defined as:

$$(h^{\alpha}_{1,\nu,\mu}f)(y) = \hat{f}^{\alpha}_{1,\nu,\mu}(y) = \int_0^{\infty} K^{\alpha}_1(x,y)f(x)dx, \tag{1}$$

where,

$$K^{\alpha}_1(x,y) = \begin{cases} \gamma^{\alpha}_{\nu,\mu} C_{\nu,\mu}(xy\csc^2\alpha)e^{i(x+y)\cot\alpha}y^{\mu}, & \alpha \neq n\pi, \\ C_{\nu,\mu}(xy)y^{\mu}, & \alpha = \frac{\pi}{2}, \\ \delta(x-y), & \alpha = n\pi, \end{cases} \tag{2}$$

where $n \in \mathbb{Z}$, $\gamma^{\alpha}_{\nu,\mu} = \frac{e^{i(\nu+1)(\alpha-\frac{\pi}{2})}}{(\sin\alpha)^{\mu+1}}$, $C_{\nu,\mu}(x) = x^{-\mu/2}J_{\nu}(2\sqrt{x})$ and J_{ν} is the Bessel function of first kind of order ν.

Analogously, the fractional powers of α $(0 < \alpha < \pi)$ of the second Hankel-Clifford transformation is defined by:

$$(h^{\alpha}_{2,\nu,\mu}g)(y) = \tilde{g}^{\alpha}_{2,\nu,\mu}(y) = \int_0^{\infty} K^{\alpha}_2(x,y)g(x)dx = y^{-\mu}(h^{\alpha}_{1,\nu,\mu}(x^{\mu}g))(y), \tag{3}$$

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where,

$$K_2^\alpha(x,y)=\begin{cases}\gamma^\alpha_{\nu,\mu}C_{\nu,\mu}(xy\csc^2\alpha)e^{i(x+y)\cot\alpha}x^\mu, & \alpha\neq n\pi,\\ C_{\nu,\mu}(xy)x^\mu, & \alpha=\frac{\pi}{2},\\ \delta(x-y), & \alpha=n\pi,\end{cases}\tag{4}$$

where n and $\gamma^\alpha_{\nu,\mu}$ as above.

The fractional powers of first and second Hankel-Clifford transformation are reduced to a pair of Hankel-Clifford transformation [1, 7, 10] by choosing $\nu=\mu$ and $\alpha=\pi/2$. The first and the second Hankel-Clifford (or fractional Hankel-Clifford) transformations are adjoint of each other.

For $\mu=0$ and $\alpha=\pi/2$, the transformations defined in (1) and (3) coincide and is denoted by h_ν and for $\varphi\in L^1_\nu(I)$, it is defined by

$$(h_\nu\varphi)(y)=\hat{\varphi}_\nu(y)=\int_0^\infty J_\nu(2\sqrt{xy})\varphi(x)dx,\quad 0<y<\infty,$$

which is adjoint of itself. Hence h_ν is known as Hankel-Clifford transformation.

The inverse of (1) and (3) respectively are defined as follows:

$$f(x)=\left((h^\alpha_{1,\nu,\mu})^{-1}\hat{f}^\alpha_{1,\nu,\mu}\right)(x)=\int_0^\infty K_1^{*\alpha}(y,x)\hat{f}^\alpha_{1,\nu,\mu}(y)dy\tag{5}$$

and

$$g(x)=\left((h^\alpha_{2,\nu,\mu})^{-1}\tilde{g}^\alpha_{2,\nu,\mu}\right)(x)=\int_0^\infty K_2^{*\alpha}(y,x)\tilde{g}^\alpha_{2,\nu,\mu}(y)dy,\tag{6}$$

where $K_1^{*\alpha}(y,x)$ and $K_2^{*\alpha}(y,x)$ are same as $K_1^{-\alpha}(y,x)$ and $K_2^{-\alpha}(y,x)$ respectively. Throughout this paper we denote complex conjugate by '$*$'. We note that $(h^{\pi/2}_{1,\nu,\mu})^{-1}=h^{\pi/2}_{1,\nu,\mu}$ and $(h^{\pi/2}_{2,\nu,\mu})^{-1}=h^{\pi/2}_{2,\nu,\mu}$.

We shall need the following operational formulas [7],

$$D_x^rC_\mu(x)=(-1)^rC_{\mu+r}(x),\tag{7}$$

$$D_x^r[x^{\mu+r}C_{\mu+r}(x)]=x^\mu C_\mu(x),\quad\forall\, r\in\mathbb{N}_0,\tag{8}$$

where $C_\mu(x)=x^{-\mu/2}J_\mu(2\sqrt{x})$.

We have the following differential and integral operators [9]:

$$R_{1,\nu,\mu,\alpha}=e^{ix\cot\alpha}x^{\frac{\mu+\nu+1}{2}}D_xx^{-\frac{(\mu+\nu)}{2}}e^{-ix\cot\alpha},\quad D_x=\frac{d}{dx},\tag{9}$$

$$S_{1,\nu,\mu,\alpha}=e^{ix\cot\alpha}x^{\frac{\mu-\nu}{2}}D_xx^{\frac{\nu-\mu+1}{2}}e^{-ix\cot\alpha},\tag{10}$$

$$\begin{aligned}\Delta_{1,\nu,\mu,\alpha}&=S_{1,\nu,\mu,\alpha}R_{1,\nu,\mu,\alpha}\\&=e^{ix\cot\alpha}x^{\frac{\mu-\nu}{2}}D_xx^{\nu+1}D_xx^{-\frac{(\mu+\nu)}{2}}e^{-ix\cot\alpha}\\&=xD_x^2+\left[(1-\mu)-2ix\cot\alpha\right]D_x-\left[(1-\mu)i\cot\alpha+x\cot^2\alpha+\frac{\nu^2-\mu^2}{4x}\right],\end{aligned}\tag{11}$$

$$R^{-1}_{1,\nu,\mu,\alpha}\varphi(x)=e^{ix\cot\alpha}x^{\frac{\mu+\nu}{2}}\int_\infty^x x_1^{-\frac{(\mu+\nu+1)}{2}}e^{-ix_1\cot\alpha}\varphi(x_1)dx_1,\tag{12}$$

$$R_{2,\nu,\mu,\alpha}=-e^{ix\cot\alpha}x^{-\frac{(\mu+\nu)}{2}}D_xx^{\frac{(\mu+\nu+1)}{2}}e^{-ix\cot\alpha},\tag{13}$$

$$S_{2,\nu,\mu,\alpha}=-e^{ix\cot\alpha}x^{\frac{\nu-\mu+1}{2}}D_xx^{\frac{\mu-\nu}{2}}e^{-ix\cot\alpha},\tag{14}$$

$$\begin{aligned}\Delta_{2,\nu,\mu,\alpha}&=R_{2,\nu,\mu,\alpha}S_{2,\nu,\mu,\alpha}\\&=e^{ix\cot\alpha}x^{-\frac{(\mu+\nu)}{2}}D_xx^{\nu+1}D_xx^{\frac{\mu-\nu}{2}}e^{-ix\cot\alpha}\\&=xD_x^2+\left[(1+\mu)-2ix\cot\alpha\right]D_x-\left[(1+\mu)i\cot\alpha+x\cot^2\alpha+\frac{\nu^2-\mu^2}{4x}\right],\end{aligned}\tag{15}$$

$$S^{-1}_{2,\nu,\mu,\alpha}\varphi(x)=e^{ix\cot\alpha}x^{\frac{\nu-\mu}{2}}\int_x^\infty x_1^{-\frac{(-\mu+\nu+1)}{2}}e^{-ix_1\cot\alpha}\varphi(x_1)dx_1.\tag{16}$$

From (12) and (16) respectively, we have

$$R^{-1}_{1,\nu,\mu,\alpha}...R^{-1}_{1,\nu+m-1,\mu,\alpha}\varphi(x) = e^{ix\cot\alpha}x^{\frac{\mu+\nu}{2}}\int_{\infty}^{x}...\int_{\infty}^{x_{m-1}} x_m^{-\frac{(\mu+\nu+m)}{2}}e^{-ix_m\cot\alpha}\varphi(x_m)dx_m...dx_1, \quad (17)$$

$$S^{-1}_{2,\nu,\mu,\alpha}...S^{-1}_{2,\nu+m-1,\mu,\alpha}\varphi(x) = (-1)^m e^{ix\cot\alpha}x^{\frac{\nu-\mu}{2}}\int_{\infty}^{x}...\int_{\infty}^{x_{m-1}} x_m^{-\frac{(\nu-\mu+m)}{2}}e^{-ix_m\cot\alpha}\varphi(x_m)dx_m...dx_1. \quad (18)$$

We observe that $\Delta^*_{1,\nu,\mu,\alpha}$ and $\Delta^*_{2,\nu,\mu,\alpha}$ are adjoint of $\Delta_{2,\nu,\mu,\alpha}$ and $\Delta_{1,\nu,\mu,\alpha}$ respectively.

1.1. *The spaces $\mathcal{H}^\alpha_{1,\nu,\mu}(I)$ and $\mathcal{H}^\alpha_{2,\nu,\mu}(I)$ and their dual*

A complex valued C^∞-function φ defined on $I = (0,\infty)$ is in $\mathcal{H}^\alpha_{1,\nu,\mu}(I)$ if

$$\Upsilon^{1,\nu,\mu}_{q,k,\alpha}(\varphi) = \sup_{x\in I}\left|x^q D^k_x x^{-\frac{(\mu+\nu)}{2}}e^{-ix\cot\alpha}\varphi(x)\right| = \sup_{x\in I}\left|x^q D^k_x x^{-\frac{(\mu+\nu)}{2}}e^{ix\cot\alpha}\varphi(x)\right| < \infty, \quad (19)$$

for each pair of non-negative integers q and k. The topology over $\mathcal{H}^\alpha_{1,\nu,\mu}(I)$ is generated by the family $\{\Upsilon^{1,\nu,\mu}_{q,k,\alpha}\}_{q,k\in\mathbb{N}_0}$ of semi-norms.

On the other hand, $\mathcal{H}^\alpha_{2,\nu,\mu}(I)$ consists of all complex valued C^∞-functions ψ defined on I which satisfies

$$\Upsilon^{2,\nu,\mu}_{q,k,\alpha}(\psi) = \sup_{x\in I}\left|x^q D^k_x x^{\frac{(\mu-\nu)}{2}}e^{-ix\cot\alpha}\psi(x)\right| = \sup_{x\in I}\left|x^q D^k_x x^{\frac{(\mu-\nu)}{2}}e^{ix\cot\alpha}\psi(x)\right| < \infty, \quad (20)$$

for each pair of non-negative integers q and k. The topology over $\mathcal{H}^\alpha_{2,\nu,\mu}(I)$ is generated by the family $\{\Upsilon^{2,\nu,\mu}_{q,k,\alpha}\}_{q,k\in\mathbb{N}_0}$ of semi-norms.

Also, $(\mathcal{H}^\alpha_{1,\nu,\mu})'(I)$ and $(\mathcal{H}^\alpha_{2,\nu,\mu})'(I)$ represent the dual of $\mathcal{H}^\alpha_{1,\nu,\mu}(I)$ and $\mathcal{H}^\alpha_{2,\nu,\mu}(I)$ respectively and their members are generalized functions of slow growth. Hence, $(\mathcal{H}^\alpha_{1,\nu,\mu})'(I)$ and $(\mathcal{H}^\alpha_{2,\nu,\mu})'(I)$ are too complete.

Main goal of this paper is to define a pair of fractional powers of Hankel-Clifford transformations for all real values of the order ν and real parameter μ and α $(0 < \alpha < \pi)$ according to the method developed in [4, 5, 8, 11] for Hankel transformations.

2. Fractional Powers of First Hankel-Clifford Transformation of Arbitrary Order

Let ν, μ be any real numbers and α $(0 < \alpha < \pi)$ and m be a positive integer such that $\nu + \mu + m \geq 0$. We define the extended fractional powers of first Hankel-Clifford transformation $h^\alpha_{1,\nu,\mu,m}$ of any $\varphi \in \mathcal{H}^\alpha_{1,\nu,\mu}(I)$ by

$$\Phi(y) = \left(h^\alpha_{1,\nu,\mu,m}\varphi\right)(y) = (-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}\left[h^\alpha_{1,\nu+m,\mu}(R^*_{1,\nu+m-1,\mu,\alpha}...R^*_{1,\nu,\mu,\alpha}\varphi)\right](y). \quad (21)$$

The inverse transformation $(h^\alpha_{1,\nu,\mu,m})^{-1}$ of any $\Phi \in \mathcal{H}^\alpha_{1,\nu,\mu}(I)$ is defined by

$$\varphi(x) = \left((h^\alpha_{1,\nu,\mu,m})^{-1}\Phi(y)\right)(x) = (-1)^m e^{i(\alpha-\frac{\pi}{2})m}R^{*^{-1}}_{1,\nu,\mu,\alpha}...R^{*^{-1}}_{1,\nu+m-1,\mu,\alpha}\left[(h^\alpha_{1,\nu+m,\mu})^{-1}((y\csc^2\alpha)^{m/2}\Phi)\right](x). \quad (22)$$

Theorem 2.1. *The extended fractional powers of first Hankel-Clifford transformation $h^\alpha_{1,\nu,\mu,m}$, as defined by* (21), *is an isomorphism from $\mathcal{H}^\alpha_{1,\nu,\mu}(I)$ onto itself whatever be the real number ν. Moreover, $h^\alpha_{1,\nu,\mu,m}$ coincides with $h^\alpha_{1,\nu,\mu}$ if $\nu + \mu \geq 0$.*

Proof. The theorem follows from the fact that $\varphi \to R^*_{1,\nu+m-1,\mu,\alpha}...R^*_{1,\nu,\mu,\alpha}\varphi$ is an isomorphism from $\mathcal{H}^\alpha_{1,\nu,\mu}(I)$ onto $\mathcal{H}^\alpha_{1,\nu+m,\mu}(I)$, $\varphi \to h^\alpha_{1,\nu+m,\mu}\varphi$ is an isomorphism on $\mathcal{H}^\alpha_{1,\nu+m,\mu}(I)$ and $\varphi \to (y\csc^2\alpha)^{-m/2}\varphi$ is an isomorphism from $\mathcal{H}^\alpha_{1,\nu+m,\mu}(I)$ onto $\mathcal{H}^\alpha_{1,\nu,\mu}(I)$. (See Ref. [9], Proposition 3.6(a) and first part of the Theorem 4.2).

Now, we prove the last part of theorem. By definition (21) for $m = 1$, we have

$$(h^\alpha_{1,\nu,\mu,1}\varphi)(y) = (-1)e^{-i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{-1/2}(h^\alpha_{1,\nu+1,\mu}R^*_{1,\nu,\mu,\alpha}\varphi)(y).$$

Using the relations [9, Proposition 3.4],

$$h^{\alpha}_{1,\nu+1,\mu}\left(R^{*}_{1,\nu,\mu,\alpha}\varphi\right)(y) = -e^{i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}\left(h^{\alpha}_{1,\nu,\mu}\varphi\right)(y),$$

we have

$$(h^{\alpha}_{1,\nu,\mu,1}\varphi)(y) = (h^{\alpha}_{1,\nu,\mu}\varphi)(y).$$

Similarly for $m = 2$,

$$\begin{aligned}(h^{\alpha}_{1,\nu,\mu,2}\varphi)(y) &= (-1)^2 e^{-2i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{-1}(h^{\alpha}_{1,\nu+2,\mu}R^{*}_{1,\nu+1,\mu,\alpha}R^{*}_{1,\nu,\mu,\alpha}\varphi)(y)\\ &= (h^{\alpha}_{1,\nu,\mu}\varphi)(y).\end{aligned}$$

Proceeding in this way, we have

$$(h^{\alpha}_{1,\nu,\mu,m}\varphi)(y) = (h^{\alpha}_{1,\nu,\mu}\varphi)(y).$$

This completes the proof of theorem. □

Theorem 2.2. *The extended inverse transformation* $(h^{\alpha}_{1,\nu,\mu,m})^{-1}$*, as defined by* (22)*, is an isomorphism from* $\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$ *onto itself whatever be the real number* ν*. Moreover,* $(h^{\alpha}_{1,\nu,\mu,m})^{-1}$ *coincides with* $(h^{\alpha}_{1,\nu,\mu})^{-1}$ *if* $\nu+\mu \geq 0$*.*

Proof. The theorem follows from the fact that $\varphi \to (y\csc^2\alpha)^{m/2}\varphi$ is an isomorphism from $\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$ onto $\mathcal{H}^{\alpha}_{1,\nu+m,\mu}(I)$ and $R^{*^{-1}}_{1,\nu,\mu,\alpha}...R^{*^{-1}}_{1,\nu+m-1,\mu,\alpha}(y\csc^2\alpha)^{m/2}\varphi \to \varphi$ is an isomorphism from $\mathcal{H}^{\alpha}_{1,\nu+m,\mu}(I)$ onto $\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$. Hence, $\varphi \to (h^{\alpha}_{1,\nu+m,\mu})^{-1}\varphi$ is an isomorphism on $\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$ and (See Ref. [9], Proposition 3.6(a) and second part of the Theorem 4.2).

Now, we prove the last part of theorem. From definition (22) and (17), we have

$$\begin{aligned}&\left((h^{\alpha}_{1,\nu,\mu,m})^{-1}\varphi\right)(y)\\ &= (-1)^m e^{i(\alpha-\frac{\pi}{2})m} e^{-iy\cot\alpha} y^{\frac{\mu+\nu}{2}} \int_{\infty}^{y} ... \int_{\infty}^{y_{m-1}} y_m^{-\frac{(\mu+\nu+m)}{2}} e^{iy_m\cot\alpha}(h^{\alpha}_{1,\nu+m,\mu})^{-1}\left((x\csc^2\alpha)^{m/2}\varphi\right)(y_m)dy_m...dy_1\\ &= (-1)^m \gamma^{*\alpha}_{\nu,\mu} e^{-iy\cot\alpha} y^{\frac{\mu+\nu}{2}} \int_{\infty}^{y} ... \int_{\infty}^{y_{m-1}} \int_{0}^{\infty} e^{-ix\cot\alpha}(x\csc^2\alpha)^{\frac{\nu-\mu}{2}+m} C_{\nu+m}(y_m x\csc^2\alpha)\varphi(x)dxdy_m...dy_1.\end{aligned}$$

Interchanging the order of integration between y and x_m and using (7), we have

$$\begin{aligned}\left((h^{\alpha}_{1,\nu,\mu,m})^{-1}\varphi\right)(y) &= (-1)^m \gamma^{*\alpha}_{\nu,\mu} e^{-iy\cot\alpha} y^{\frac{\mu+\nu}{2}} \int_{\infty}^{y} ... \int_{\infty}^{y_{m-2}} \int_{0}^{\infty} e^{-ix\cot\alpha}(x\csc^2\alpha)^{\frac{\nu-\mu}{2}+m}\\ &\quad \times(-1)C_{\nu+m-1}(y_{m-1}x\csc^2\alpha)(x\csc^2\alpha)^{-1}\varphi(x)dxdy_{m-1}...dy_1.\end{aligned} \tag{23}$$

Proceeding in this way, we have

$$[(h^{\alpha}_{1,\nu,\mu,m})^{-1}\varphi](y) = [(h^{\alpha}_{1,\nu,\mu})^{-1}\varphi)](y).$$

This completes the proof of theorem. □

Lemma 2.3. *For any positive integers m and n both greater than* $-(\nu+\mu)$*, we have* $h^{\alpha}_{1,\nu,\mu,m} = h^{\alpha}_{1,\nu,\mu,n}$ *on* $\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$*.*

Proof. Note that the definition $h^{\alpha}_{1,\nu,\mu,m}$ is independent of choice of m so long as $\nu+\mu+m\ge 0$. Indeed if $m>n\ge-(\nu+\mu)$, then $h^{\alpha}_{1,\nu+n,\mu,m-n}=h^{\alpha}_{1,\nu+n,\mu}$ by Theorem 2.1.
Hence,

$$\begin{aligned}
&\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y)\\
&=(-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}\left[h^{\alpha}_{1,\nu+m,\mu}(R^*_{1,\nu+m-1,\mu,\alpha}...R^*_{1,\nu,\mu,\alpha}\varphi)\right](y)\\
&=(-1)^n e^{-i(\alpha-\frac{\pi}{2})n}(y\csc^2\alpha)^{-n/2}(-1)^{m-n}e^{-i(\alpha-\frac{\pi}{2})(m-n)}(y\csc^2\alpha)^{-(m-n)/2}\\
&\quad\times\left[h^{\alpha}_{1,\nu+n+m-n,\mu}(R^*_{1,\nu+n+m-n-1,\mu,\alpha}...R^*_{1,\nu+n,\mu,\alpha}R^*_{1,\nu+n-1,\mu,\alpha}...R^*_{1,\nu,\mu,\alpha}\varphi)\right](y)\\
&=(-1)^n e^{-i(\alpha-\frac{\pi}{2})n}(y\csc^2\alpha)^{-n/2}\left[h^{\alpha}_{1,\nu+n,\mu,m-n}(R^*_{1,\nu+n-1,\mu,\alpha}...R^*_{1,\nu,\mu,\alpha}\varphi)\right](y)\\
&=(-1)^n e^{-i(\alpha-\frac{\pi}{2})n}(y\csc^2\alpha)^{-n/2}\left[h^{\alpha}_{1,\nu+n,\mu}(R^*_{1,\nu+n-1,\mu,\alpha}...R^*_{1,\nu,\mu,\alpha}\varphi)\right](y)\\
&=\left(h^{\alpha}_{1,\nu,\mu,n}\varphi\right)(y).
\end{aligned}$$

This completes the proof. □

Now, we obtained some interesting operational formulae for the transformation $h^{\alpha}_{1,\nu,\mu,m}$ as:

Proposition 2.4. *Let ν and μ be the real numbers and m be a positive integer such that $\nu+\mu+m\ge 0$. Then for $\varphi\in\mathcal{H}^{\alpha}_{1,\nu,\mu,m}(I)$, we have*

$$\Delta_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y) = h^{\alpha}_{1,\nu,\mu,m}\left(-(x\csc^2\alpha)\varphi\right)(y), \tag{24}$$

$$h^{\alpha}_{1,\nu+1,\mu,m}\left(R^*_{1,\nu,\mu,\alpha}\varphi\right)(y) = -e^{i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y), \tag{25}$$

$$h^{\alpha}_{1,\nu,\mu,m}\left(\Delta^*_{1,\nu,\mu,\alpha}\varphi\right)(y) = -(y\csc^2\alpha)\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y). \tag{26}$$

If $\varphi\in\mathcal{H}^{\alpha}_{1,\nu+1,\mu}(I)$, then

$$h^{\alpha}_{1,\nu,\mu,m}\left(S^*_{1,\nu,\mu,\alpha}\varphi\right)(y) = e^{-i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}\left(h^{\alpha}_{1,\nu+1,\mu,m}\varphi\right)(y). \tag{27}$$

Proof. First we prove (24). Since $\varphi\in\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$, then $x\varphi\in\mathcal{H}^{\alpha}_{1,\nu+2,\mu}(I)\subset\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$. Moreover,

$$R^*_{1,\nu+m-1,\mu,\alpha}...R^*_{1,\nu,\mu,\alpha}=e^{-ix\cot\alpha}x^{\frac{\nu+\mu+m}{2}}D^m_x x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}. \tag{28}$$

Now, from definition (21), relation (28) and using (7), we have

$$\begin{aligned}
&\Delta_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y)=S_{1,\nu,\mu,\alpha}R_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y)\\
&=(-1)^m e^{iy\cot\alpha}y^{\frac{\mu-\nu}{2}}D_y y^{\nu+1}D_y y^{-\frac{(\nu+\mu)}{2}}e^{-iy\cot\alpha}e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}\\
&\times\gamma^{\alpha}_{\nu+m,\mu}y^{\mu}\int_0^{\infty}C_{\nu+m,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}x^{\frac{\nu+\mu+m}{2}}D^m_x x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&=(-1)^m(\csc^2\alpha)^{-m/2}e^{iy\cot\alpha}y^{\frac{\mu-\nu}{2}}D_y y^{\nu+1}D_y y^{-\frac{(\nu+\mu+m)}{2}}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^{\infty}C_{\nu+m,\mu}(xy\csc^2\alpha)x^{\frac{\nu+\mu+m}{2}}\\
&\quad\times D^m_x x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&=(-1)^m(\csc^2\alpha)^{(\nu-\mu)/2}e^{iy\cot\alpha}y^{\frac{\mu-\nu}{2}}D_y y^{\nu+1}\gamma^{\alpha}_{\nu,\mu}\int_0^{\infty}D_y\{C_{\nu+m}(xy\csc^2\alpha)\}x^{\nu+m}D^m_x x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&=(-1)^{m+1}(\csc^2\alpha)^{(\nu-\mu)/2+1}e^{iy\cot\alpha}y^{\frac{\mu-\nu}{2}}D_y y^{\nu+1}\gamma^{\alpha}_{\nu,\mu}\int_0^{\infty}C_{\nu+m+1}(xy\csc^2\alpha)x^{\nu+m+1}\\
&\quad\times D^m_x x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx.
\end{aligned}$$

Now, differentiating under the sign of integration and then using (8), the right-hand side of above equation can be written as

$$(-1)^{m+1}(\csc^2\alpha)^{-(\nu+\mu+m)/2}e^{iy\cot\alpha}y^{\frac{\mu-\nu}{2}}\gamma^{\alpha}_{\nu,\mu}\int_0^\infty D_y\{y^{-m}(xy\csc^2\alpha)^{\nu+m+1}C_{\nu+m+1}(xy\csc^2\alpha)\}D_x^m x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx$$
$$= (-1)^{m+1}(\csc^2\alpha)^{-(\nu+\mu+m)/2}e^{iy\cot\alpha}y^{\frac{\mu-\nu}{2}}\gamma^{\alpha}_{\nu,\mu}\int_0^\infty (-1)my^{-m-1}(xy\csc^2\alpha)^{\nu+m+1}C_{\nu+m+1}(xy\csc^2\alpha)$$
$$\times D_x^m x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx + (-1)^{m+1}(\csc^2\alpha)^{-(\nu+\mu+m)/2}e^{iy\cot\alpha}y^{\frac{\mu-\nu}{2}}\gamma^{\alpha}_{\nu,\mu}\int_0^\infty y^{-m}(x\csc^2\alpha)$$
$$\times(xy\csc^2\alpha)^{\nu+m}C_{\nu+m}(xy\csc^2\alpha)D_x^m x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx.$$

The first of the two integrals, in the above relation, is integrated by parts yield

$$\Delta_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y)$$
$$= (-1)^{m+1}m(\csc^2\alpha)(y\csc^2\alpha)^{-m/2}e^{iy\cot\alpha}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^\infty C_{\nu+m,\mu}(xy\csc^2\alpha)x^{\frac{\nu+\mu+m}{2}}D_x^m x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx$$
$$+ (-1)^{m+1}(\csc^2\alpha)(y\csc^2\alpha)^{-m/2}e^{iy\cot\alpha}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^\infty C_{\nu+m,\mu}(xy\csc^2\alpha)x^{\frac{\nu+\mu+m}{2}+1}D_x^m x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx. \quad (29)$$

From [9], we have

$$D_x^m x x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x) = mD_x^{m-1}x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x) + xD_x^m x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x). \quad (30)$$

We now consider the right-hand side of (24), to which we invoke (21) and using (30), we have

$$\begin{aligned} h^{\alpha}_{1,\nu,\mu,m}\Big((-x\csc^2\alpha)\varphi\Big)(y) &= (-1)^{m+1}e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu+m,\mu}y^{\mu}\int_0^\infty C_{\nu+m,\mu}(xy\csc^2\alpha) \\ &\quad \times e^{iy\cot\alpha}x^{\frac{\nu+\mu+m}{2}}D_x^m x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}(x\csc^2\alpha)\varphi(x)dx \\ &= (-1)^{m+1}(\csc^2\alpha)(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^\infty C_{\nu+m,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha} \\ &\quad \times x^{\frac{\nu+\mu+m}{2}}\Big[mD_x^{m-1}x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x) + xD_x^m x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)\Big]dx, \end{aligned}$$

which is equivalent to (29). This proves (24).

To prove (25), we employ Lemma 2.3 to obtain

$$h^{\alpha}_{1,\nu+1,\mu,m}\left(R^*_{1,\nu,\mu,\alpha}\varphi\right)(y)$$
$$= (-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}h^{\alpha}_{1,\nu+m+1,\mu}\left(R^*_{1,\nu+m,\mu,\alpha}...R^*_{1,\nu,\mu,\alpha}\varphi\right)(y)$$
$$= -e^{i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}(-1)^{m+1}e^{-i(\alpha-\frac{\pi}{2})(m+1)}(y\csc^2\alpha)^{-(m+1)/2}h^{\alpha}_{1,\nu+m+1,\mu}\left(R^*_{1,\nu+m+1-1,\mu,\alpha}...R^*_{1,\nu,\mu,\alpha}\varphi\right)(y)$$
$$= -e^{i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}\left(h^{\alpha}_{1,\nu,\mu,m+1}\varphi\right)(y).$$

This proves (25).

Next, we prove (27). Let $\varphi \in \mathcal{H}^{\alpha}_{1,\nu+1,\mu}(I)$, then we have

$$\begin{aligned}
&h^{\alpha}_{1,\nu+1,\mu,m}\left(S^{*}_{1,\nu,\mu,\alpha}\varphi\right)(y)\\
&= (-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2} h^{\alpha}_{1,\nu+m,\mu}\Big(R^{*}_{1,\nu+m-1,\mu,\alpha}...R^{*}_{1,\nu,\mu,\alpha}S^{*}_{1,\nu,\mu,\alpha}\varphi\Big)(y)\\
&= (-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu+m,\mu}y^{\mu}\int_0^{\infty} C_{\nu+m,\mu}(xy\csc^2\alpha)\\
&\quad\times e^{iy\cot\alpha}x^{\frac{\nu+\mu+m}{2}}D^m_x x^{-\nu}D_x x^{\frac{\nu-\mu+1}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&= (-1)^m(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu+m,\mu}y^{\mu}\int_0^{\infty} C_{\nu+m,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}\\
&\quad\times x^{\frac{\nu+\mu+m}{2}}D^{m+1}_x\int_{\infty}^{x} t^{-\nu}D_t t^{\frac{\nu-\mu+1}{2}}e^{it\cot\alpha}\varphi(t)dtdx\\
&= (-1)^m(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu+m,\mu}y^{\mu}\int_0^{\infty} C_{\nu+m,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}x^{\frac{\nu+\mu+m}{2}}D^{m+1}_x x^{-\frac{(\nu+\mu-1)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&\quad+(-1)^m\nu(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu+m,\mu}y^{\mu}\int_0^{\infty} C_{\nu+m,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}x^{\frac{\nu+\mu+m}{2}}D^{m}_x x^{-\frac{(\nu+\mu+1)}{2}}e^{ix\cot\alpha}\varphi(x)dx. \quad (31)
\end{aligned}$$

In view of (30), (31) can be obtained in the form

$$\begin{aligned}
h^{\alpha}_{1,\nu+1,\mu,m}\left(S^{*}_{1,\nu,\mu,\alpha}\varphi\right)(y) &= (-1)^m(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu+m,\mu}y^{\mu}\int_0^{\infty} C_{\nu+m,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}\\
&\quad\times x^{\frac{\nu+\mu+m}{2}}xD^{m+1}_x x^{-\frac{(\nu+\mu+1)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&+ (-1)^m(\nu+m+1)(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu+m,\mu}y^{\mu}\int_0^{\infty} C_{\nu+m,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}\\
&\quad\times x^{\frac{\nu+\mu+m}{2}}D^{m}_x x^{-\frac{(\nu+\mu+1)}{2}}e^{ix\cot\alpha}\varphi(x)dx. \quad (32)
\end{aligned}$$

Further, continuing the proceedings to prove the relation (27), We prove that $e^{-i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}(h^{\alpha}_{1,\nu+1,\mu,m}\varphi)(y)$ is equivalent to (32). Since $\varphi \in \mathcal{H}^{\alpha}_{1,\nu+1,\mu}(I)$. Then, we have

$$\begin{aligned}
&e^{-i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}(h^{\alpha}_{1,\nu+1,\mu,m}\varphi)(y)\\
&= e^{-i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}(-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}h^{\alpha}_{1,\nu+m+1,\mu}\left(R^{*}_{1,\nu+m,\mu,\alpha}...R^{*}_{1,\nu+1,\mu,\alpha}\varphi\right)(y)\\
&= e^{-i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}(-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}\gamma_{\nu+m+1,\alpha}y^{\mu}\\
&\quad\times\int_0^{\infty} C_{\nu+m+1,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}x^{\frac{\nu+\mu+m+1}{2}}D^m_x x^{-\frac{(\nu+\mu+1)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&= (-1)^m(y\csc^2\alpha)^{(\nu-\mu)/2}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^{\infty}(y\csc^2\alpha)C_{\nu+m+1}(xy\csc^2\alpha)\\
&\quad\times e^{iy\cot\alpha}x^{\nu+m+1}D^m_x x^{-\frac{(\nu+\mu+1)}{2}}e^{ix\cot\alpha}\varphi(x)dx.
\end{aligned}$$

Now, using the formula

$$D_x[C_{\nu+m}(xy\csc^2\alpha)] = -(y\csc^2\alpha)C_{\nu+m+1}(xy\csc^2\alpha),$$

and integrating by parts, we have

$$
\begin{aligned}
&e^{-i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}(h^{\alpha}_{1,\nu+1,\mu,m}\varphi)(y)\\
&=(-1)^{m+1}(y\csc^2\alpha)^{(\nu-\mu)/2}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^{\infty}D_x[C_{\nu+m}(xy\csc^2\alpha)]e^{iy\cot\alpha}x^{\nu+m+1}D^m_x x^{-\frac{(\nu+\mu+1)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&=(-1)^{m}(y\csc^2\alpha)^{(\nu-\mu)/2}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^{\infty}C_{\nu+m}(xy\csc^2\alpha)e^{iy\cot\alpha}D_x\{x^{\nu+m+1}D^m_x x^{-\frac{(\nu+\mu+1)}{2}}e^{ix\cot\alpha}\varphi(x)\}dx\\
&=(-1)^{m}(y\csc^2\alpha)^{(\nu-\mu)/2}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^{\infty}C_{\nu+m}(xy\csc^2\alpha)e^{iy\cot\alpha}\Big[x^{\nu+m+1}D^{m+1}_x x^{-\frac{(\nu+\mu+1)}{2}}e^{ix\cot\alpha}\varphi(x)\\
&\quad+(\nu+m+1)x^{\nu+m}D^m_x x^{-\frac{(\nu+\mu+1)}{2}}e^{ix\cot\alpha}\varphi(x)\Big]dx. \qquad (33)
\end{aligned}
$$

By using the formula $C_{\nu,\mu}(x)=x^{(\nu-\mu)/2}C_\nu(x)$, relation (33) can be made equivalent to (32). This proves (27). Finally, combining (25) and (27), we obtain (26). □

Problem 2.5. *With ν, μ and m as Proposition 2.4 and for all $\varphi\in\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$, prove that*

$$R_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y) = e^{-i(\alpha-\frac{\pi}{2})}h^{\alpha}_{1,\nu+1,\mu,m}\left(-(x\csc^2\alpha)^{1/2}\varphi\right)(y), \qquad (34)$$

$$S_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu+1,\mu,m}\varphi\right)(y) = e^{i(\alpha-\frac{\pi}{2})}h^{\alpha}_{1,\nu,\mu,m}\left((x\csc^2\alpha)^{1/2}\varphi\right)(y). \qquad (35)$$

Proof. Applying $R_{1,\nu,\mu,\alpha}$ to both sides of (21) and then using (28) with formula (7), we have

$$
\begin{aligned}
&R_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y)\\
&=e^{iy\cot\alpha}y^{\frac{\mu+\nu+1}{2}}D_y y^{-\frac{(\mu+\nu)}{2}}e^{-iy\cot\alpha}(-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu+m,\mu}\\
&\quad\times y^{\mu}\int_0^{\infty}C_{\nu+m,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}x^{\frac{\nu+\mu+m}{2}}D^m_x x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&=(-1)^m e^{iy\cot\alpha}y^{\frac{\mu+\nu+1}{2}}\gamma^{\alpha}_{\nu,\mu}(\csc^2\alpha)^{(\nu-\mu)/2}\int_0^{\infty}D_y\{C_{\nu+m}(xy\csc^2\alpha)\}x^{\nu+m}D^m_x x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&=(-1)^{m+1}e^{iy\cot\alpha}y^{\frac{\mu+\nu+1}{2}}\gamma^{\alpha}_{\nu,\mu}(\csc^2\alpha)^{(\nu-\mu)/2+1}\int_0^{\infty}C_{\nu+m+1}(xy\csc^2\alpha)x^{\nu+m+1}D^m_x x^{-\frac{(\nu+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&=(-1)^m e^{-i(\alpha-\frac{\pi}{2})(m+1)}(y\csc^2\alpha)^{-m/2}h^{\alpha}_{1,\nu+m+1,\mu}\Big(R^*_{1,\nu+m,\mu,\alpha}...R^*_{1,\nu+1,\mu,\alpha}(-(x\csc^2\alpha)^{1/2}\varphi)\Big)(y)\\
&=e^{-i(\alpha-\frac{\pi}{2})}h^{\alpha}_{1,\nu+1,\mu,m}\Big(-(x\csc^2\alpha)^{1/2}\varphi)\Big)(y).
\end{aligned}
$$

This proves (34).

To prove (35), we have form (21) and (28)

$$
\begin{aligned}
&S_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu+1,\mu,m}\varphi\right)(y)\\
&=e^{iy\cot\alpha}y^{\frac{\mu-\nu}{2}}D_y y^{\frac{(\nu-\mu+1)}{2}}e^{-iy\cot\alpha}(-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}\gamma_{\nu+m+1,\alpha}\\
&\times y^{\mu}\int_0^{\infty}C_{\nu+m+1,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}x^{\frac{\nu+1+\mu+m}{2}}D^m_x x^{-\frac{(\nu+1+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx\\
&=(-1)^m e^{iy\cot\alpha}y^{\frac{\mu-\nu}{2}}\gamma^{\alpha}_{\nu+1,\mu}(\csc^2\alpha)^{-(\mu+\nu+1)/2-m}\int_0^{\infty}D_y\{y^{-m}\\
&\quad\times(xy\csc^2\alpha)^{\nu+m+1}C_{\nu+m+1}(xy\csc^2\alpha)\}D^m_x x^{-\frac{(\nu+1+\mu)}{2}}e^{ix\cot\alpha}\varphi(x)dx
\end{aligned}
$$

$$= (-1)^m e^{iy\cot\alpha} y^{\frac{\mu-\nu}{2}} \gamma^{\alpha}_{\nu+1,\mu} (\csc^2\alpha)^{-(\mu+\nu+1)/2-m} \int_0^\infty (-m) y^{-m-1}$$
$$\times (xy\csc^2\alpha)^{\nu+m+1} C_{\nu+m+1}(xy\csc^2\alpha) D_x^m x^{-\frac{(\nu+1+\mu)}{2}} e^{ix\cot\alpha}\varphi(x)dx$$
$$+(-1)^m e^{iy\cot\alpha} y^{\frac{\mu-\nu}{2}} \gamma^{\alpha}_{\nu+1,\mu} (\csc^2\alpha)^{-(\mu+\nu+1)/2-m} \int_0^\infty y^{-m}(x\csc^2\alpha)$$
$$\times (xy\csc^2\alpha)^{\nu+m} C_{\nu+m}(xy\csc^2\alpha) D_x^m x^{-\frac{(\nu+1+\mu)}{2}} e^{ix\cot\alpha}\varphi(x)dx.$$

Now, integrating the first integral by parts and using (30), we have

$$S_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu+1,\mu,m}\varphi\right)(y)$$
$$= (-1)^m e^{iy\cot\alpha} y^{\frac{\mu-\nu}{2}} \gamma^{\alpha}_{\nu+1,\mu} (\csc^2\alpha)^{-(\mu+\nu+1)/2-m} \int_0^\infty y^{-m}(\csc^2\alpha)(xy\csc^2\alpha)^{\nu+m} C_{\nu+m}(xy\csc^2\alpha)$$
$$\times\left\{ m D_x^{m-1} x^{-\frac{(\nu+1+\mu)}{2}} e^{ix\cot\alpha}\varphi(x) \;+\; x D_x^m x^{-\frac{(\nu+1+\mu)}{2}} e^{ix\cot\alpha}\varphi(x)\right\}dx$$
$$= (-1)^m e^{iy\cot\alpha} y^{\frac{\mu-\nu}{2}} \gamma^{\alpha}_{\nu+1,\mu} (\csc^2\alpha)^{-(\mu+\nu+1)/2-m} \int_0^\infty y^{-m}(\csc^2\alpha)$$
$$\times (xy\csc^2\alpha)^{\nu+m} C_{\nu+m}(xy\csc^2\alpha) D_x^m x^{-\frac{(\nu+\mu)}{2}} e^{ix\cot\alpha} x^{1/2}\varphi(x)dx$$
$$= (-1)^m e^{i(\alpha-\frac{\pi}{2})(-m+1)} (y\csc^2\alpha)^{-m/2} h^{\alpha}_{1,\nu+m,\mu}\Big(R^*_{1,\nu+m-1,\mu,\alpha}\dots R^*_{1,\nu,\mu,\alpha}((x\csc^2\alpha)^{1/2}\varphi)\Big)(y)$$
$$= e^{i(\alpha-\frac{\pi}{2})} h^{\alpha}_{1,\nu,\mu,m}\Big((x\csc^2\alpha)^{1/2}\varphi)\Big)(y).$$

This proves (35). □

Theorem 2.6. *Let ν be any real number, μ and α $(0 < \alpha < \pi)$ are real parameters. Then for any positive integer m such that $\nu + \mu + m \geq 0$,*

$$R_{1,\nu+m-1,\mu,\alpha}\dots R_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y) = (-1)^m e^{-i(\alpha-\frac{\pi}{2})m} h^{\alpha}_{1,\nu+m,\mu}\left((x\csc^2\alpha)^{m/2}\varphi\right)(y). \tag{36}$$

Moreover, $h^{\pi/2}_{1,\nu.\mu,m} = [h^{\pi/2}_{1,\nu.\mu,m}]^{-1}$.

Proof. Applying $R_{1,\nu+1,\mu,\alpha}$ to both side of (34), we have

$$\begin{aligned} R_{1,\nu+1,\mu,\alpha}R_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y) &= e^{-i(\alpha-\frac{\pi}{2})} R_{1,\nu+1,\mu,\alpha} h^{\alpha}_{1,\nu+1,\mu,m}\left(-(x\csc^2\alpha)^{1/2}\varphi\right)(y) \\ &= e^{-2i(\alpha-\frac{\pi}{2})} h^{\alpha}_{1,\nu+2,\mu,m}\left((x\csc^2\alpha)\varphi\right)(y). \end{aligned}$$

Repeating this process, we have

$$R_{1,\nu+m-1,\mu,\alpha}\dots R_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y) = e^{-i(\alpha-\frac{\pi}{2})m} h^{\alpha}_{1,\nu+m,\mu,m}\left((-1)^m(x\csc^2\alpha)^{m/2}\varphi\right)(y).$$

Using Theorem 2.1, we have

$$R_{1,\nu+m-1,\mu,\alpha}\dots R_{1,\nu,\mu,\alpha}\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y) = e^{-i(\alpha-\frac{\pi}{2})m} h^{\alpha}_{1,\nu+m,\mu}\left((-1)^m(x\csc^2\alpha)^{m/2}\varphi\right)(y).$$

This proves (36).

If $\alpha = \pi/2$, we have

$$R_{1,\nu+m-1,\mu,\pi/2}\dots R_{1,\nu,\mu,\pi/2}(h^{\pi/2}_{1,\nu,\mu,m}\varphi)(y) = (-1)^m\left(h^{\pi/2}_{1,\nu+m,\mu}(x^{m/2}\varphi)\right)(y).$$

Using the fact that $h^{\pi/2}_{1,\nu,\mu} = [h^{\pi/2}_{1,\nu,\mu}]^{-1}$ and from (22), we get

$$\left((h^{\pi/2}_{1,\nu,\mu,m})^{-1}\varphi\right)(y) = (-1)^m R^{-1}_{1,\nu,\mu,\pi/2}\dots R^{-1}_{1,\nu+m-1,\mu,\pi/2}\left(h^{\pi/2}_{1,\nu+m,\mu}(x^{m/2}\varphi)\right)(y).$$

Hence, we conclude that

$$h^{\pi/2}_{1,\nu.\mu,m} = [h^{\pi/2}_{1,\nu.\mu,m}]^{-1}.$$

This completes the proof of theorem. □

3. Fractional Powers of Second Hankel-Clifford Transformation of Arbitrary Order

Let ν, μ be any real numbers and α $(0 < \alpha < \pi)$ and n be a positive integer such that $\nu + \mu + n \geq 0$. We define the extended fractional powers of second Hankel-Clifford transformation $h^{\alpha}_{2,\nu,\mu,n}$ of any $\psi \in \mathcal{H}^{\alpha}_{2,\nu,\mu}(I)$ by

$$\Psi(y) = \left(h^{\alpha}_{2,\nu,\mu,n}\psi\right)(y) = (-1)^n e^{i(\alpha-\frac{\pi}{2})n}(y\csc^2\alpha)^{-n/2}\left[h^{\alpha}_{2,\nu+n,\mu}(S^*_{2,\nu+n-1,\mu,\alpha}...S^*_{2,\nu,\mu,\alpha}\psi)\right](y). \tag{37}$$

The inverse transformation $(h^{\alpha}_{2,\nu,\mu,n})^{-1}$ of any $\Psi \in \mathcal{H}^{\alpha}_{2,\nu,\mu}(I)$ is defined by

$$\psi(x) = \left((h^{\alpha}_{2,\nu,\mu,n})^{-1}\Psi(y)\right)(x) = (-1)^n e^{-i(\alpha-\frac{\pi}{2})n} S^{*^{-1}}_{2,\nu,\mu,\alpha}...S^{*^{-1}}_{2,\nu+n-1,\mu,\alpha}\left[(h^{\alpha}_{2,\nu+n,\mu})^{-1}(y\csc^2\alpha)^{n/2}\Psi)\right](x). \tag{38}$$

Theorem 3.1. *The fractional powers of second Hankel-Clifford transformation $h^{\alpha}_{2,\nu,\mu,n}$, as defined by* (37)*, is an isomorphism from $\mathcal{H}^{\alpha}_{2,\nu,\mu}(I)$ onto itself whatever be the real number ν. Moreover, $h^{\alpha}_{2,\nu,\mu,n}$ coincides with $h^{\alpha}_{2,\nu,\mu}$ if $\nu + \mu \geq 0$.*

Proof. The proof of theorem is similar to that of Theorem 2.1. □

Now, we obtained some interesting operational formulae for the transformation $h^{\alpha}_{2,\nu,\mu,n}$ as:

Proposition 3.2. *Let ν and μ be the real numbers and n be a positive integer such that $\nu + \mu + n \geq 0$. Then for $\psi \in \mathcal{H}^{\alpha}_{2,\nu,\mu}(I)$, we have*

$$S_{2,\nu,\mu,\alpha}\left(h^{\alpha}_{2,\nu,\mu,n}\psi\right)(y) = e^{-i(\alpha-\frac{\pi}{2})}h^{\alpha}_{2,\nu+1,\mu,n}\left(-(x\csc^2\alpha)^{1/2}\psi\right)(y), \tag{39}$$

$$h^{\alpha}_{2,\nu+1,\mu,n}\left(S^*_{2,\nu,\mu,\alpha}\psi\right)(y) = -e^{i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}\left(h^{\alpha}_{2,\nu,\mu,n}\psi\right)(y), \tag{40}$$

$$h^{\alpha}_{2,\nu,\mu,n}\left(\Delta^*_{2,\nu,\mu,\alpha}\psi\right)(y) = -(y\csc^2\alpha)\left(h^{\alpha}_{2,\nu,\mu,n}\psi\right)(y), \tag{41}$$

$$\Delta_{2,\nu,\mu,\alpha}\left(h^{\alpha}_{2,\nu,\mu,n}\psi\right)(y) = h^{\alpha}_{2,\nu,\mu,n}\left(-(x\csc^2\alpha)\psi\right)(y). \tag{42}$$

If $\psi \in \mathcal{H}^{\alpha}_{2,\nu+1,\mu}(I)$, then

$$h^{\alpha}_{2,\nu,\mu,n}\left(R^*_{2,\nu,\mu,\alpha}\psi\right)(y) = e^{-i(\alpha-\frac{\pi}{2})}(y\csc^2\alpha)^{1/2}\left(h^{\alpha}_{2,\nu+1,\mu,n}\psi\right)(y), \tag{43}$$

$$R_{2,\nu,\mu,\alpha}\left(h^{\alpha}_{2,\nu+1,\mu,n}\psi\right)(y) = e^{i(\alpha-\frac{\pi}{2})}h^{\alpha}_{2,\nu,\mu,n}\left((x\csc^2\alpha)^{1/2}\psi\right)(y). \tag{44}$$

Remark 3.3. *Similar results can be proved as Lemma 2.3 and Theorem 2.6 for $h^{\alpha}_{2,\nu,\mu,n}$ and $\psi \in \mathcal{H}^{\alpha}_{2,\nu,\mu}(I)$.*

4. Generalized Fractional Powers of Hankel-Clifford Transformation of Arbitrary Order

In this section, we have investigated a pair of generalized fractional powers of Hankel-Clifford transformation of arbitrary order on the dual spaces of $\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$ and $\mathcal{H}^{\alpha}_{2,\nu,\mu}(I)$.

As before n is any positive integer such that $n \geq -(\mu + \nu)$. The generalized fractional powers of first Hankel-Clifford transformation of arbitrary order $(h^{\alpha}_{1,\nu,\mu})'$ is defined on $(\mathcal{H}^{\alpha}_{2,\nu,\mu})'(I)$, as the adjoint of $h^{\alpha}_{2,\nu,\mu,n}$ on $\mathcal{H}^{\alpha}_{2,\nu,\mu}(I)$, by

$$\left\langle (h^{\alpha}_{1,\nu,\mu})'f,\ \Psi\right\rangle = \left\langle f,\ h^{\alpha}_{2,\nu,\mu,n}\Psi\right\rangle, \tag{45}$$

for $f \in (\mathcal{H}^{\alpha}_{2,\nu,\mu})'(I)$, $\Psi \in \mathcal{H}^{\alpha}_{2,\nu,\mu}(I)$.

Hence from (45) and Theorem 3.1, we have the following theorem:

Theorem 4.1. *The generalized fractional powers of first Hankel-Clifford transformation of arbitrary order ν, defined in* (45)*, is an isomorphism from* $(\mathcal{H}^{\alpha}_{2,\nu,\mu})'(I)$ *into itself.*

This leads to the following transformation formulae:

Proposition 4.2. *For any real number ν and $f \in (\mathcal{H}^{\alpha}_{2,\nu,\mu})'(I)$, we have*

$$(h^{\alpha}_{1,\nu,\mu})'\left(\Delta^{*}_{1,\nu,\mu,\alpha}f\right)(y) = -(y\csc^2\alpha)\left((h^{\alpha}_{1,\nu,\mu})'f\right)(y), \tag{46}$$

$$\Delta_{1,\nu,\mu,\alpha}\left((h^{\alpha}_{1,\nu,\mu})'f\right)(y) = (h^{\alpha}_{1,\nu,\mu})'\left(-(x\csc^2\alpha)f\right)(y). \tag{47}$$

Proof. Let $\Psi \in \mathcal{H}^{\alpha}_{2,\nu,\mu}(I)$. Then form (45) and (42), we have

$$\begin{aligned}\left\langle (h^{\alpha}_{1,\nu,\mu})'\left(\Delta^{*}_{1,\nu,\mu,\alpha}f\right),\ \Psi\right\rangle &= \left\langle \Delta^{*}_{1,\nu,\mu,\alpha}f,\ h^{\alpha}_{2,\nu,\mu,n}\Psi\right\rangle = \left\langle f,\ \Delta_{2,\nu,\mu,\alpha}\left(h^{\alpha}_{2,\nu,\mu,n}\Psi\right)\right\rangle \\ &= \left\langle f,\ h^{\alpha}_{2,\nu,\mu,n}\left(-(y\csc^2\alpha)\Psi\right)\right\rangle = \left\langle -(y\csc^2\alpha)\left((h^{\alpha}_{1,\nu,\mu})'f\right),\ \Psi\right\rangle.\end{aligned}$$

In the sense of equality in distributions, we conclude the proof of (46). By the similar arguments, we can prove (47). □

Analogously, the generalized fractional powers of second Hankel-Clifford transformation of arbitrary order $(h^{\alpha}_{2,\nu,\mu})'$ is defined on $(\mathcal{H}^{\alpha}_{1,\nu,\mu})'(I)$, as the adjoint of $h^{\alpha}_{1,\nu,\mu,m}$ on $\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$, by

$$\left\langle (h^{\alpha}_{2,\nu,\mu})'f,\ \Phi\right\rangle = \left\langle f,\ h^{\alpha}_{1,\nu,\mu,m}\Phi\right\rangle, \tag{48}$$

for $f \in (\mathcal{H}^{\alpha}_{1,\nu,\mu})'(I)$, $\Phi \in \mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$.

Remark 4.3. *Similar results can also be proved as Theorem 4.1 and Proposition 4.2 for $(h^{\alpha}_{2,\nu,\mu})'$ and $\Phi \in \mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$.*

5. Applications

In this section, applications of a pair of fractional powers of Hankel-Clifford transformations of arbitrary order are given.

Problem 5.1. *If the generalized function $\delta(x-a)$, $a > 0$ is defined on $\mathcal{H}^{\alpha}_{1,\nu,\mu}(I)$, then*

(i) $\left(h^{\alpha}_{1,\nu,\mu,m}\delta(x-a)\right)(y) = \gamma^{\alpha}_{\nu,\mu}y^{\mu}e^{i(y+a)\cot\alpha}C_{\nu,\mu}(ay\csc^2\alpha)$,

(ii) $\left((h^{\alpha}_{1,\nu,\mu,m})^{-1}\delta(x-a)\right)(y) = \gamma^{*\alpha}_{\nu,\mu}y^{\mu}e^{-i(y+a)\cot\alpha}C_{\nu,\mu}(ay\csc^2\alpha)$.

Proof. By definition (21) and (28), we have

$$\begin{aligned}&\left(h^{\alpha}_{1,\nu,\mu,m}\delta(x-a)\right)(y)\\ &= (-1)^m e^{-i(\alpha-\frac{\pi}{2})m}(y\csc^2\alpha)^{-m/2}\left[h^{\alpha}_{1,\nu+m,\mu}(R^{*}_{1,\nu+m-1,\mu,\alpha}\ldots R^{*}_{1,\nu,\mu,\alpha}\delta(x-a))\right](y)\\ &= (-1)^m(y\csc^2\alpha)^{-m/2}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^{\infty}C_{\nu+m,\mu}(xy\csc^2\alpha)e^{iy\cot\alpha}x^{\frac{\mu+\nu+m}{2}}D^m_x x^{-\frac{(\mu+\nu)}{2}}e^{ix\cot\alpha}\delta(x-a)dx.\end{aligned}$$

Integrating by parts repeatedly m times and using the formula (8), we obtain

$$\begin{aligned}\left(h^{\alpha}_{1,\nu,\mu,m}\delta(x-a)\right)(y) &= (y\csc^2\alpha)^{-(\mu+\nu)/2-m}\gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^{\infty}D^m_x\{(xy\csc^2\alpha)^{\nu+m}C_{\nu+m}(xy\csc^2\alpha)\}\\ &\quad\times e^{iy\cot\alpha}x^{-\frac{(\mu+\nu)}{2}}e^{ix\cot\alpha}\delta(x-a)dx\\ &= \gamma^{\alpha}_{\nu,\mu}y^{\mu}\int_0^{\infty}C_{\nu,\mu}(xy\csc^2\alpha)e^{i(y+x)\cot\alpha}\delta(x-a)dx.\end{aligned}$$

Hence, by the properties of $\delta(x-a)$, we have the required result (i).

Next, we prove (ii). From (22) and (17), we have

$$\begin{aligned}\left((h^{\alpha}_{1,\nu,\mu,m})^{-1}\delta(x-a)\right)(y) &= (-1)^m e^{i(\alpha-\frac{\pi}{2})m}\Big[R^{*^{-1}}_{1,\nu,\mu,\alpha}...R^{*^{-1}}_{1,\nu+m-1,\mu,\alpha}(h^{\alpha}_{1,\nu+m,\mu})^{-1}\left((x\csc^2\alpha)^{m/2}\delta(x-a)\right)\Big](y)\\ &= (-1)^m e^{i(\alpha-\frac{\pi}{2})m}e^{-iy\cot\alpha}y^{\frac{\mu+\nu}{2}}\int_{\infty}^{y}\int_{\infty}^{y_1}...\int_{\infty}^{y_{m-1}} y_m^{-\frac{(\mu+\nu+m)}{2}}e^{iy_m\cot\alpha}\\ &\quad\times\left[(h^{\alpha}_{1,\nu+m,\mu})^{-1}((x\csc^2\alpha)^{m/2}\delta(x-a))\right](y_m)dy_m...dy_2dy_1\\ &= (-1)^m\gamma^{*\alpha}_{\nu,\mu}e^{-iy\cot\alpha}y^{\frac{\mu+\nu}{2}}\int_{\infty}^{y}\int_{\infty}^{y_1}...\int_{\infty}^{y_{m-1}}\int_0^{\infty}C_{\nu+m}(xy_m\csc^2\alpha)\\ &\quad\times(x\csc^2\alpha)^{\frac{\nu-\mu}{2}+m}e^{-ix\cot\alpha}\delta(x-a)dxdy_m...dy_2dy_1.\end{aligned}$$

Now, by properties of $\delta(x-a)$ and then using (7), we have

$$\begin{aligned}\left((h^{\alpha}_{1,\nu,\mu,m})^{-1}\delta(x-a)\right)(y) &= (-1)^m\gamma^{*\alpha}_{\nu,\mu}e^{-i(y+a)\cot\alpha}(a\csc^2\alpha)^{\frac{\nu-\mu}{2}+m}y^{\frac{\mu+\nu}{2}}\int_{\infty}^{y}\int_{\infty}^{y_1}\\ &\quad ...\int_{\infty}^{y_{m-2}}(-1)C_{\nu+m-1}(ay_{m-1}\csc^2\alpha)(a\csc^2\alpha)^{-1}dy_{m-1}...dy_2dy_1.\end{aligned}$$

Proceeding in this way, we get

$$\left((h^{\alpha}_{1,\nu,\mu,m})^{-1}\delta(x-a)\right)(y) = \gamma^{*\alpha}_{\nu,\mu}e^{-i(y+a)\cot\alpha}y^{\mu}C_{\nu,\mu}(ay\csc^2\alpha).$$

This proves (ii). □

The fractional powers of Hankel-Clifford transformations of arbitrary order can also be utilized in solving the some partial differential equations. Consider the general equation [3]:

$$a(x,y)\frac{\partial^2\varphi}{\partial x^2}+b(x,y)\frac{\partial^2\varphi}{\partial x\partial y}+c(x,y)\frac{\partial^2\varphi}{\partial y^2}+d(x,y)\frac{\partial\varphi}{\partial x}+e(x,y)\frac{\partial\varphi}{\partial y}+f(x,y)\varphi = G(x,y), \tag{49}$$

when

$$\begin{aligned}&a(x,y)=x,\ \ b(x,y)=c(x,y)=e(x,y)=0,\ \ d(x,y)=(1-\mu)+2ix\cot\alpha,\\ &\quad f(x,y)=(1-\mu)i\cot\alpha - x\cot^2\alpha-\frac{\nu^2-\mu^2}{4x}\text{ and }G(x,y)=\delta(x-a),\end{aligned}$$

then (49) is reduced as

$$\varphi(x)-\Delta^{*}_{1,\nu,\mu,\alpha}\varphi(x)=\delta(x-a). \tag{50}$$

Applying $h^{\alpha}_{1,\nu,\mu,m}$ to the both sides and using (26) and Problem 5.1(i), we get

$$(1+y\csc^2\alpha)\left(h^{\alpha}_{1,\nu,\mu,m}\varphi\right)(y)=\gamma^{\alpha}_{\nu,\mu}y^{\mu}e^{i(y+a)\cot\alpha}C_{\nu,\mu}(ay\csc^2\alpha).$$

Therefore,

$$\begin{aligned}\varphi(x) &= \left(h^{\alpha}_{1,\nu,\mu,m}\right)^{-1}\left[\gamma^{\alpha}_{\nu,\mu}y^{\mu}e^{i(y+a)\cot\alpha}C_{\nu,\mu}(ay\csc^2\alpha)(1+y\csc^2\alpha)^{-1}\right](x)\\ &= (-1)^m\gamma^{\alpha}_{\nu,\mu}e^{i(\alpha-\frac{\pi}{2})m}\Big[R^{*^{-1}}_{1,\nu,\mu,\alpha}...R^{*^{-1}}_{1,\nu+m-1,\mu,\alpha}\left(h^{\alpha}_{1,\nu+m,\mu}\right)^{-1}\Big((y\csc^2\alpha)^{m/2}\\ &\quad\times y^{\mu}e^{i(y+a)\cot\alpha}C_{\nu,\mu}(ay\csc^2\alpha)(1+y\csc^2\alpha)^{-1}\Big)\Big](x)\\ &= (-1)^m\gamma^{\alpha}_{\nu,\mu}e^{i(\alpha-\frac{\pi}{2})m}e^{-ix\cot\alpha}x^{\frac{\mu+\nu}{2}}\int_{\infty}^{x}\int_{\infty}^{x_1}...\int_{\infty}^{x_{m-1}}x_m^{-\frac{(\mu+\nu+m)}{2}}e^{ix_m\cot\alpha}\\ &\quad\times\left(h^{\alpha}_{1,\nu+m,\mu}\right)^{-1}\left((y\csc^2\alpha)^{m/2}y^{\mu}e^{i(y+a)\cot\alpha}C_{\nu,\mu}(ay\csc^2\alpha)(1+y\csc^2\alpha)^{-1}\right)(x_m)dx_m...dx_2dx_1.\end{aligned}$$

Interchanging the order of integration as (23), we obtain

$$\begin{aligned}\varphi(x) &= e^{i(a-x)\cot\alpha}x^{\frac{\mu+\nu}{2}}\int_0^\infty \frac{C_\nu(xy\csc^2\alpha)(y\csc^2\alpha)^{\frac{\nu-\mu}{2}}y^\mu C_{\nu,\mu}(ay\csc^2\alpha)}{(1+y\csc^2\alpha)}dy\\ &= a^{-\mu/2}e^{i(a-x)\cot\alpha}x^{\mu/2}\int_0^\infty \frac{J_\nu(2\sqrt{xy\csc^2\alpha})(y\csc^2\alpha)^{-\mu}y^\mu J_\nu(2\sqrt{ay\csc^2\alpha})}{(1+y\csc^2\alpha)}dy,\end{aligned}$$

which on putting $y\csc^2\alpha = t$, we have

$$\varphi(x) = \frac{e^{i(1-x)\cot\alpha}x^{\mu/2}}{a^{\mu/2}(\csc^2\alpha)^{\mu+1}}\int_0^\infty \frac{J_\nu(2\sqrt{xt})J_\nu(2\sqrt{at})}{(1+t)}dt,$$

then from Erdelyi [2, p. 49],

$$\varphi(x) = \frac{2e^{i(1-x)\cot\alpha}x^{\mu/2}}{a^{\mu/2}(\csc^2\alpha)^{\mu+1}}\begin{cases} I_\nu(2\sqrt{x})K_\nu(2a), & 0<x<1,\\ I_\nu(2a)K_\nu(2\sqrt{x}), & 1<x<\infty,\end{cases}$$

where I_ν and K_ν are known as modified Bessel function of first and third kind respectively.

Similarly, if $a(x,y), b(x,y), c(x,y), d(x,y), e(x,y)$ and $f(x,y)$ are as above and $G(x,y) = e^{-ix\cot\alpha}x^\mu C_{\nu,\mu}(ax\csc^2\alpha)$, then we have

$$\varphi(x) - \Delta^*_{1,\nu,\mu,\alpha}\varphi(x) = e^{-ix\cot\alpha}x^\mu C_{\nu,\mu}(ax\csc^2\alpha). \tag{51}$$

Now, applying $h^\alpha_{1,\nu,\mu,m}$ to the both sides and using (26) and Problem 5.1(ii), we have

$$(1+y\csc^2\alpha)\left(h^\alpha_{1,\nu,\mu,m}\varphi\right)(y) = \gamma^\alpha_{\nu,\mu}e^{ia\cot\alpha}\delta(y-a).$$

Therefore,

$$\begin{aligned}\varphi(x) &= \gamma^\alpha_{\nu,\mu}e^{ia\cot\alpha}(h^\alpha_{1,\nu,\mu,m})^{-1}\left(\delta(y-a)(1+y\csc^2\alpha)^{-1}\right)(x)\\ &= (-1)^m\gamma^\alpha_{\nu,\mu}e^{ia\cot\alpha}e^{i(\alpha-\frac{\pi}{2})m}\Big[R^{*^{-1}}_{1,\nu,\mu,\alpha}...R^{*^{-1}}_{1,\nu+m-1,\mu,\alpha}\left(h^\alpha_{1,\nu+m,\mu}\right)^{-1}\left((y\csc^2\alpha)^{m/2}\delta(y-a)(1+y\csc^2\alpha)^{-1}\right)\Big](x)\\ &= \gamma^\alpha_{\nu,\mu}e^{i(\alpha-\frac{\pi}{2})m}e^{i(a-x)\cot\alpha}x^{\frac{\mu+\nu}{2}}\int_\infty^x\int_\infty^{x_1}...\int_\infty^{x_{m-1}} x_m^{-\frac{(\mu+\nu+m)}{2}}e^{ix_m\cot\alpha}\\ &\times(-1)^m\left(h^\alpha_{1,\nu+m,\mu}\right)^{-1}\left((y\csc^2\alpha)^{m/2}\delta(y-a)(1+y\csc^2\alpha)^{-1}\right)(x_m)dx_m...dx_2dx_1.\end{aligned}$$

Proceeding similar as Problem 5.1(ii), we obtain

$$\varphi(x) = (1+a\csc^2\alpha)^{-1}e^{-ix\cot\alpha}x^\mu C_{\nu,\mu}(ax\csc^2\alpha).$$

This solve our problems.

Remark 5.2. *Analogously, applying the theory of fractional powers of second Hankel-Clifford transformation of arbitrary order* $h^\alpha_{2,\nu,\mu,n}$*, we can solve some differential equation associated with Bessel type operator* $\Delta^*_{2,\nu,\mu,\alpha}$.

Remark 5.3. *Similar results of all theorems of Sects. 2 and 3 may be proved using the technique* (23) *for* $(h^\alpha_{1,\nu,\mu,m})^{-1}$ *and* $(h^\alpha_{2,\nu,\mu,n})^{-1}$.

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