Second Order Alternating Harmonic Number Sums

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Abstract. We develop new closed form representations of sums of alternating harmonic numbers of order two and reciprocal binomial coefficients. Moreover we develop new integral representations in terms of harmonic numbers of order two.

1. Introduction and Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers and let $\mathbb{N} := \{1, 2, 3, \cdots\}$ be the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A generalized binomial coefficient $\binom{\lambda}{\mu}$ $(\lambda, \mu \in \mathbb{C})$ is defined, in terms of the familiar (Euler's) gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda + 1)}{\Gamma(\mu + 1)\Gamma(\lambda - \mu + 1)}, \qquad (\lambda, \mu \in \mathbb{C}),$$

which, in the special case when $\mu = n$, $n \in \mathbb{N}_0$, yields

$$\binom{\lambda}{0} := 1 \quad \text{and} \quad \binom{\lambda}{n} := \frac{\lambda(\lambda - 1) \cdots (\lambda - n + 1)}{n!} = \frac{(-1)^n (-\lambda)_n}{n!} \quad (n \in \mathbb{N}),$$

where $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is the Pochhammer symbol defined, also in terms of the gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \ \lambda \in \mathbb{C} \setminus \{0\}) \\ \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed that the Γ-quotient exists. Let

$$H_n = \sum_{r=1}^n \frac{1}{r} = \gamma + \psi(n + 1) = \int_0^1 \frac{1 - t^n}{1 - t} \, dt \qquad (H_0 := 0) \tag{1}$$

be the *nth* harmonic number. Here, as usual, γ denotes the Euler-Mascheroni constant and $\psi(z)$ is the Psi (or Digamma) function defined by

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \psi(t)\, dt.$$

A *generalized harmonic number* $H_n^{(m)}$ of order m is defined, for positive integers n and m, as follows:

$$H_n^{(m)} := \sum_{r=1}^n \frac{1}{r^m}, \quad (m, n \in \mathbb{N}) \qquad \text{and} \qquad H_0^{(m)} := 0 \qquad (m \in \mathbb{N}).$$

In the case of *non-integer* values of n such as (for example) a value $\rho \in \mathbb{R}$, the generalized harmonic numbers $H_\rho^{(m+1)}$ may be defined, in terms of the Polygamma functions

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} = \frac{d^{n+1}}{dz^{n+1}}\{\log \Gamma(z)\} \qquad (n \in \mathbb{N}_0),$$

by

$$H_\rho^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(\rho + 1) \tag{2}$$

$$(\rho \in \mathbb{R} \setminus \{-1, -2, -3, \cdots\};\ m \in \mathbb{N}),$$

where $\zeta(z)$ is the *Riemann zeta function*. Whenever we encounter harmonic numbers of the form $H_\rho^{(m)}$ at admissible real values of ρ, they may be evaluated by means of this known relation (2). In the exceptional case of (2) when $m = 0$, we may define $H_\rho^{(1)}$ by

$$H_\rho^{(1)} = H_\rho = \gamma + \psi(\rho + 1) \qquad (\rho \in \mathbb{R} \setminus \{-1, -2, -3, \cdots\}).$$

We assume (as above) that

$$H_0^{(m)} = 0 \qquad (m \in \mathbb{N}).$$

In the case of non integer values of the argument $z = \frac{r}{q}$, we may write the generalized harmonic numbers, $H_z^{(\alpha+1)}$, in terms of polygamma functions

$$H_{\frac{r}{q}}^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^\alpha}{\alpha!}\psi^{(\alpha)}\left(\frac{r}{q}+1\right), \ \frac{r}{q} \neq \{-1, -2, -3, \ldots\},$$

where $\zeta(z)$ is the zeta function. When we encounter harmonic numbers at possible rational values of the argument, of the form $H_{\frac{r}{q}}^{(\alpha)}$ they maybe evaluated by an available relation in terms of the polygamma function $\psi^{(\alpha)}(z)$ or, for rational arguments $z = \frac{r}{q}$, and we also define

$$H_{\frac{r}{q}}^{(1)} = \gamma + \psi\left(\frac{r}{q}+1\right), \text{ and } H_0^{(\alpha)} = 0.$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [10], or Choi and Cvijovic [2] in terms of the Polylogarithmic or other special functions. Some specific values are listed in the books[16], [19] and [20]. In this paper we

will develop identities, closed form representations of alternating harmonic numbers of order two and reciprocal binomial coefficients of the form:

$$S_k^{(2)}(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n^p \binom{n+k}{k}}, \tag{3}$$

for $p = 0$ and 1. While there are many results for sums of harmonic numbers with positive terms, there are fewer results for sums of the type (3). Let us define the alternating zeta function

$$\bar{\zeta}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = \left(1 - 2^{1-z}\right) \zeta(z)$$

with $\bar{\zeta}(1) = \ln 2$, and

$$S_{p,q}^{+-} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(p)}}{n^q}.$$

SitaramachandraRao [12] gave, for $1 + q$ an odd integer,

$$2S_{1,q}^{+-} = (1 + q) \bar{\zeta}(1 + q) - \zeta(1 + q) - 2 \sum_{j=1}^{\frac{q}{2}-1} \bar{\zeta}(2j)\, \zeta(1 + q - 2j)$$

and in another special case, gave the integral

$$S_{1,1+2q}^{+-} = \int_0^1 \frac{\ln^{2q}(x) \ln(1+x)}{x(1+x)} dx.$$

In the case where p and q are both positive integers and $p + q$ is an odd integer, Flajolet and Salvy [9] gave the identity:

$$2S_{p,q}^{+-} = (1 - (-1)^p) \zeta(p) \bar{\zeta}(q) + 2 \sum_{i+2k=q} \binom{p+i-1}{p-1} \zeta(p+i)\, \bar{\zeta}(2k)$$

$$+ \bar{\zeta}(p+q) - 2 \sum_{j+2k=p} \binom{q+j-1}{q-1} (-1)^j \bar{\zeta}(q+j)\, \bar{\zeta}(2k) ,$$

where $\bar{\zeta}(0) = \frac{1}{2}$, $\bar{\zeta}(1) = \ln 2$, $\zeta(1) = 0$, and $\zeta(0) = -\frac{1}{2}$ in accordance with the analytic continuation of the Riemann zeta function. Flajolet and Salvy [9] further, gave some specific examples, such as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{2n+1} = \frac{\pi \ln 2}{2} - G$$

where

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx .91596 \text{ is Catalan's constant.}$$

Some other interesting cases are given by Coffey [6]

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{(n+1)^2} = \frac{65}{16}\zeta(4) + \zeta(2)\ln^2 2 - \frac{1}{6}\ln^4 2 - \frac{7}{2}\zeta(3)\ln 2 - 4L_{i_4}\left(\frac{1}{2}\right)$$

where

$$L_{i_p}(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^p}, \ p \in \mathbb{C} \text{ when } |z| < 1; \ \mathbb{R}(p) > 1 \text{ when } |z| = 1$$

is the Polylogarithm, or deJonquière's function, Coffey, [7] also gave the expression

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \psi^{(p)}(n+a+1)}{n^q} = (-1)^q \int_0^1 \int_0^1 \frac{x^{a+1}\ln^{q-1}(y)\ln^p(x)}{(1+xy)(1-x)}dxdy,$$

where $\psi^{(p)}(\cdot)$ is the Polygamma function. Some results for finite sums of alternating harmonic numbers may be seen in the works of [1], [3], [4], [8], [11], [13], [14], [15], [18], [21], [22], [23] and references therein. Chu [5] has also given some closed form representations of mainly finite alternating sums of first order harmonic numbers. The following lemma will be useful in the development of the main theorems.

Lemma 1.1. *Let r be a positive integer and $p \in \mathbb{N}$. Then:*

$$\sum_{j=1}^{r} \frac{(-1)^j}{j^p} = \frac{1}{2^p}\left(H_{[\frac{r}{2}]}^{(p)} + H_{[\frac{r-1}{2}]}^{(p)}\right) - H_{2[\frac{r+1}{2}]-1}^{(p)} \tag{4}$$

where [x] is the integer part of x, and when $p = 1$,

$$\sum_{j=1}^{r} \frac{(-1)^j}{j} = H_{[\frac{r}{2}]} - H_r. \tag{5}$$

For $p = 2$,

$$\sum_{j=1}^{r} \frac{(-1)^j}{j^2} = \frac{1}{4}\left(H_{[\frac{r}{2}]}^{(2)} - H_{[\frac{r+1}{2}]-\frac{1}{2}}^{(2)}\right) - \frac{1}{2}\zeta(2). \tag{6}$$

Proof. To prove (4) we write,

$$\sum_{j=1}^{r} \frac{(-1)^j}{j^p} = \frac{1}{2^p}\sum_{j=1}^{[\frac{r}{2}]} \frac{1}{j^p} - \sum_{j=1}^{[\frac{r+1}{2}]} \frac{1}{(2j-1)^p}$$

$$= \frac{1}{2^p}H_{[\frac{r}{2}]}^{(p)} - H_{2[\frac{r-1}{2}]-1}^{(p)} + \frac{1}{2^p}H_{[\frac{r-1}{2}]}^{(p)}$$

$$= \frac{1}{2^p}\left(H_{[\frac{r}{2}]}^{(p)} + H_{[\frac{r-1}{2}]}^{(p)}\right) - H_{2[\frac{r+1}{2}]-1}^{(p)}.$$

For $p = 1$,

$$\sum_{j=1}^{r} \frac{(-1)^j}{j} = H_r + H_{[\frac{r}{2}]} - 2H_{[\frac{2r+1}{2}]}, \text{ since } r \text{ is a positive integer}$$

$$= H_{[\frac{r}{2}]} - H_r.$$

For $p = 2$,

$$\sum_{j=1}^{r} \frac{(-1)^j}{j^2} = \frac{1}{4} \sum_{j=1}^{\left[\frac{r}{2}\right]} \frac{1}{j^2} - \sum_{j=1}^{\left[\frac{r+1}{2}\right]} \frac{1}{(2j-1)^2}$$

$$= \frac{1}{4} H_{\left[\frac{r}{2}\right]}^{(2)} - \frac{3}{4} \zeta(2) + \frac{1}{4} \left(\zeta(2) - H_{\left[\frac{r+1}{2}\right]-\frac{1}{2}}^{(2)} \right)$$

$$= \frac{1}{4} \left(H_{\left[\frac{r}{2}\right]}^{(2)} - H_{\left[\frac{r+1}{2}\right]-\frac{1}{2}}^{(2)} \right) - \frac{1}{2} \zeta(2).$$

$\square$

Lemma 1.2. *The following identity holds.*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n} = \zeta(3) - \frac{1}{2} \zeta(2) \ln 2. \tag{7}$$

Proof. From the definition of the trigamma function, or from (1),

$$\ln(\Gamma(n+1))'' = \psi'(n+1) = \zeta(2) - H_n^{(2)} = - \int_0^1 \frac{x^n \ln x}{1-x} dx$$

then

$$H_n^{(2)} = - \int_0^1 \frac{(1-x^n) \ln x}{1-x} dx$$

or more generally, for $p \in \mathbb{N}_0$,

$$H_n^{(p+1)} = \frac{(-1)^p}{p!} \int_0^1 \frac{(1-x^n) (\ln x)^p}{1-x} dx.$$

For $p = 1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n} = \int_0^1 \frac{\ln x}{1-x} \sum_{n=1}^{\infty} \frac{(-1)^n (1-x^n)}{n} dx$$

$$= \int_0^1 \frac{\ln x}{1-x} (\log(1+x) - \log 2) dx$$

$$= \zeta(3) - \frac{3}{2} \zeta(2) \ln 2 + \zeta(2) \ln 2,$$

hence (7) follows. By a change of counter we also have the identity

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n+1} = -\frac{1}{4} \zeta(3) + \frac{1}{2} \zeta(2) \ln 2. \tag{8}$$

$\square$

Lemma 1.3. *Let r be a positive integer, then*

$$S_r^{(2)} := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n+r} \tag{9}$$

$$= (-1)^r \left(\frac{1}{4} \zeta(3) - \frac{1}{2} \zeta(2) \ln 2 \right) + \frac{(-1)^r \zeta(2)}{2} \left(H_{r-1} - H_{\left[\frac{r-1}{2} \right]} \right)$$

$$+ (-1)^{r+1} \left(H_{r-1}^{(2)} - \frac{1}{4} \left(H_{\left[\frac{r-1}{2} \right]}^{(2)} + H_{\left[\frac{r-2}{2} \right]}^{(2)} \right) + H_{2\left[\frac{r}{2} \right]-1}^{(2)} \right) \ln 2$$

$$+ (-1)^r H_{r-1}^{(2)} \left(H_r - H_{\left[\frac{r}{2} \right]} \right) + (-1)^r \sum_{j=1}^{r-1} (-1)^{j+1} \left(\frac{H_{\left[\frac{r-j}{2} \right]} - H_{r-j}}{j^2} + \frac{H_j^{(2)}}{j+1} \right)$$

$$+ (-1)^{r+1} \sum_{j=1}^{r-1} \frac{(-1)^j}{j} \left(\frac{1}{4} \left(H_{\left[\frac{r-j}{2} \right]}^{(2)} + H_{\left[\frac{r-j-1}{2} \right]}^{(2)} \right) - H_{2\left[\frac{r-j+1}{2} \right]-1}^{(2)} \right).$$

Proof. by a change of counter

$$S_r^{(2)} \quad : \quad = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n+r} = \sum_{n=r}^{\infty} \frac{(-1)^{n-r} H_{n-(r-1)}^{(2)}}{n+1}$$

$$= \sum_{n=r}^{\infty} \frac{(-1)^{n-r}}{n+1} \left(H_n^{(2)} - \sum_{j=1}^{r-1} \frac{1}{(n+1-j)^2} \right)$$

$$= (-1)^{r+1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n+1} - (-1)^{r+1} \sum_{n=1}^{r-1} \frac{(-1)^{n+1} H_n^{(2)}}{n+1}$$

$$- \sum_{n=r}^{\infty} \frac{(-1)^{n-r}}{n+1} \sum_{j=1}^{r-1} \frac{1}{(n+1-j)^2}.$$

From lemma 1.1 and using the known results,

$$S_r^{(2)} = (-1)^r \left(\frac{1}{4} \zeta(3) - \frac{1}{2} \zeta(2) \ln 2 \right) - (-1)^{r+1} \sum_{n=1}^{r-1} \frac{(-1)^{n+1} H_n^{(2)}}{n+1}$$

$$- \sum_{n=r}^{\infty} \frac{(-1)^{n-r}}{n+1} \sum_{j=1}^{r-1} \frac{1}{(n+1-j)^2}. \tag{10}$$

Let us now consider the last term in (10).

$$\sum_{n=r}^{\infty} \frac{(-1)^{n-r}}{n+1} \sum_{j=1}^{r-1} \frac{1}{(n+1-j)^2} \tag{11}$$

$$= \sum_{n=r}^{\infty} \frac{(-1)^{n-r}}{n+1} \left(\frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{(n+2-r)^2} \right)$$

$$= - \sum_{n=r}^{\infty} (-1)^{n-r} \left[\begin{array}{c} \frac{1}{n^2} - \left(\frac{1}{n} - \frac{1}{n+1} \right) + \frac{1}{2(n-1)^2} - \frac{1}{4} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \\ + \frac{1}{3(n-2)^2} - \frac{1}{3^2} \left(\frac{1}{n2-1} - \frac{1}{n+1} \right) + \cdots + \frac{1}{(r-1)(n+2-r)^2} \\ - \frac{1}{(r-1)^2} \left(\frac{1}{n+2-r} - \frac{1}{n+1} \right) \end{array} \right]$$

$$= -\sum_{n=r}^{\infty} \frac{(-1)^{n-r}}{n+1} \sum_{j=1}^{r-1} \frac{1}{j^2} + \sum_{n=r}^{\infty} (-1)^{n-r} \sum_{j=1}^{r-1} \frac{1}{j^2(n+1-j)}$$

$$-\sum_{n=r}^{\infty} (-1)^{n-r} \sum_{j=1}^{r-1} \frac{1}{j(n+1-j)^2} \tag{12}$$

Consider the first term in (12).

$$T_r := -\sum_{n=r}^{\infty} \frac{(-1)^{n-r}}{n+1} \sum_{j=1}^{r-1} \frac{1}{j^2} = (-1)^r \sum_{n=r}^{\infty} \frac{(-1)^{n+1}}{n+1} H_{r-1}^{(2)}$$

$$= (-1)^r H_{r-1}^{(2)} \sum_{n=r+1}^{\infty} \frac{(-1)^{n+1}}{n} = (-1)^r H_{r-1}^{(2)} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{r} \frac{(-1)^{n+1}}{n} \right)$$

$$= (-1)^r H_{r-1}^{(2)} \left(\ln 2 + \sum_{j=1}^{r} \frac{(-1)^j}{j} \right) = (-1)^r H_{r-1}^{(2)} \left(\ln 2 + \left(H_{\left[\frac{r}{2}\right]} - H_r \right) \right)$$

and from Lemma 1.1

$$T_r := (-1)^r \left(H_{r-1}^{(2)} \ln 2 + H_{r-1}^{(2)} \left(H_{\left[\frac{r}{2}\right]} - H_r \right) \right).$$

The second term in (12) we write as

$$W_r := \sum_{n=r}^{\infty} (-1)^{n-r} \sum_{j=1}^{r-1} \frac{1}{j^2(n+1-j)} = \sum_{n=r}^{\infty} (-1)^{n-r} \left(\begin{array}{c} \frac{1}{n} + \frac{1}{2^2(n-1)} + \frac{1}{3^2(n-2)} \\ +... + \frac{1}{(r-1)^2(n-(r-2))} \end{array} \right),$$

changing the counter term by term, we have

$$W_r = (-1)^{r+1} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{r-1} \frac{(-1)^{n+1}}{n} \right) + \frac{(-1)^r}{2^2} \left(\begin{array}{c} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \\ -\sum_{n=1}^{r-2} \frac{(-1)^{n+1}}{n} \end{array} \right)$$

$$+... + \frac{(-1)^{r-(r-1)}}{(r-1)^2} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{1} \frac{(-1)^{n+1}}{n} \right)$$

$$= (-1)^{r+1} \left(\ln 2 - \sum_{j=1}^{r-1} \frac{(-1)^{j+1}}{j} \right) + \frac{(-1)^r}{2^2} \left(\ln 2 - \sum_{j=1}^{r-2} \frac{(-1)^{j+1}}{j} \right)$$

$$+... + \frac{(-1)^{r-(r-1)}}{(r-1)^2} \left(\ln 2 - \sum_{j=1}^{1} \frac{(-1)^{j+1}}{j} \right)$$

and collecting like terms

$$W_r = (-1)^{r+1} \sum_{j=1}^{r-1} \frac{(-1)^{j+1}}{j^2} \ln 2 + (-1)^r \sum_{j=1}^{r-1} \sum_{m=1}^{r-j} \frac{(-1)^{m+j+1}}{j^2 m}$$

$$= (-1)^{r+1} \sum_{j=1}^{r-1} \frac{(-1)^{j+1}}{j^2} \left(\ln 2 - \left(H_{\left[\frac{r-j}{2}\right]} - H_{r-j} \right) \right).$$

The last term in (12) is

$$X_r := -\sum_{n=r}^{\infty} (-1)^{n-r} \sum_{j=1}^{r-1} \frac{1}{j(n+1-j)^2} = \sum_{n=r}^{\infty} (-1)^{n+1-r} \left(\begin{array}{c} \frac{1}{n^2} + \frac{1}{2(n-1)^2} + \cdots \\ \\ + \frac{1}{(r-1)(n+2-r)^2} \end{array} \right)$$

$$= \frac{(-1)^r}{1} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} - \sum_{n=1}^{r-1} \frac{(-1)^{n+1}}{n^2} \right) + \frac{(-1)^{r+1}}{2} \left(\begin{array}{c} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \\ \\ - \sum_{n=1}^{r-2} \frac{(-1)^{n+1}}{n^2} \end{array} \right)$$

$$+ \frac{(-1)^r}{3} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} - \sum_{n=1}^{r-3} \frac{(-1)^{n+1}}{n^2} \right) + \cdots$$

$$\cdots + \frac{(-1)^{r-(r-1)}}{(r-1)} \left(\begin{array}{c} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \\ \\ - \sum_{n=1}^{r-(r-1)} \frac{(-1)^{n+1}}{n^2} \end{array} \right).$$

Replacing T_r W_r and X_r in (12) and then from (10) we can write

$$S_r^{(2)} = (-1)^r \left(\frac{1}{4} \zeta(3) - \frac{1}{2} \zeta(2) \ln 2 \right) - (-1)^{r+1} \sum_{j=1}^{r-1} \frac{(-1)^{j+1} H_j^{(2)}}{j+1}$$

$$+ T_r + W_r + X_r$$

and after simplification (9) follows. $\square$

Remark 1.4. *The alternating harmonic sum, $S_r^{(2)}$ admits a recurrence relation as follows.*

$$S_r^{(2)} + S_{r-1}^{(2)} = \frac{\zeta(2)}{2(r-1)} - \frac{\ln 2}{(r-1)^2} + \frac{H_{\frac{r-1}{2}} - H_{\frac{r-2}{2}}}{2(r-1)^2}, \text{for } r > 1$$

with

$$S_0^{(2)} = \zeta(3) - \frac{1}{2} \zeta(2) \ln 2, \text{ and}$$

$$S_1^{(2)} = -\frac{1}{4} \zeta(3) + \frac{1}{2} \zeta(2) \ln 2.$$

By successive iteration the recurrence relation admits the solution

$$S_r^{(2)} = (-1)^{r+1} S_1^{(2)} + \frac{(-1)^r \zeta(2)}{2} \left(H_{r-1} - H_{\left[\frac{r-1}{2} \right]} \right)$$

$$+ (-1)^{r+1} \left(H_{2\left[\frac{r}{2} \right]-1}^{(2)} - \frac{1}{4} \left(H_{\left[\frac{r-1}{2} \right]}^{(2)} + H_{\left[\frac{r-2}{2} \right]}^{(2)} \right) \right)$$

$$+ (-1)^r \sum_{j=1}^{r-1} (-1)^j \left(\frac{H_{\frac{j}{2}} - H_{\frac{j-1}{2}}}{2j^2} \right). \tag{13}$$

Analytically (9) and (13) are identical although all the $\ln 2$ coefficients in (13) have not been isolated, however they are embedded in the harmonic numbers of the form $H_{\frac{j}{2}}$.

It is of some interest to note that $S_r^{(2)}$ may be expanded in a slightly different way so that it gives rise to another unexpected harmonic series identity. This is pursued in the next lemma.

Lemma 1.5. *For a positive integer $r > 1$, we have the identity*

$$V_r^{(2)} := \sum_{j=1}^{\infty} \frac{\left(8j^2 - 4j + 1\right)\left(H_{j+\frac{r-2}{2}} - H_{j+\frac{r-3}{2}}\right)}{j^2 \left(2j - 1\right)^2}$$

$$= 4\left(-1\right)^r \left(\frac{1}{2}\zeta\left(3\right) - \zeta\left(2\right)\ln 2\right) + 4\left(-1\right)^r \zeta\left(2\right)\left(H_{r-1} - H_{\left[\frac{r-1}{2}\right]}\right) - \frac{4H_{\frac{r-1}{2}}}{\left(r-1\right)^2}$$

$$+ \frac{2\zeta\left(2\right)}{r-1} + 8\left(-1\right)^{r+1}\left[H_{r-1}^{(2)} - \frac{1}{4}\left(H_{\left[\frac{r-1}{2}\right]}^{(2)} + H_{\left[\frac{r-2}{2}\right]}^{(2)}\right) + H_{2\left[\frac{r}{2}\right]-1}^{(2)}\right]\ln 2$$

$$+ 8\left(-1\right)^r H_{r-1}^{(2)}\left(H_r - H_{\left[\frac{r}{2}\right]}\right) + 8\left(-1\right)^r \sum_{j=1}^{r-1}\left(-1\right)^{j+1}\left(\frac{H_{\left[\frac{r-j}{2}\right]} - H_{r-j}}{j^2} + \frac{H_j^{(2)}}{j+1}\right)$$

$$+ 8\left(-1\right)^{r+1}\sum_{j=1}^{r-1}\frac{\left(-1\right)^j}{j}\left(\frac{1}{4}\left(H_{\left[\frac{r-j}{2}\right]}^{(2)} + H_{\left[\frac{r-j-1}{2}\right]}^{(2)}\right) - H_{2\left[\frac{r-j+1}{2}\right]-1}^{(2)}\right). \tag{14}$$

For $r = 1$,

$$V_1^{(2)} = \sum_{j=1}^{\infty} \frac{\left(8j^2 - 4j + 1\right)\left(H_{j-\frac{1}{2}} - H_{j-1}\right)}{j^2 \left(2j - 1\right)^2} = -\zeta\left(3\right) + 4\zeta\left(2\right)\ln 2,$$

and for $r = 0$,

$$V_0^{(2)} = \sum_{j=1}^{\infty} \frac{\left(8j^2 - 4j + 1\right)\left(H_{j-1} - H_{j-\frac{3}{2}}\right)}{j^2 \left(2j - 1\right)^2} = 8\zeta\left(3\right) - 4\zeta\left(2\right)\ln 2 - 2\zeta\left(2\right) + 8\ln 2.$$

Proof. By expansion

$$S_r^{(2)} := \sum_{n=1}^{\infty} \frac{\left(-1\right)^{n+1} H_n^{(2)}}{n+r} = \sum_{n=1}^{\infty}\left(\frac{H_{2n}^{(2)}}{\left(2n+r\right)\left(2n+r-1\right)} - \frac{1}{\left(2n\right)^2\left(2n+r-1\right)}\right),$$

since we know that $H_{2n}^{(2)} = \frac{1}{4}H_n^{(2)} + \sum_{j=1}^{n}\frac{1}{\left(2j-1\right)^2}$ then,

$$S_r^{(2)} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{\left(2n+r\right)\left(2n+r-1\right)} + \sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{1}{\left(2j-1\right)^2\left(2n+r\right)\left(2n+r-1\right)}$$

$$- \sum_{n=1}^{\infty}\frac{1}{\left(2n\right)^2\left(2n+r-1\right)}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{1}{j^2\left(2n+r\right)\left(2n+r-1\right)} + \sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{1}{\left(2j-1\right)^2\left(2n+r\right)\left(2n+r-1\right)}.$$

$$+ \frac{H_{\frac{r-1}{2}}}{2\left(r-1\right)^2} - \frac{\zeta\left(2\right)}{4\left(r-1\right)}. \tag{15}$$

For an arbitrary double sequence $Y_{k,l}$ we have that

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k} Y_{k,l} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} Y_{k,l+k},$$

and from (15)

$$
\begin{aligned}
V_r^{(2)} \quad &= \quad \sum_{n=1}^{\infty}\sum_{j=1}^{n} \frac{1}{(2j-1)^2\,(2n+r)\,(2n+r-1)} + \frac{1}{4}\sum_{n=1}^{\infty}\sum_{j=1}^{n} \frac{1}{j^2\,(2n+r)\,(2n+r-1)} \\[2mm]
&= \quad \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\left(\begin{array}{c} \frac{1}{(2j-1)^2(2n+2j+r)(2n+2j+r-1)} \\[2mm] +\frac{1}{4j^2(2n+2j+r)(2n+2j+r-1)} \end{array} \right) \\[2mm]
&= \quad \sum_{j=1}^{\infty} \frac{\left(8j^2-4j+1\right)\left(H_{j+\frac{r-2}{2}} - H_{j+\frac{r-3}{2}}\right)}{j^2\,(2j-1)^2}.
\end{aligned}
$$

Upon utilizing (15) and the known result (9) for $S_r^{(2)}$, we are able to write

$$V_r^{(2)} \quad : \quad = \sum_{j=1}^{\infty} \frac{\left(8j^2-4j+1\right)\left(H_{j+\frac{r-2}{2}} - H_{j+\frac{r-3}{2}}\right)}{j^2\,(2j-1)^2} = 8S_r^{(2)} + \frac{2\zeta(2)}{r-1} - \frac{4H_{\frac{r-1}{2}}}{(r-1)^2}.$$

Substituting for $S_r^{(2)}$ and upon simplification we have the result (14) for $V_r^{(2)}$. $\quad\square$

Example 1.6. *We have some examples as follows:*

$$
\begin{aligned}
V_0^{(2)} \quad &= \quad \sum_{j=1}^{\infty} \frac{\left(8j^2-4j+1\right)\left(H_{j-1} - H_{j-\frac{3}{2}}\right)}{j^2\,(2j-1)^2} = 8\zeta(3) - 4\zeta(2)\ln 2 - 2\zeta(2) \\
&\quad +8\ln 2,
\end{aligned}
$$

$$V_1^{(2)} \quad = \quad \sum_{j=1}^{\infty} \frac{\left(8j^2-4j+1\right)\left(H_{j-\frac{1}{2}} - H_{j-1}\right)}{j^2\,(2j-1)^2} = -\zeta(3) + 4\zeta(2)\ln 2,$$

$$
\begin{aligned}
V_2^{(2)} \quad &= \quad \sum_{j=1}^{\infty} \frac{\left(8j^2-4j+1\right)\left(H_{j} - H_{j-\frac{1}{2}}\right)}{j^2\,(2j-1)^2} = 2\zeta(3) - 4\zeta(2)\ln 2 + 6\zeta(2) \\
&\quad -8\ln 2, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
V_3^{(2)} \quad &= \quad \sum_{j=1}^{\infty} \frac{\left(8j^2-4j+1\right)\left(H_{j+\frac{1}{2}} - H_{j}\right)}{j^2\,(2j-1)^2} = -2\zeta(3) + 4\zeta(2)\ln 2 - \zeta(2) \\
&\quad +16\ln 2 - 10.
\end{aligned}
$$

The next few theorems relate the main results of this investigation, namely the closed form and integral representation of (3).

2. Closed Form and Integral Identities

We now prove the following theorems.

Theorem 2.1. *Let k be real positive integer, then from (3) with p = 0 we have*

$$S_k^{(2)}(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{\binom{n+k}{k}} = 2^{k-2}k\left(\zeta(2)\ln 2 - \frac{1}{2}\zeta(3)\right)$$

(16)

$$+ \sum_{r=1}^{k} r\binom{k}{r}\begin{pmatrix} +H_{r-1}^{(2)}\left(H_{\left[\frac{r}{2}\right]} - H_r\right) + \frac{\zeta(2)}{2}\left(H_{\left[\frac{r-1}{2}\right]} - H_{r-1}\right) \\ \\ +\left(H_{r-1}^{(2)} - \frac{1}{4}\left(H_{\left[\frac{r-1}{2}\right]}^{(2)} + H_{\left[\frac{r-2}{2}\right]}^{(2)}\right) + H_{2\left[\frac{r}{2}\right]-1}^{(2)}\right)\ln 2 \\ \\ -\sum_{j=1}^{r-1}(-1)^{j+1}\left(\frac{H_{\left[\frac{r-j}{2}\right]} - H_{r-j}}{j^2} + \frac{H_j^{(2)}}{j+1}\right) \\ \\ +\sum_{j=1}^{r-1}\frac{(-1)^{j+1}}{j}\left(\frac{1}{4}\left(H_{\left[\frac{r-j}{2}\right]}^{(2)} + H_{\left[\frac{r-j-1}{2}\right]}^{(2)}\right) - H_{2\left[\frac{r-j+1}{2}\right]-1}^{(2)}\right)\end{pmatrix}.$$

Proof. Consider the expansion

$$S_k^{(2)}(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{\binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} k!\, H_n^{(2)}}{(n+1)_{1+k}}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} k!\, H_n^{(2)} \sum_{r=1}^{k} \frac{\Omega_r}{n+r}$$

where

$$\Omega_r = \lim_{n\to -r}\left\{\frac{n+r}{\prod_{r=1}^{k} n+r}\right\} = \frac{(-1)^{r+1}\, r}{k!}\binom{k}{r},$$

hence

$$S_k^{(2)}(0) = \sum_{r=1}^{k}(-1)^{r+1} r\binom{k}{r}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} H_n^{(2)}}{n+r}$$

$$= \sum_{r=1}^{k}(-1)^{r+1} r\binom{k}{r}S_r^{(2)}.$$

(17)

From lemma 1.3, (9) substituting into (17), yields the result (16). □

The other case of $S_k^{(2)}(1)$, can be evaluated in a similar fashion. We list the result in the next corollary.

Corollary 2.2. *Under the assumptions of Theorem 2.1, then from (3) with $p = 1$*

$$S_k^{(2)}(1) = \left(\frac{3 + 2^k}{4}\right)\zeta(3) - 2^{k-1}\zeta(2)\ln 2 \tag{18}$$

$$-\sum_{r=1}^{k}\binom{k}{r}\begin{pmatrix} +H_{r-1}^{(2)}\left(H_{\left[\frac{r}{2}\right]} - H_r\right) + \frac{\zeta(2)}{2}\left(H_{\left[\frac{r-1}{2}\right]} - H_{r-1}\right) \\[2mm] +\left(H_{r-1}^{(2)} - \frac{1}{4}\left(H_{\left[\frac{r-1}{2}\right]}^{(2)} + H_{\left[\frac{r-2}{2}\right]}^{(2)}\right) + H_{2\left[\frac{r}{2}\right]-1}^{(2)}\right)\ln 2 \\[2mm] -\sum_{j=1}^{r-1}(-1)^{j+1}\left(\frac{H_{\left[\frac{r-j}{2}\right]} - H_{r-j}}{j^2} + \frac{H_j^{(2)}}{j+1}\right) \\[2mm] +\sum_{j=1}^{r-1}\frac{(-1)^{j+1}}{j}\left(\frac{1}{4}\left(H_{\left[\frac{r-j}{2}\right]}^{(2)} + H_{\left[\frac{r-j-1}{2}\right]}^{(2)}\right) - H_{2\left[\frac{r-j+1}{2}\right]-1}^{(2)}\right) \end{pmatrix}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,H_n^{(2)}}{n\binom{n+k}{k}}.$$

Proof. The proof follows directly from theorem 2.1 and using the same technique. $\square$

It is possible to represent the alternating harmonic number sums (16)and (18) in terms of an integral, this is developed in the next theorem. The following integral representations are new identities.

Theorem 2.3. *Let k be a positive integer, then we have:*

$$\frac{1}{1+k}\int_0^1 \frac{x\ln x \; {}_2F_1\left[\begin{matrix} 1,2 \\ 2+k \end{matrix}\middle| -x\right]}{1-x}dx \tag{19}$$

$$= \; S_k^{(2)}(0) + \zeta(2)\left(2^{k-1}k\ln 2 + \sum_{r=1}^{k}r\binom{k}{r}\left(H_{\left[\frac{r}{2}\right]} - H_r\right)\right),$$

where $S_k^{(2)}(0)$ is given by (16).

Proof. From lemma 1.2,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(2)}}{\binom{n+k}{k}} = \int_0^1 \frac{\ln x}{1-x}\sum_{n=1}^{\infty}\frac{(-1)^n(1-x^n)}{\binom{n+k}{k}}dx$$

$$= \; \frac{1}{1+k}\int_0^1 \frac{\ln x}{1-x}\left(x\;{}_2F_1\left[\begin{matrix} 1,2 \\ 2+k \end{matrix}\middle| -x\right] - {}_2F_1\left[\begin{matrix} 1,2 \\ 2+k \end{matrix}\middle| -1\right]\right)dx$$

$$= \; {}_2F_1\left[\begin{matrix} 1,2 \\ 2+k \end{matrix}\middle| -1\right]\frac{\zeta(2)}{1+k} + \frac{1}{1+k}\int_0^1 \frac{x\ln x}{1-x}\;{}_2F_1\left[\begin{matrix} 1,2 \\ 2+k \end{matrix}\middle| -x\right]dx.$$

Using the techniques developed in [17], we can write

$$\frac{1}{1+k} \; {}_2F_1 \left[\begin{array}{c} 1,2 \\ 2+k \end{array} \middle| -1 \right] = -2^{k-1} k \ln 2 - \sum_{r=1}^{k} r \binom{k}{r} \left(H_{[\frac{r}{2}]} - H_r \right)$$

and by substitution (19) follows. □

A Similar integral representation can be evaluated for $S_k^{(2)}(1)$, the results are recorded in the next theorem.

Theorem 2.4. *Let the conditions of theorem 2.3 hold, then we have:*

$$\frac{1}{1+k} \int_0^1 \frac{x \ln x \; {}_2F_1 \left[\begin{array}{c} 1,1 \\ 2+k \end{array} \middle| -x \right]}{1-x} dx$$

$$= \zeta(2) \left(2^k \ln 2 + \sum_{r=1}^{k} \binom{k}{r} \left(H_{[\frac{r}{2}]} - H_r \right) \right) - S_k^{(2)}(1),$$

where $S_k^{(2)}(1)$ is given by (18).

Proof. The proof follows the same pattern as that employed in theorem 2.3. □

Example 2.5. *Some illustrative examples follow.*

$$S_5^{(2)}(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{\binom{n+5}{5}} = 40\zeta(2)\ln 2 + \frac{1400}{9}\ln 2 - 20\zeta(3)$$
$$- \frac{655}{24}\zeta(2) - \frac{145795}{1728},$$

$$S_5^{(2)}(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n\binom{n+5}{5}} = \frac{35}{4}\zeta(3) - 16\zeta(2)\ln 2 - \frac{160}{3}\ln 2$$
$$+ \frac{79}{8}\zeta(2) - \frac{16469}{576}.$$

3. Concluding Remarks

The alternating sums of harmonic numbers $S_k^{(2)}(p)$, for $p = 0$ and 1 have been successfully represented in integral form and in terms of zeta functions, harmonic numbers and ln functions. It may also be possible to represent the sums

$$S_k^{(r)}(p,q,r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(r)}}{n^p \binom{n+k}{k}^q}$$

in closed form, this work is currently under investigation. It does appear however, that there is an impasse with the representation of

$$S_k^{(2)}(2,1,2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n^2 \binom{n+k}{k}}$$

in closed form. In the evaluation of $S_k^{(2)}(2)$ we require the closed form representation of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{n^2} = \frac{5}{4}\zeta(4) - \int_0^1 \frac{\ln x}{1-x} L_{i_2}(-x)\, dx.$$

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