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Multiplicative Perturbation Bounds of the Group Inverse and Oblique Projection

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Abstract. In this paper, the multiplicative perturbation bounds of the group inverse and related oblique projection under general unitarily invariant norm are presented by using the decompositions of $B^\# - A^\#$ and $BB^\# - AA^\#$.

1. Introduction

Let $C^{n\times n}$ be the set of all $n\times n$ complex matrices. For a given matrix $A \in C^{n\times n}$, the symbols A^*, $\|A\|_2$, $\|A\|$, $A^\#$, and $\mathrm{Ind}(A)$ will stand for the conjugate transpose, the spectral norm, general unitarily invariant norm, the group inverse, and the index of A, respectively. I denotes the $n\times n$ identity matrix. Also, for the sake of the simplicity in the later discussion, we will adopt the following notations with $A \in C^{n\times n}$:

$$P_A = AA^\# = A^\# A, \qquad \tilde{P}_A = I - P_A.$$

In addition, we always assume that D_1 and D_2 are $n\times n$ nonsingular matrices throughout this paper.

We recall that the group inverse $A^\#$ of a matrix $A \in C^{n\times n}$ is the unique solution X, if exists, to the following three equations [1]:

$$(1)\ AXA = A, \qquad (2)\ XAX = X, \qquad (3)\ AX = XA.$$

A square matrix A has a group inverse if and only if $\mathrm{rank}(A) = \mathrm{rank}(A^2)$, i.e., $\mathrm{Ind}(A) = 1$. When $A^\#$ exists, $P_A = AA^\#$ is the projector which projects a vector on $R(A)$ along $N(A)$, i.e., $AA^\# = P_{R(A),N(A)}$, where $R(A)$ and $N(A)$ are the range and null space of A, respectively. For the details of group inverse, we refer the readers to the book by Ben-Israel and Greville [1].

The group inverse plays an important role in numerical analysis, Markov chains, etc. see [2, 8, 9]. However, in most numerical applications the elements of A will seldom be known exactly, so it is necessary to know how its group inverse is perturbed when A is perturbed. There are extensive studies in this regard, e.g., [6, 9, 10], for the so-called additive perturbations, namely A is perturbed to $B = A + E$. Since the matrix scaling technique is often used to yield better-conditioned problems [3], the multiplicative perturbation model $B = D_1^* A D_2$ with both D_1 and D_2 are nonsingular matrices and near the identities has been received much attention. Notice that $B = D_1^* A D_2$ can be rewritten as $B = A + E$ with $E = -(I - D_1^*)A - D_1^* A(I - D_2)$

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or $E = -A(I - D_2) - (I - D_1^*)AD_2$, the multiplicative perturbation is a special additive perturbation. Hence the existing additive perturbation bounds can be applied directly to give the multiplicative perturbation bounds. But in general this technique may produce unideal perturbation bounds because it overlooks the nature of the multiplicative perturbation. Various multiplicative perturbation analysis have been done to many problems, such as the polar decomposition [4], the singular value decomposition [5], and the Moore-Penrose inverse [7] when A is multiplicatively perturbed. In this paper, we will study the multiplicative perturbation bounds to the group inverse and the related oblique projection under unitarily invariant norm.

2. Multiplicative Perturbation Bounds of the Oblique Projection

In this section, we study the multiplicative perturbation bounds of the oblique projection related to group inverse under general unitarily invariant norm. Let $A \in C^{n\times n}$, $B = D_1^*AD_2$ and $\mathrm{Ind}(A) = 1$, then $B^\#$ may not exist (see Example 1). After a simple analysis, we can get the following theorem.

Theorem 2.1. *Let $A \in C^{n\times n}$ and $B = D_1^*AD_2$, then $B^\#$ exists if and only if $rank(AD_2D_1^*A) = rank(A)$.*

Example 1. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $D_1 = I_2$ and $D_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$B = D_1^*AD_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $\mathrm{rank}(A^2) = \mathrm{rank}(A) = 1$ and $0 = \mathrm{rank}(B^2) < \mathrm{rank}(B) = 1$. Hence, $B^\#$ does not exist.

In order to get the multiplicative perturbation bounds of the oblique projection, we fist give a decomposition of $BB^\# - AA^\#$.

Lemma 2.2. *Let $A \in C^{n\times n}$, $B = D_1^*AD_2$ and $Ind(A) = 1$. If $rank(AD_2D_1^*A) = rank(A)$, then*

$$P_B - P_A = P_B(I - D_2^{-1})\tilde{P}_A + \tilde{P}_B(D_1^* - I)P_A. \tag{2.1}$$

Proof. Since

$$P_B - P_A = B^\#(B - A)(I - AA^\#) + (I - BB^\#)(B - A)A^\# \tag{2.2}$$

and

$$B - A = B(I - D_2^{-1}) + (D_1^* - I)A, \tag{2.3}$$

combining (2.2) and (2.3), we immediately get (2.1). □

Based on the above lemma, we can give a multiplicative perturbation bound for $\|P_B - P_A\|$. But before stating this result, we mention another lemma which will be used.

Lemma 2.3. [10] *Let $A \in C^{n\times n}$ with $Ind(A) = 1$. Then*

$$\|AA^\#\|_2 = \|I - AA^\#\|_2.$$

Theorem 2.4. *Let $A \in C^{n\times n}$, $B = D_1^*AD_2$ and $Ind(A) = 1$. If $rank(AD_2D_1^*A) = rank(A)$, then*

$$\|P_B - P_A\| \le \|P_A\|_2\|P_B\|_2(\|I - D_1\| + \|I - D_2^{-1}\|). \tag{2.4}$$

If the used norm is the spectral norm, we have

$$\frac{\|P_B - P_A\|_2}{\|P_A\|_2} \le \|P_B\|_2(\|I - D_1\|_2 + \|I - D_2^{-1}\|_2). \tag{2.5}$$

Proof. Note that the spectral norm is a special unitarily invariant norm and (2.5) is a direct consequence of (2.4). So we only prove (2.4).

Taking the unitarily invariant norm on both sides of (2.1) and using Lemma 2.2, we get the estimate

$$\begin{aligned}\|P_B - P_A\| &\leq \|P_B\|_2\|\tilde{P}_A\|_2\|I - D_2^{-1}\| + \|P_A\|_2\|\tilde{P}_B\|_2\|I - D_1\| \\ &= \|P_A\|_2\|P_B\|_2(\|I - D_1\| + \|I - D_2^{-1}\|).\end{aligned}$$

The proof is completed. □

Corollary 2.5. *Under the assumptions of Theorem 2.4, if* $rank(AD_2D_1^*A) = rank(A)$ *and* $\gamma = \|P_A\|_2(\|I - D_1\|_2 + \|I - D_2^{-1}\|_2) < 1$, *then we have*

$$\frac{\|P_A\|_2}{1+\gamma} \leq \|P_B\|_2 \leq \frac{\|P_A\|_2}{1-\gamma} \tag{2.6}$$

and

$$\|P_B - P_A\| \leq \frac{\|P_A\|_2^2}{1-\gamma}(\|I - D_1\| + \|I - D_2^{-1}\|). \tag{2.7}$$

Proof. From (2.4), we have

$$\|P_B\|_2 - \|P_A\|_2 \leq \|P_B - P_A\|_2 \leq \|P_A\|_2\|P_B\|_2(\|I - D_1\|_2 + \|I - D_2^{-1}\|_2),$$

i.e.,

$$(1-\gamma)\|P_B\|_2 \leq \|P_A\|_2,$$

since $1 - \gamma > 0$, we get the right inequality of (2.6). Similarly, from $\|P_A\|_2 - \|P_B\|_2 \leq \|P_B - P_A\|_2$, we can get the left inequality of (2.6). It is obvious that, (2.7) is the direct consequence of (2.4) and (2.6). □

3. Multiplicative Perturbation Bounds of the Group Inverse

In this section, we will provide multiplicative perturbation bounds of the group inverse under general unitarily invariant norm. To this end, we need the following lemma.

Lemma 3.1. *Let* $A \in C^{n\times n}$, $B = D_1^*AD_2$ *such that* $Ind(A) = 1$ *and* $rank(AD_2D_1^*A) = rank(A)$, *then we have*

$$\begin{aligned}B^\# &= P_BD_2^{-1}A^\#D_1^{-*}P_B &(3.1)\\ &= (I + \Theta_1(D_1, D_2))A^\#(I + \Theta_2(D_1, D_2)), &(3.2)\end{aligned}$$

where $\Theta_1(D_1, D_2) = \tilde{P}_B(D_1^* - I) - P_B(I - D_2^{-1})$, $\Theta_2(D_1, D_2) = (D_2 - I)\tilde{P}_B - (I - D_1^{-*})P_B$ *and* D_1^{-*} *denotes the inverse of the conjugate transpose of* D_1.

Proof. Let Z be the matrix on the right side of equation (3.1). From $B = D_1^*AD_2$, we have

$$BZ = BD_2^{-1}A^\#D_1^{-*}BB^\# = D_1^*AA^\#AD_2B^\# = BB^\#$$

and

$$ZB = B^\#BD_2^{-1}A^\#D_1^{-*}B = B^\#D_1^*AD_2 = B^\#B.$$

Hence, $BZB = B$, $ZBZ = Z$ and $BZ = ZB$, i.e., $B^\# = Z$. To prove (3.2), it is only needed to prove

$$(I + \Theta_1(D_1, D_2))A^\# = P_BD_2^{-1}A^\# \text{ and } A^\#(I + \Theta_2(D_1, D_2)) = A^\#D_1^{-*}P_B.$$

In fact, we have

$$\begin{aligned}(I+\Theta_1(D_1,D_2))A^{\#} &= [I+(I-B^{\#}B)(D_1^{*}-I)-B^{\#}B(I-D_2^{-1})]AA^{\#}A^{\#}\\ &= A^{\#}-(I-B^{\#}B)A^{\#}-B^{\#}B(I-D_2^{-1})A^{\#}\\ &= P_BD_2^{-1}A^{\#}.\end{aligned}$$

Similarly, we can prove $A^{\#}(I+\Theta_2(D_1,D_2)) = A^{\#}D_1^{-*}P_B$. □

Obviously, Lemma 3.1 is valid both for full rank and rank deficient matrices. Combining (2.6) and (3.1), we can get the following corollary.

Corollary 3.2. *With the same assumptions as in Lemma 3.1. If* $\gamma = \|P_A\|_2(\|I-D_1\|_2+\|I-D_2^{-1}\|_2) < 1$ *and* $\max\{\|I-D_1\|_2, \|I-D_2\|_2\} < 1$, *we have*

$$\|B^{\#}\| \le \frac{\|P_A\|_2^2}{(1-\gamma)^2\Phi(D_1,D_2)}\|A^{\#}\|,$$

where $\Phi(D_1,D_2) = (1-\|I-D_1\|_2)(1-\|I-D_2\|_2)$.

The following corollary presents an expression for $B^{\#}-A^{\#}$ that follows directly from (3.2), which is very important to get the perturbation bound of group inverse.

Corollary 3.3. *Let* $\Theta_1(D_1,D_2)$ *and* $\Theta_2(D_1,D_2)$ *be the same as in Lemma 3.1. With the same assumptions as in Lemma 3.1, we have*

$$B^{\#}-A^{\#} = \Theta_1(D_1,D_2)A^{\#}+A^{\#}\Theta_2(D_1,D_2)+\Theta_1(D_1,D_2)A^{\#}\Theta_2(D_1,D_2). \tag{3.3}$$

Next, we state the main result in this section.

Theorem 3.4. *With the same assumptions as in Lemma 3.1, then*

$$\begin{aligned}\|B^{\#}-A^{\#}\| \le\ & \|A^{\#}\|_2[\|P_B\|_2(\|I-D_1\|+\|I-D_2^{-1}\|+\|I-D_1^{-1}\|+\|I-D_2\|)\\ &+\|P_B\|_2^2(\|I-D_1\|+\|I-D_2^{-1}\|)(\|I-D_1^{-1}\|+\|I-D_2\|)].\end{aligned} \tag{3.4}$$

If $\gamma = \|P_A\|_2(\|I-D_1\|_2+\|I-D_2^{-1}\|_2) < 1$, *we have*

$$\begin{aligned}\|B^{\#}-A^{\#}\| \le\ & \|A^{\#}\|_2[\frac{\|P_A\|_2}{1-\gamma}(\|I-D_1\|+\|I-D_2^{-1}\|+\|I-D_1^{-1}\|+\|I-D_2\|)\\ &+\frac{\|P_A\|_2^2}{(1-\gamma)^2}(\|I-D_1\|+\|I-D_2^{-1}\|)(\|I-D_1^{-1}\|+\|I-D_2\|)].\end{aligned} \tag{3.5}$$

Proof. From Lemma 2.2, we can get

$$\|\Theta_1(D_1,D_2)\| \le \|P_B\|_2(\|I-D_1\|+\|I-D_2^{-1}\|) \tag{3.6}$$

and

$$\|\Theta_2(D_1,D_2)\| \le \|P_B\|_2(\|I-D_2\|+\|I-D_1^{-1}\|). \tag{3.7}$$

Hence, (3.4) follows directly from (3.3), (3.6) and (3.7). Substituting (2.6) into (3.4), we can get (3.5) immediately. □

4. Numerical Comparison

In this section, we give an example to illustrate that the multiplicative perturbation bound is much better, in some cases, than the additive perturbation bound.

To compare the multiplicative perturbation bound with the additive perturbation bound, we first mention the additive perturbation bounds of group inverse and related oblique projection which were obtained in [10]. Let $A, B = A + E \in C^{n\times n}$ and $\mathrm{Ind}(A) = \mathrm{Ind}(B) = 1$, then

$$\|B^{\#} - A^{\#}\|_2 \leq (1 + 2\max\{\|P_A\|_2, \|P_B\|_2\}) \max\{\|A^{\#}\|_2^2, \|B^{\#}\|_2^2\}\|E\|_2 \tag{4.1}$$

and

$$\|P_B - P_A\|_2 \leq 2\max\{\|P_A\|_2, \|P_B\|_2\} \max\{\|A^{\#}\|_2, \|B^{\#}\|_2\}\|E\|_2. \tag{4.2}$$

Example 2. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $D_2 = \begin{pmatrix} 1.001 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $D_1 = I_3$. Then

$$B = D_1^* A D_2 = \begin{pmatrix} 2.002 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } E = B - A = \begin{pmatrix} 0.002 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The estimates of the additive and multiplicative perturbation bounds of group inverse and associated oblique projector can be found in the following table (we choose the spectral norm):

	Exact value	Additive bound (4.1)	Multiplicative bound (3.4)
$\|B^{\#} - A^{\#}\|_2$	0.0005	60	0.2000
	Exact value	Additive bound (4.2)	Multiplicative bound (2.4)
$\|P_B - P_A\|_2$	0	0.4	0.0010

Obviously, the multiplicative perturbation bounds of the group inverse and the oblique projection are much better than the additive perturbation bounds for the above-mentioned example.

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