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Some Properties of New Classes of Analytic Functions

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Dedicated to Professor H.M. Srivastava on the occasion of his 75th birthday

Abstract. The authors introduce two new classes $H_k(\lambda, A, B)$ and $M_k(\lambda, A, B)$ of analytic functions. Distortion bounds, inclusion relations and integral transforms properties for these classes are investigated.

1. Introduction and preliminaries

In the present paper the following assumptions are given.

$$N = \{1, 2, 3, \cdots\}, k \in N \setminus \{1\}, -1 \leq B \leq 0, B < A \leq 1 \text{ and } 0 \leq \lambda \leq 1. \tag{1.1}$$

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

which are analytic in $U = \{z : |z| < 1\}$.

For two functions f and g analytic in U, the function f is said to be subordinate to g, written $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$.

Let

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \in \mathcal{A} \quad (j = 1, 2).$$

The Hadamard product (or convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = f_1(z) * f_2(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n.$$

Lemma. Let $f(z) \in \mathcal{A}$ defined by (1.2) satisfy

$$\sum_{n=2}^{\infty} [(1 - \lambda + \lambda n)(1 - B) - (1 - A)\delta_{n,k}]|a_n| \leq A - B. \tag{1.3}$$

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Then

$$\frac{(1-\lambda)f(z)+\lambda zf'(z)}{f_k(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{1.4}$$

where

$$\delta_{n,k} = \begin{cases} 0 & \left(\frac{n-1}{k} \notin N\right), \\ 1 & \left(\frac{n-1}{k} \in N\right) \end{cases} \tag{1.5}$$

for $n \geq 2$,

$$f_k(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-j} f(\varepsilon_k^j z) \quad \text{and} \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right). \tag{1.6}$$

Proof. For $f(z) \in \mathcal{A}$ defined by (1.2), the function $f_k(z)$ in (1.6) can be expressed as

$$f_k(z) = z + \sum_{n=2}^{\infty} \delta_{n,k} a_n z^n, \tag{1.7}$$

where

$$\delta_{n,k} = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{j(n-1)} = \begin{cases} 0 & \left(\frac{n-1}{k} \notin N\right), \\ 1 & \left(\frac{n-1}{k} \in N\right) \end{cases}$$

for $n \geq 2$. Also, by (1.1) and (1.5), we see that

$$A\delta_{n,k} - B(1-\lambda+\lambda n) \geq 0 \quad (n \geq 2). \tag{1.8}$$

Let the inequality (1.3) hold. Then from (1.7) and (1.8) we deduce that

$$\begin{aligned} \left| \frac{\frac{(1-\lambda)f(z)+\lambda zf'(z)}{f_k(z)} - 1}{A - B\frac{(1-\lambda)f(z)+\lambda zf'(z)}{f_k(z)}} \right| &= \left| \frac{\sum_{n=2}^{\infty}(1-\lambda+\lambda n-\delta_{n,k})a_n z^{n-1}}{A-B+\sum_{n=2}^{\infty}[A\delta_{n,k}-B(1-\lambda+\lambda n)]a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty}(1-\lambda+\lambda n-\delta_{n,k})|a_n|}{A-B-\sum_{n=2}^{\infty}[A\delta_{n,k}-B(1-\lambda+\lambda n)]|a_n|} \\ &\leq 1 \quad (|z|=1). \end{aligned}$$

Thus, by the maximum modulus theorem, we have (1.4).

Now we introduce the following two subclasses of $\mathcal{A}$.

Definition 1. A function $f(z) \in \mathcal{A}$ defined by (1.2) is said to be in the class $H_k(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (1.3).

It follows from Lemma that, if $f(z) \in H_k(\lambda, A, B)$, then the subordination relation (1.4) holds.

Definition 2. A function $f(z) \in \mathcal{A}$ defined by (1.2) is said to be in the class $M_k(\lambda, A, B)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} n[(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}]|a_n| \leq A-B. \tag{1.9}$$

It is clear that for $f(z) \in \mathcal{A}$,

$$f(z) \in M_k(\lambda, A, B) \Longleftrightarrow zf'(z) \in H_k(\lambda, A, B). \tag{1.10}$$

If we write

$$\alpha_n = \alpha_{n,k}(\lambda, A, B) = \frac{(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}}{A-B} \text{ and } \beta_n = n\alpha_n > \alpha_n \quad (n \geq 2),$$

then it is easy to see that

$$\frac{\partial \beta_n}{\partial \lambda} = n\frac{\partial \alpha_n}{\partial \lambda} > 0, \ \frac{\partial \beta_n}{\partial A} = n\frac{\partial \alpha_n}{\partial A} \leq 0 \text{ and } \frac{\partial \beta_n}{\partial B} = n\frac{\partial \alpha_n}{\partial B} \geq 0.$$

Therefore we have the following inclusion relations: If

$$0 \leq \lambda_0 \leq \lambda \leq 1, -1 \leq B_0 \leq B < A \leq A_0 \leq 1 \text{ and } B \leq 0,$$

then

$$M_k(\lambda, A, B) \subset H_k(\lambda, A, B) \subseteq H_k(\lambda_0, A_0, B_0) \subseteq H_k(0, 1, -1).$$

In particular, by taking $\lambda = 1$ and the Lemma, we see that each function in the classes $H_k(1, A, B)$ and $M_k(1, A, B)$ is starlike with respect to k-symmetric points. Analytic (and meromorphic) functions which are starlike with respect to symmetric points and related functions have been extensively studied by several authors (see, e.g., [1] to [8], [10] to [18] and [20] to [22]; see also the recent works [6], [12] and [19]).

In the present paper, we obtain distortion bounds, inclusion relations and integral transforms for each of the above-defined classes $H_k(\lambda, A, B)$, $M_k(\lambda, A, B)$. Our results are motivated by a number of recent works (see, for example, [1] to [22]).

2. Distortion Bounds

Theorem 1. Let $\frac{1-A}{(k-1)(1-B)} \leq \lambda \leq 1$.
(i) If $f(z) \in H_k(\lambda, A, B)$, then for $z \in U$,

$$|z| - \frac{A-B}{(1+\lambda)(1-B)}|z|^2 \leq |f(z)| \leq |z| + \frac{A-B}{(1+\lambda)(1-B)}|z|^2. \tag{2.1}$$

The bounds in (2.1) are sharp for the function

$$f(z) = z - \frac{A-B}{(1+\lambda)(1-B)}z^2 \in H_k(\lambda, A, B). \tag{2.2}$$

(ii) If $f(z) \in M_k(\lambda, A, B)$, then for $z \in U$,

$$1 - \frac{A-B}{(1+\lambda)(1-B)} \leq |f'(z)| \leq 1 + \frac{A-B}{(1+\lambda)(1-B)}|z|. \tag{2.3}$$

The bounds in (2.3) are sharp for the function

$$f(z) = z - \frac{A-B}{2(1+\lambda)(1-B)}z^2 \in M_k(\lambda, A, B). \tag{2.4}$$

Proof. For $n \geq 2$ and $\frac{n-1}{k} \notin N$, we have $\delta_{n,k} = 0$, $\delta_{1+m,k} = 0$ $(1 \leq m \leq k-1)$ and

$$(1-\lambda+\lambda n)(1-B) - (1-A)\delta_{n,k} \geq (1+\lambda)(1-B). \tag{2.5}$$

For $n \geq 2$ and $\frac{n-1}{k} \in N$, we have $n = 1 + mk$ $(m \in N)$, $\delta_{n,k} = 1$ and

$$(1-\lambda+\lambda n)(1-B) - (1-A)\delta_{n,k} \geq (1+\lambda k)(1-B) - (1-A). \tag{2.6}$$

If $\frac{1-A}{(k-1)(1-B)} \leq \lambda \leq 1$, then

$$(1+\lambda k)(1-B)-(1-A) \geq (1+\lambda)(1-B). \tag{2.7}$$

(i) If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H_k(\lambda, A, B),$$

then it follows from (2.5) to (2.7) that

$$(1+\lambda)(1-B)\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}[(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}]|a_n| \leq A-B.$$

Hence we obtain

$$|f(z)| \leq |z| + |z|^2\sum_{n=2}^{\infty}|a_n| \leq |z| + \frac{A-B}{(1+\lambda)(1-B)}|z|^2$$

and

$$|f(z)| \geq |z| - |z|^2\sum_{n=2}^{\infty}|a_n| \geq |z| - \frac{A-B}{(1+\lambda)(1-B)}|z|^2 \geq 0$$

for $z \in U$.

(ii) If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_k(\lambda, A, B),$$

then (2.5) to (2.7) yield

$$(1+\lambda)(1-B)\sum_{n=2}^{\infty} n|a_n| \leq A-B.$$

From this we easily have (2.3).

Theorem 2. Let

$$0 \leq \lambda < \frac{1-A}{(k-1)(1-B)}. \tag{2.8}$$

(i) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H_k(\lambda, A, B)$, then for $z \in U$,

$$|f(z)| \leq |z| + \sum_{n=2}^{k}|a_n||z|^n + \frac{(A-B)-(1-B)\sum_{n=2}^{k}(1-\lambda+\lambda n)|a_n|}{(1+\lambda k)(1-B)-(1-A)}|z|^{k+1} \tag{2.9}$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{k}|a_n||z|^n - \frac{(A-B)-(1-B)\sum_{n=2}^{k}(1-\lambda+\lambda n)|a_n|}{(1+\lambda k)(1-B)-(1-A)}|z|^{k+1}. \tag{2.10}$$

Equalities in (2.9) and (2.10) are attained, for example, by the function

$$f(z) = z - \frac{A-B}{(1+\lambda k)(1-B)-(1-A)}z^{k+1} \in H_k(\lambda, A, B). \tag{2.11}$$

(ii) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_k(\lambda, A, B)$, then for $z \in U$,

$$|f'(z)| \leq 1 + \sum_{n=2}^{k} n|a_n||z|^{n-1} + \frac{(A-B)-(1-B)\sum_{n=2}^{k} n(1-\lambda+\lambda n)|a_n|}{(1+\lambda k)(1-B)-(1-A)}|z|^k \tag{2.12}$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^{k} n|a_n||z|^{n-1} - \frac{(A-B) - (1-B)\sum_{n=2}^{k} n(1-\lambda+\lambda n)|a_n|}{(1+\lambda k)(1-B) - (1-A)}|z|^k. \tag{2.13}$$

Equalities in (2.12) and (2.13) are attained, for example, by the function

$$f(z) = z - \frac{A-B}{(1+k)[(1+\lambda k)(1-B) - (1-A)]}z^{k+1} \in M_k(\lambda, A, B). \tag{2.14}$$

Proof. (i) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H_k(\lambda, A, B)$, then from (2.5), (2.6) and (2.8) we find that

$$\begin{aligned} A - B &\geq \sum_{n=2}^{\infty} [(1-\lambda+\lambda n)(1-B) - (1-A)\delta_{n,k}]|a_n| \\ &\geq \sum_{n=2}^{k} (1-\lambda+\lambda n)(1-B)|a_n| + [(1+\lambda k)(1-B) - (1-A)] \sum_{n=k+1}^{\infty} |a_n|. \end{aligned}$$

From this we easily have (2.9) and (2.10).

(ii) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_k(\lambda, A, B)$, then we have

$$\begin{aligned} A - B &\geq \sum_{n=2}^{\infty} n[(1-\lambda+\lambda n)(1-B) - (1-A)\delta_{n,k}]|a_n| \\ &\geq \sum_{n=2}^{k} n(1-\lambda+\lambda n)(1-B)|a_n| + [(1+\lambda k)(1-B) - (1-A)] \sum_{n=k+1}^{\infty} n|a_n|. \end{aligned}$$

This leads to (2.12) and (2.13).

Theorem 3. Let $f(z)$ given by (2.1) be in the class $H_k(\lambda, A, B)$.

(i) If $0 < \lambda \leq \frac{A-B}{1-B}$, then for $z \in U$,

$$1 - \frac{A-B}{\lambda(1-B)}|z| \leq |f'(z)| \leq 1 + \frac{A-B}{\lambda(1-B)}|z|. \tag{2.15}$$

The bounds in (2.15) are sharp for the function

$$f(z) = z - \frac{A-B}{2\lambda(1-B)}z^2 \in H_k(\lambda, A, B). \tag{2.16}$$

(ii) If $\frac{A-B}{1-B} < \lambda \leq 1$, then for $z \in U$,

$$|f'(z)| \leq 1 + \sum_{n=2}^{k} n|a_n||z|^{n-1} + \frac{(A-B) - \lambda(1-B)\sum_{n=2}^{k} n|a_n|}{\lambda k(1-B) + A - B}(1+k)|z|^k \tag{2.17}$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^{k} n|a_n||z|^{n-1} - \frac{(A-B) - \lambda(1-B)\sum_{n=2}^{k} n|a_n|}{\lambda k(1-B) + A - B}(1+k)|z|^k. \tag{2.18}$$

The bounds in (2.17) and (2.18) are sharp for the function

$$f(z) = z - \frac{A-B}{\lambda k(1-B) + A - B}z^{k+1} \in H_k(\lambda, A, B).$$

Proof. For $n \geq 2$ and $\frac{n-1}{k} \notin N$, we have $\delta_{n,k} = 0$ and

$$\frac{(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}}{n} = \lambda(1-B) + \frac{(1-\lambda)(1-B)}{n} \geq \lambda(1-B) \tag{2.19}$$

For $n \geq 2$ and $\frac{n-1}{k} \in N$, we have $\delta_{n,k} = 1$, $n = 1 + mk$ ($m \in N$) and

$$\frac{(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}}{n} = \lambda(1-B) + \frac{(A-B)-\lambda(1-B)}{n}. \tag{2.20}$$

(i) If $0 < \lambda \leq \frac{A-B}{1-B}$, then it is seen from (2.19) and (2.20) that

$$\frac{(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}}{n} \geq \lambda(1-B) \tag{2.21}$$

for all $n \geq 2$. Using (2.21) we obtain

$$\begin{aligned} A - B &\geq \sum_{n=2}^{\infty}[(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}]|a_n| \\ &\geq \lambda(1-B)\sum_{n=2}^{\infty} n|a_n|. \end{aligned}$$

From this we easily have (2.15).

(ii) If $\frac{A-B}{1-B} < \lambda \leq 1$, then it is seen from (2.19) and (2.20) that

$$\frac{(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}}{n} \geq \lambda(1-B) - \frac{\lambda(1-B)-(A-B)}{1+k} \tag{2.22}$$

for $n \geq 1 + k$. Using (2.22) we obtain

$$\begin{aligned} A - B &\geq \sum_{n=2}^{\infty}[(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}]|a_n| \\ &\geq \lambda(1-B)\sum_{n=2}^{k} n|a_n| + \left[\lambda(1-B) - \frac{\lambda(1-B)-(A-B)}{1+k}\right]\sum_{n=1+k}^{\infty} n|a_n|. \end{aligned}$$

From this we easily have (2.17) and (2.18).

3. Inclusion Relation between $H_k(\lambda, C, D)$ and $M_k(\lambda, A, B)$

In this section we generalize and improve the above mentioned inclusion relation

$$M_k(\lambda, A, B) \subset H_k(\lambda, A, B). \tag{3.1}$$

Theorem 4. If $-1 \leq D \leq 0$, then

$$M_k(\lambda, A, B) \subset H_k(\lambda, C(D), D), \tag{3.2}$$

where

$$C(D) = D + \frac{(1-D)(A-B)}{2(1-B)}, \tag{3.3}$$

and the number $C(D)$ cannot be decreased for each D.

Proof. Obviously $D < C(D) < 1$. Let $f(z) \in M_k(\lambda, A, B)$. In order to prove that $f(z) \in H_k(\lambda, C(D), D)$, we need only to find the smallest C $(D < C \leq 1)$ such that

$$\frac{(1-\lambda+\lambda n)(1-D)-(1-C)\delta_{n,k}}{C-D} \leq \frac{n[(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}]}{A-B} \tag{3.4}$$

for all $n \geq 2$, that is , that

$$\frac{(1-D)(1-\lambda+\lambda n-\delta_{n,k})}{C-D} + \delta_{n,k} \leq n\left\{\frac{(1-B)(1-\lambda+\lambda n-\delta_{n,k})}{A-B} + \delta_{n,k}\right\} \quad (n \geq 2). \tag{3.5}$$

For $n \geq 2$ and $\frac{n-1}{k} \in N$, (3.5) is equivalent to

$$C \geq D + \frac{\lambda(1-D)}{\frac{\lambda n(1-B)}{A-B}+1} = \varphi(n) \quad \text{(say)}. \tag{3.6}$$

The function $\varphi(n)(n \geq 2)$ is decreasing in n and hence

$$\varphi(n) \leq \varphi(1+k) = D + \frac{\lambda(1-D)}{\frac{\lambda(1+k)(1-B)}{A-B}+1}.$$

For $n \geq 2$ and $\frac{n-1}{k} \notin N$, (3.5) reduces to

$$C \geq D + \frac{1-D}{\frac{n(1-B)}{A-B}} = \psi(n) \quad \text{(say)} \tag{3.7}$$

and we have

$$\psi(n) \leq \psi(2) = D + \frac{(1-D)(A-B)}{2(1-B)}. \tag{3.8}$$

Noting that (1.1), a simple calculation shows that $\varphi(1+k) \leq \psi(2)$. Consequently, by taking $C = \psi(2) = C(D)$, it follows from (3.4) to (3.8) that $f(z) \in H_k(\lambda, C(D), D)$.

Furthermore, for $D < C_0 < C(D)$, we have

$$\frac{(1+\lambda)(1-D)}{C_0-D} \cdot \frac{A-B}{2(1+\lambda)(1-B)} > \frac{(1+\lambda)(1-D)}{C(D)-D} \cdot \frac{A-B}{2(1+\lambda)(1-B)} = 1,$$

which implies that the function $f(z) \in M_k(\lambda, A, B)$ defined by (2.4) is not in the class $H_k(\lambda, C_0, D)$. The proof of Theorem 4 is thus completed.

Setting $D = B$, Theorem 4 reduces to the following result.

Corollary 1. $M_k(\lambda, A, B) \subset H_k(\lambda, C(B), B)$, where

$$C(B) = \frac{A+B}{2} \in (B, A)$$

cannot be decreased for each B.

Note that Corollary 1 refines the inclusion relation (3.1).

4. Integral Transforms

Theorem 5. Let $f(z) \in H_k(\lambda, A, B)$ and

$$I_\mu(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}f(t)dt \quad (\mu > -1). \tag{4.1}$$

Then $I_\mu(z) \in H_k(\lambda, C_1(D), D)$, where $-1 \le D \le 0$ and

$$C_1(D) = D + \frac{(\mu+1)(1-D)(A-B)}{(\mu+2)(1-B)}. \tag{4.2}$$

The number $C_1(D)$ cannot be decreased for each D.

Proof. Note that $D < C_1(D) < 1$. For

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H_k(\lambda, A, B),$$

it follows from (4.1) that

$$I_\mu(z) = z + \sum_{n=2}^{\infty} \frac{\mu+1}{\mu+n} a_n z^n. \tag{4.3}$$

In order to prove that $I_\mu(z) \in H_k(\lambda, C_1(D), D)$, we need only to find the smallest C $(D < C \le 1)$ such that

$$\frac{(1-\lambda+\lambda n)(1-D)-(1-C)\delta_{n,k}}{C-D} \cdot \frac{\mu+1}{\mu+n} \le \frac{(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}}{A-B} \tag{4.4}$$

for all $n \ge 2$.

For $n \ge 2$ and $\frac{n-1}{k} \in N$, (4.4) becomes

$$C \ge D + \frac{\lambda(1-D)}{\frac{\lambda(1-B)(\mu+n)}{(A-B)(\mu+1)} + \frac{1}{\mu+1}} = \varphi_1(n) \quad \text{(say)} \tag{4.5}$$

and we have

$$\varphi_1(n) \le \varphi_1(1+k) = D + \frac{\lambda(1-D)}{\frac{\lambda(1-B)(\mu+1+k)}{(A-B)(\mu+1)} + \frac{1}{\mu+1}}.$$

For $n \ge 2$ and $\frac{n-1}{k} \notin N$, (4.5) reduces to

$$C \ge D + \frac{1-D}{\frac{(1-B)(\mu+n)}{(A-B)(\mu+1)}} = \psi_1(n) \quad \text{(say)} \tag{4.6}$$

and we have

$$\psi_1(n) \le \psi_1(2) = D + \frac{(\mu+1)(1-D)(A-B)}{(\mu+2)(1-B)}. \tag{4.7}$$

A simple calculation shows that $\varphi_1(1+k) \le \psi_1(2)$. Therefore, by taking $C = \psi_1(2) = C_1(D)$, it follows from (4.4) to (4.7) that $I_\mu(z) \in H_k(\lambda, C_1(D), D)$.

Furthermore the number $C_1(D)$ is best possible for the function $f(z)$ defined by (2.2).

Theorem 6. Let $I_\mu(z)$ and $C_1(D)$ be the same as in Theorem 5. If $f(z) \in M_k(\lambda, A, B)$, then $I_\mu(z) \in M_k(\lambda, C_1(D), D)$ and the number $C_1(D)$ cannot be decreased for each D.

Proof. By (4.3) we have

$$I_\mu(z) = \left(z + \sum_{n=2}^{\infty} \frac{\mu+1}{\mu+n} z^n\right) * f(z)$$

and so

$$zI'_\mu(z) = \left(z + \sum_{n=2}^{\infty} \frac{\mu+1}{\mu+n} z^n\right) * zf'(z). \tag{4.8}$$

In view of (4.8) and (1.10), an application of Theorem 5 yields Theorem 6.

Corollary 2. Let $f(z) \in M_k(\lambda, A, B)$ and $I_\mu(z)$ be the same as in Theorem 5. Then $I_\mu(z) \in H_k(\lambda, C_2(D), D)$, where $-1 \leq D \leq 0$ and

$$C_2(D) = D + \frac{(\mu+1)(1-D)(A-B)}{2(\mu+2)(1-B)}.$$

The number $C_2(D)$ cannot be decreased for each D.

Proof. Note that $D < C_2(D) < 1$. Let $f(z) \in M_k(\lambda, A, B)$. Then it follows from Theorem 6 and Corollary 1 that $I_\mu(z) \in H_k(\lambda, C(D), D)$, where $-1 \leq D \leq 0$ and

$$C(D) = D + \frac{C_1(D) - D}{2} = D + \frac{(\mu+1)(1-D)(A-B)}{2(\mu+2)(1-B)} = C_2(D).$$

Furthermore, for the function $f(z) \in M_k(\lambda, A, B)$ given by (2.4) and $D < C_0 < C_2(D)$, we have

$$I_\mu(z) = z - \frac{(\mu+1)(A-B)}{2(\mu+2)(1+\lambda)(1-B)} z^2$$

and

$$\frac{(1+\lambda)(1-D)}{C_0 - D} \cdot \frac{(\mu+1)(A-B)}{2(\mu+2)(1+\lambda)(1-B)} > \frac{(1+\lambda)(1-D)}{C_2(D) - D} \cdot \frac{(\mu+1)(A-B)}{2(\mu+2)(1+\lambda)(1-B)} = 1.$$

Hence $I_\mu(z) \notin H_k(\lambda, C_0, D)$ and the proof of Corollary 2 is completed.

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