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An Inequality for Similarity Condition Numbers of Unbounded Operators with Schatten - von Neumann Hermitian Components

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Abstract. Let H be a linear unbounded operator in a separable Hilbert space. It is assumed the resolvent of H is a compact operator and $H - H^*$ is a Schatten - von Neumann operator. Various integro-differential operators satisfy these conditions. Under certain assumptions it is shown that H is similar to a normal operator and a sharp bound for the condition number is suggested.

We also discuss applications of that bound to spectrum perturbations and operator functions.

1. Introduction and Statement of the Main Result

Let $\mathfrak{H}$ be a separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and unit operator I. Two operators A and $\tilde{A}$ acting in $\mathfrak{H}$ are said to be similar if there exists a boundedly invertible bounded operator T such that $\tilde{A} = T^{-1}AT$. The constant $\kappa_T = \|T^{-1}\|\|T\|$ is called the condition number. The condition number is important in applications. We refer the reader to [5], where condition number estimates are suggested for combined potential boundary integral operators in acoustic scattering and [23], where condition numbers are estimated for second-order elliptic operators. Conditions that provide the similarity of various operators to normal and selfadjoint ones were considered by many mathematicians, cf. [1, 4, 7], [14, 15], [17]-[21], and references given therein. In many cases, the condition number must be numerically calculated, e.g. [2, 20]. The interesting generalizations of condition numbers of bounded linear operators in Banach spaces were explored in the paper [13].

In the present paper we consider a class of unbounded operators in a Hilbert space with Schatten - von Neumann Hermitian components. Numerous integro-differential operators belong to that class. We suggest a sharp bound for the condition numbers of the considered operators. It generalizes and improves the bound for the condition numbers of operators with Hilbert-Schmidt Hermitian components from [11]. We also discuss applications of the obtained bound to spectrum perturbations and norm estimates for operator functions.

Introduce the notations. For a linear operator A in $\mathfrak{H}$, $Dom(A)$ is the domain, A^* is the adjoint of A; $\sigma(A)$ denotes the spectrum of A and A^{-1} is the inverse to A; $R_\lambda(A)= (A - I\lambda)^{-1}$ $(\lambda \notin \sigma(A))$ is the resolvent; $A_I := (A - A^*)/2i$; $\lambda_k(A)$ $(k = 1, 2, ...)$ are the eigenvalues of A taken with their multiplicities and enumerated as $|\lambda_j(A)| \le |\lambda_{j+1}(A)|$, and $\rho(A, \lambda) = \inf_k |\lambda - \lambda_k(A)|$. By SN_r $(1 \le r < \infty)$ we denote the Schatten - von Neumann ideal of compact operators K with the finite norm $N_r(K) := [Trace(KK^*)^{r/2}]^{1/r}$.

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Everywhere below H is an invertible operator in $\mathfrak{H}$, with the following properties: $Dom(H) = Dom(H^*)$, and there are an $r \in [1, \infty)$ and an integer $p \geq 1$, such that

$$H^{-1} \in SN_r \text{ and } H_I \in SN_{2p}. \tag{1.1}$$

Note that instead of the condition $H^{-1} \in SN_r$, in our reasonings below, one can require the condition $(H - aI)^{-1} \in SN_r$ for some point $a \notin \sigma(H)$. Since H^{-1} is compact, $\sigma(H)$ is purely discrete. It is assumed that *all the eigenvalues $\lambda_j(H)$ of H are different*. For a fixed integer m put

$$\delta_m(H) = \inf_{j=1,2,...;\ j\neq m} |\lambda_j(H) - \lambda_m(H)|.$$

It is further supposed that

$$\zeta_q(H) := \left[\sum_{j=1}^{\infty} \frac{1}{\delta_j^q(H)}\right]^{1/q} < \infty \ \ (\frac{1}{q} + \frac{1}{2p} = 1) \tag{1.2}$$

for an integer $p \geq 1$. Hence it follows that

$$\hat{\delta}(H) := \inf_m \delta_m(H) = \inf_{j\neq k; j,k=1,2,...} |\lambda_j(H) - \lambda_k(H)| > 0. \tag{1.3}$$

Denote also

$$u_p(H) := \sqrt{2}\zeta_q(H) \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \frac{\beta_p^{kp+m} N_{2p}^{kp+m+1}(H_I)}{\hat{\delta}^{kp+m}(H)\sqrt{k!}},$$

where

$$\beta_p := 2\left(1 + \frac{2p}{e^{2/3} ln 2}\right). \tag{1.4}$$

Now we are in a position to formulate our main result.

Theorem 1.1. *Let conditions (1.1) and (1.2) be fulfilled. Then there are an invertible operator T and a normal operator D acting in $\mathfrak{H}$, such that*

$$THx = DTx \ \ (x \in Dom(H)). \tag{1.5}$$

Moreover,

$$\kappa_T := \|T^{-1}\|\|T\| \leq e^{2u_p(H)} \tag{1.6}$$

The proof of this theorem is divided into a series of lemmas which are presented in the next three sections. The theorem is sharp: if H is selfadjoint, then $u_p(H) = 0$ and we obtain $\kappa_T = 1$.

As it is shown below, one can replace (1.6) by the inequality

$$\kappa_T \leq e^{2\hat{u}_p(H),} \tag{1.7}$$

where

$$\hat{u}_p(H) := \sqrt{2e}\ \zeta_q(H) \sum_{m=0}^{p-1} \frac{\beta_p^m N_{2p}^{m+1}(H_I)}{\hat{\delta}^m(A)}\ exp\left[\frac{(\beta_p N_{2p}(H_I))^{2p}}{2\hat{\delta}^{2p}(A)}\right].$$

In addition, below we show that in our considerations instead of β_p defined by (1.4) in the case

$$p = 2^{m-1},\ m = 2, 3, ...,\ \text{one can take } \hat{\beta}_p = 2(1 + \text{ctg}(\frac{\pi}{4p})) \text{ and } \hat{\beta}_1 = \sqrt{2} \tag{1.8}$$

instead of β_1.

To illustrate Theorem 1.1, consider the operator $H = S + K$, where $K \in SN_{2p}$ and S is a positive definite selfadjoint operator with a discrete spectrum, whose eigenvalues are different and

$$\lambda_{j+1}(S) - \lambda_j(S) \geq b_0 j^{\alpha} \ \ (b_0 = const > 0; \alpha > 1/q = (2p-1)/(2p); j = 1, 2, ...) \tag{1.9}$$

Since S is selfadjoint we have

$$\sup_k \inf_j |\lambda_k(H) - \lambda_j(S)| \le \|K\|,$$

cf. [16]. Thus, if

$$2\|K\| < \inf_j (\lambda_{j+1}(S) - \lambda_j(S)), \tag{1.10}$$

then $\hat{\delta}(H) \ge \inf_j(\lambda_{j+1}(S) - \lambda_j(S) - 2\|K\|)$ and (1.2) holds with

$$\zeta_q(H) \le \zeta_q(S,K), \text{ where } \zeta_q(S,K) := [\sum_{j=1}^{\infty} (\lambda_{j+1}(S) - \lambda_j(S) - 2\|K\|)^{-q}]^{1/q} < \infty.$$

Example 1.2. *Consider in $L^2(0,1)$ the spectral problem*

$$u^{(4)}(x) + (Ku)(x) = \lambda u(x) \ \ (\lambda \in \mathbb{C}, 0 < x < 1); \ u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $K \in SN_{2p}, p \ge 1$ for an arbitrary $p \ge 1$. So H is defined by $H = d^4/dx^4 + K$ with

$$Dom\ (H) = \{v \in L^2(0,1) : v^{(4)} \in L^2(0,1),\ v(0) = v(1) = v''(0) = v''(1) = 0\}.$$

Take $S = d^4/dx^4$ with $Dom\ (S) = Dom\ (H)$. Then $\lambda_j(S) = \pi^4 j^4$ $(j = 1, 2, ...)$ and $\lambda_{j+1}(S) - \lambda_j(S) \ge 4\pi^4 j^3$. If $\|K\| < 2\pi^4$, then $\hat{\delta}(H) \ge 4\pi^4 - 2\|K\|$ and

$$\zeta_q^q(H) \le \sum_{j=1}^{\infty} (4\pi^4 j^3 - 2\|K\|)^{-q} < \infty.$$

Now one can directly apply Theorem 1.1.

2. Auxiliary Results

Let B_0 be a bounded linear operator in $\mathfrak{H}$ having a finite chain of invariant projections P_k $(k = 1, ..., n;\ n < \infty)$:

$$0 \subset P_1\mathfrak{H} \subset P_2\mathfrak{H} \subset ... \subset P_n\mathfrak{H} = \mathfrak{H} \tag{2.1}$$

and

$$P_k B_0 P_k = B_0 P_k \ \ (k = 1, ..., n). \tag{2.2}$$

That is, B_0 maps $P_k\mathfrak{H}$ into $P_k\mathfrak{H}$ for each k. Put

$$\Delta P_k = P_k - P_{k-1} \ \ (P_0 = 0) \text{ and } A_k = \Delta P_k B_0 \Delta P_k.$$

It is assumed that the spectra $\sigma(A_k)$ of A_k in $\Delta P_k\mathfrak{H}$ satisfy the condition

$$\sigma(A_k) \cap \sigma(A_j) = \emptyset \ \ (j \ne k;\ j, k = 1, ..., n). \tag{2.3}$$

Lemma 2.1. *One has*

$$\sigma(B_0) = \cup_{k=1}^n \sigma(A_k).$$

For the proof see [11].

Under conditions (2.1), (2.2) put

$$Q_k = I - P_k, B_k = Q_k B_0 Q_k \text{ and } C_k = \Delta P_k B_0 Q_k.$$

Since B_j is a a block triangular operator matrix, according to the previous lemma we have

$$\sigma(B_j) = \cup_{k=j+1}^n \sigma(A_k) \ \ (j = 0, ..., n).$$

Under this condition, according to the Rosenblum theorem from [22], the equation

$$A_j X_j - X_j B_j = -C_j \ \ (j = 1, ..., n-1) \tag{2.4}$$

has a unique solution (see also [6, Section I.3] and [3]). We need also the following result proved in [11].

Lemma 2.2. *Let condition (2.3) hold and X_j be a solution to (2.4). Then*

$$(I - X_{n-1})(I - X_{n-2}) \cdots (I - X_1)\, B_0\, (I + X_1)(I + X_2) \cdots (I + X_{n-1}) =$$
$$A_1 + A_2 + ... + A_n. \tag{2.5}$$

Take

$$\hat{T}_n = (I + X_1)(I + X_2) \cdots (I + X_{n-1}). \tag{2.6}$$

It is simple to see that the inverse to $I + X_j$ is the operator $I - X_j$. Thus,

$$\hat{T}_n^{-1} = (I - X_{n-1})(I - X_{n-2}) \cdots (I - X_1) \tag{2.7}$$

and (2.5) can be written as

$$\hat{T}_n^{-1} B_0 \hat{T}_n = diag\ (A_k)_{k=1}^n. \tag{2.8}$$

By the inequalities between the arithmetic and geometric means we get

$$\|\hat{T}_n\| \le \prod_{k=1}^{n-1} (1 + \|X_k\|) \le \left(1 + \frac{1}{n-1} \sum_{k=1}^{n-1} \|X_k\|\right)^{n-1} \tag{2.9}$$

and

$$\|\hat{T}_n^{-1}\| \le \left(1 + \frac{1}{n-1} \sum_{k=1}^{n-1} \|X_k\|\right)^{n-1}. \tag{2.10}$$

Furthermore, we need the following result

Theorem 2.3. *Let M be a linear operator in $\mathfrak{H}$, such that $Dom\ (M) = Dom\ (M^*)$ and $M_I = (M - M^*)/2i \in SN_{2p}$ for some integer $p \ge 1$. Then*

$$\|R_\lambda(M)\| \le \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \frac{(\beta_p N_{2p}(M_I))^{kp+m}}{\rho^{pk+m+1}(M, \lambda) \sqrt{k!}} \quad (\lambda \notin \sigma(M)). \tag{2.11}$$

Moreover, one has

$$\|R_\lambda(M)\| \le \sqrt{e} \sum_{m=0}^{p-1} \frac{(\beta_p N_{2p}(M_I))^m}{\rho^{m+1}(M, \lambda)}\ exp\ \left[\frac{(\beta_p N_{2p}(M_I))^{2p}}{2\rho^{2p}(M, \lambda)}\right] \quad (\lambda \notin \sigma(M)). \tag{2.12}$$

For the proof in the case $p > 1$ see [8, Theorem 7.9.1]. The case $p = 1$ is proved in [8, Theorem 7.7.1]. Besides, β_p can be replaced by $\hat{\beta}_p$ according to (1.8).

3. The Finite Dimensional Case

In this section we apply Lemma 2.3 to an $n \times n$-matrix A whose eigenvalues are different and are enumerated in the increasing way of their absolute values. We define

$$\hat{\delta}(A) := \min_{j,k=1,\dots,n;\ k \ne j} |\lambda_j(A) - \lambda_k(A)| > 0. \tag{3.1}$$

Hence, there is an invertible matrix $T_n \in \mathbb{C}^{n \times n}$ and a normal matrix $D_n \in \mathbb{C}^{n \times n}$, such that

$$T_n^{-1} A T_n = D_n. \tag{3.2}$$

Furthermore, for a fixed $m \le n$ put

$$\delta_j(A) = \inf_{m=1,2,\dots,n;\ m \ne j} |\lambda_j(A) - \lambda_m(A)|\,.$$

Let $\{e_k\}$ be the Schur basis (the orthogonal normal basis of the triangular representation) of matrix A:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

with $a_{jj} = \lambda_j(A)$. Take $P_j = \sum_{k=1}^{j} (., e_k)e_k$. $B_0 = A$, $\Delta P_k = (., e_k)e_k$,

$$Q_j = \sum_{k=j+1}^{n} (., e_k)e_k, A_k = \Delta P_k A \Delta P_k = \lambda_k(A)\Delta P_k,$$

$$B_j = Q_j A Q_j = \begin{pmatrix} a_{j+1,j+1} & a_{j+1,j+2} & \dots & a_{j+1,n} \\ 0 & a_{j+2,j+2} & \dots & a_{j+2,n} \\ . & . & . & \dots \\ 0 & 0 & . & a_{nn} \end{pmatrix},$$

$$C_j = \Delta P_j A Q_j = \begin{pmatrix} a_{j,j+1} & a_{j,j+2} & \dots & a_{j,n} \end{pmatrix}$$

and

$$D_n = diag(\lambda_k(A)). \tag{3.4}$$

In addition,

$$A = \begin{pmatrix} \lambda_1(A) & C_1 \\ 0 & B_1 \end{pmatrix}, B_1 = \begin{pmatrix} \lambda_2(A) & C_2 \\ 0 & B_2 \end{pmatrix}, \dots, B_j = \begin{pmatrix} \lambda_{j+1}(A) & C_{j+1} \\ 0 & B_{j+1} \end{pmatrix}$$

($j < n$). So B_j is an upper-triangular $(n-j)\times(n-j)$-matrix. Equation (2.4) takes the form

$$\lambda_j(A)X_j - X_j B_j = -C_j.$$

Since $X_j = X_j Q_j$, we can write $X_j(\lambda_j(A)Q_j - B_j) = C_j$. Therefore

$$X_j = C_j\ (\lambda_j(A)Q_j - B_j)^{-1}. \tag{3.5}$$

The inverse operator is understood in the sense of subspace $Q_j\mathbb{C}^n$. Hence,

$$\|X_j\| \le \|C_j\|\|(\lambda_j(A)Q_j - B_j)^{-1}\|.$$

Besides, due to (2.11)

$$\|(\lambda_j(A)Q_j - B_j)^{-1}\| \le \sum_{m=0}^{p-1}\sum_{k=0}^{\infty} \frac{(\beta_p N_{2p}(B_{jI})^{kp+m}}{\delta_j^{kp+m+1}(A)\sqrt{k!}},$$

where B_{jI} is the imaginary Hermitian component of B_j.

But $N_{2p}(B_{jI}) = N_{2p}(Q_j A_I Q_j) \le N_{2p}(A_I)$ ($j \ge 1$). So

$$\|(\lambda_j(A)Q_j - B_j)^{-1}\| \le \frac{\tau(A)}{\delta_j(A)}$$

where

$$\tau(A) = \sum_{m=0}^{p-1}\sum_{k=0}^{\infty} \frac{(\beta_p N_{2p}(A_I))^{kp+m}}{\hat{\delta}^{kp+m}(A)\sqrt{k!}}.$$

Consequently,

$$\|X_j\| \le \tau(A)\frac{\|C_j\|}{\delta_j(A)}.$$

Take $T_n = \hat{T}_n$ as in (2.6) with X_k defined by (3.5). Besides (2.9) and (2.10) imply

$$\|T_n\| \le \left(1 + \frac{1}{n-1}\sum_{j=1}^{n-1}\|X_j\|\right)^{n-1} \le \left(1 + \frac{\tau(A)}{(n-1)}\sum_{j=1}^{n-1}\frac{\|C_j\|}{\delta_j(A)}\right)^{n-1} \tag{3.6}$$

and

$$\|T_n^{-1}\| \le \left(1 + \frac{\tau(A)}{(n-1)}\sum_{j=1}^{n-1}\frac{\|C_j\|}{\delta_j(A)}\right)^{n-1}. \tag{3.7}$$

But by the Hólder inequality,

$$\sum_{j=1}^{n-1}\frac{\|C_j\|}{\delta_j(A)} \le \left(\sum_{j=1}^{n-1}\|C_j\|^{2p}\right)^{1/2p}\zeta_q(A) \ \ (1/(2p) + 1/q = 1), \tag{3.8}$$

where

$$\zeta_q(A) := \left(\sum_{k=1}^{n-1}\frac{1}{\delta_k^q(A)}\right)^{1/q}.$$

In addition,

$$\|C_j\|^2 \le \sum_{k=j+1}^{n}|a_{jk}|^2,\ j<n;\ C_n = 0,$$

and

$$4\|A_I e_j\|^2 = \|(A - A^*)e_j\|^2 = |a_{jj} - \overline{a}_{jj}|^2 + 2\sum_{k=j+1}^{n}|a_{jk}|^2 \ge 2\|C_j\|^2;\ j<n.$$

Thus, $\|C_j\| \le \sqrt{2}\|A_I e_j\|$, $j \le n$ and therefore

$$\sum_{j=1}^{n-1}\|C_j\|^{2p} \le 2^p\sum_{j=1}^{n-1}\|A_I e_j\|^{2p}.$$

But from Lemmas II.4.1 and II.3.4 [12], it follows that

$$\sum_{j=1}^{n-1}\|A_I e_j\|^{2p} \le N_{2p}^{2p}(A_I).$$

Therefore relations (3.6)-(3.8) with the notation

$$\psi_{n,p}(A) = \left(1 + \frac{\tau(A)\sqrt{2}N_{2p}(A_I)\zeta_q(A)}{n-1}\right)^{n-1}$$

imply $\|T_n\| \le \psi_{n,p}(A)$ and $\|T_n^{-1}\| \le \psi_{n,p}(A)$.

We thus have proved the following.

Lemma 3.1. *Let condition (3.1) be fulfilled. Then there is an invertible operator T_n, such that (3.2) holds with $\kappa_{T_n} := \|T_n^{-1}\|\|T_n\| \le \psi_{n,p}^2(A)$.*

According to (2.12) one can replace $\tau(A)$ by

$$\hat{\tau}(A) := \sqrt{e}\sum_{m=0}^{p-1} \frac{(\beta_p N_{2p}(A_I))^m}{\hat{\delta}^m(A)}\; exp\left[\frac{(\beta_p N_{2p}(A_I))^{2p}}{2\hat{\delta}^{2p}(A)}\right]$$

and therefore

$$\kappa_{T_n} \le \hat{\psi}^2_{n,p}(A), \tag{3.9}$$

where

$$\hat{\psi}_{n,p}(A) = \left(1 + \frac{\hat{\tau}(A)\,\sqrt{2}N_{2p}(A_I)\zeta_q(A)}{n-1}\right)^{n-1}.$$

The previous lemma and (3.9) improve the bound from [9, 10] for the condition numbers of matrices with large n.

4. Proof of Theorem 1.1

Recall the Keldysh theorem, cf. [12, Theorem V. 8.1].

Theorem 4.1. *Let $A = S(I + K)$, where $S = S^* \in SN_r$ for some $r \in [1, \infty)$ and K is compact. In addition, let from $Af = 0$ ($f \in \mathfrak{H}$) it follows that $f = 0$. Then A has a complete system of root vectors.*

We need the following result.

Lemma 4.2. *Under the hypothesis of Theorem 1.1, operator H^{-1} has a complete system of root vectors.*

Proof. We can write $H = H_R + iH_I$ with the notation $H_R = (H + H^*)/2$. For any real c with $-c \notin \sigma(H) \cup \sigma(H_R)$ we have

$$(H + cI)^{-1} = (I + i(H_R + cI)^{-1}H_I)^{-1}(H_R + cI)^{-1}.$$

But $(I + i(H_R + cI)^{-1}H_I)^{-1} - I = K_0$, where $K_0 = -i(H_R + cI)^{-1}H_I(I + i(H_R + cI)^{-1}H_I)^{-1}$ is compact. So

$$(H + cI)^{-1} = (H_R + cI)^{-1}(I + K_0). \tag{4.1}$$

Due to (1.1) $(H + cI)^{-1} = H^{-1}(I + cH^{-1})^{-1} \in SN_r$. Hence

$$(H_R + cI)^{-1} = (I + i(H_R + cI)^{-1}H_I)(H + cI)^{-1} \in SN_r$$

and therefore by (4.1) and the Keldysh theorem operator $(H + cI)^{-1}$ has a complete system of roots vectors. Since $(H + cI)^{-1}$ and H^{-1} commute, H^{-1} has a complete system of roots vectors, as claimed. □

From the previous lemma it follows that there is an orthonormal (Schur) basis $\{\hat{e}_k\}_{k=1}^{\infty}$, in which H^{-1} is represented by a triangular matrix (see [12, Lemma I.4.1]). Denote $\hat{P}_k = \sum_{j=1}^{k}(., \hat{e}_j)\hat{e}_j$. Then

$$H^{-1}\hat{P}_k = \hat{P}_k H^{-1}\hat{P}_k \;\; (k = 1, 2, ...).$$

Besides,

$$\Delta\hat{P}_k H^{-1}\Delta\hat{P}_k = \lambda_k^{-1}(H)\Delta\hat{P}_k \;\; (\Delta\hat{P}_k = \hat{P}_k - \hat{P}_{k-1},\; k = 1, 2, ...; \hat{P}_0 = 0). \tag{4.2}$$

Put

$$D = \sum_{k=1}^{\infty} \lambda_k \Delta\hat{P}_k \;\; (\Delta\hat{P}_k = \hat{P}_k - \hat{P}_{k-1},\; k = 1, 2, ...) \text{ and } V = H - D.$$

We have

$$H\hat{P}_k f = \hat{P}_k H\hat{P}_k f \; (k = 1, 2, ...;\; f \in Dom(H)). \tag{4.3}$$

Indeed, $H^{-1}\hat{P}_k$ is an invertible $k \times k$ matrix, and therefore, $H^{-1}\hat{P}_k\mathfrak{H}$ is dense in $\hat{P}_k\mathfrak{H}$. Since $\Delta\hat{P}_j\hat{P}_k = 0$ for $j > k$, we have $0 = \Delta\hat{P}_j H H^{-1}\hat{P}_k = \Delta\hat{P}_j H\hat{P}_k H^{-1}\hat{P}_k$. Hence $\Delta\hat{P}_j Hf = 0$ for any $f \in \hat{P}_k H$. This implies (4.3).

Furthermore, put $H_n = HP_n$. Due to (4.3) we have

$$\|H_n f - Hf\| \to 0 \ \ (f \in Dom(H)) \text{ as } n \to \infty. \tag{4.4}$$

From Lemma 3.1 and (4.4) with $A = H_n$ it follows that in $\hat{P}_n\mathfrak{H}$ there is a invertible operator T_n such that $T_n H_n = \hat{P}_n D T_n$ and

$$\|T_n\| \le \psi_{n,p}(H_n) := (1 + \frac{\tau(H_n)\sqrt{2}N_{2p}(H_{nI})\zeta_q(H_n)}{n-1})^{n-1}$$

where

$$\tau(H_n) = \sum_{m=0}^{p-1}\sum_{k=0}^{\infty}\frac{(\beta_p N_{2p}(H_{nI}))^{kp+m}}{\hat{\delta}^{kp+m}(H_n)\sqrt{k!}}.$$

It is clear, that

$$\tau(H_n)\sqrt{2}N_{2p}(H_{nI})\zeta_q(H_n) \le \tau(H)\sqrt{2}N_{2p}(H_I)\zeta_q(H) = u_p(H)$$

and therefore

$$\|T_n\| \le (1 + \frac{u_p(H)}{n-1})^{n-1} \le e^{u_p(H)}.$$

Similarly, $\|T_n^{-1}\| \le e^{u_p(H)}$.

So there is a weakly convergent subsequence T_{n_j} whose limit we denote by T. It is simple to check that $T_n = P_n T$. Since projections P_n converge strongly, subsequence $\{T_{n_j}\}$ converges strongly. Thus $T_{n_j}H_{n_j}f \to THf$ strongly and, therefore $\hat{P}_{n_j}DT_{n_j}f = T_{n_j}H_{n_j}f \to THf$ strongly. Letting $n_j \to \infty$ hence we arrive at the required result. □

Inequality (1.7) follows from (3.9) according to the above arguments.

5. Operators with Hilbert - Schmidt Components

In this section in the case $p = 1$ we slightly improve Theorem 1.1. Besides, the misprint in the main result from [11] is corrected.

Denote

$$g(H) := \sqrt{2}[N_2^2(H_I) - \sum_{k=1}^{\infty}|Im\ \lambda_k(H)|^2\ ]^{1/2} \le \sqrt{2}N_2(H_I),$$

and

$$\tau_2(H) := \sum_{k=0}^{\infty}\frac{g^{k+1}(H)}{\sqrt{k!}\hat{\delta}^k(H)}.$$

Theorem 5.1. *Let conditions (1.1) and (1.2) be fulfilled with $p = 1$. Then there are an invertible operator T and a normal operator D acting in $\mathfrak{H}$, such that (1.5) holds. Moreover,*

$$\kappa_T \le e^{2\zeta_2(H)\tau_2(H)}. \tag{5.1}$$

Proof. Let A be an $n \times n$-matrix whose eigenvalues are different. Define $\hat{\delta}(A), \delta_m(A)$ and $\zeta_2(A)$ as in Section 3. We have

$$g(A) := \sqrt{2}[N_2^2(A_I) - \sum_{k=1}^{n}|Im\ \lambda_k(A)|^2\ ]^{1/2}.$$

Put

$$\tau_2(A) := \sum_{k=0}^{n-2}\frac{g^{k+1}(A)}{\sqrt{k!}\hat{\delta}^k(A)} \text{ and } \gamma_n(A) := \left(1 + \frac{\zeta_2(A)\tau_2(A)}{n-1}\right)^{2(n-1)}.$$

Due to Lemma 3.1 from [11], there are an invertible matrix $M_n \in \mathbb{C}^{n\times n}$ and a normal matrix $D_n \in \mathbb{C}^{n\times n}$, such that $M_n^{-1}AM_n = D_n$. and

$$\|M_n^{-1}\|\|M_n\| \leq \gamma_n(A). \tag{5.2}$$

Now take H_n and $\hat{P}_n$ as in the proof of Theorem 1.1 from which it follows follows that in $\hat{P}_n\mathfrak{H}$ there is a invertible operator T_n such that $T_nH_n = \hat{P}_nDT_n$. Besides, according to (5.2)

$$\|T_n^{-1}\|\|T_n\| \leq \left(1 + \frac{\zeta_2(H_n)\tau_2(H_n)}{n-1}\right)^{2(n-1)}$$

with

$$\tau_2(H_n) = \sum_{k=0}^{n-2} \frac{g^{k+1}(H_n)}{\sqrt{k!}\hat{\delta}^k(H_n).}$$

It is simple to see that $\zeta_2(H_n) \leq \zeta_2(H)$, $\tau_2(H_n) \leq \tau_2(H)$ and thus

$$\|T_n^{-1}\|\|T_n\| \leq e^{2\zeta_2(H)\tau_2(H)}.$$

Hence taking into account (4.4) and that a subsequence of $\{T_n\}$ strongly converges (see the proof of Theorem 1.1), we arrive at the required result. □

6. Applications of Theorem 1.1

Rewrite (1.5) as $Hx = T^{-1}DTx$. Let ΔP_k be the eigenprojections of the normal operator D and $E_k = T^{-1}\Delta P_kT$. Then

$$Hx = \sum_{k=1}^{\infty} \lambda_k(H)E_kx \quad (x \in Dom(H)).$$

Let $f(z)$ be a scalar function defined and bounded on the spectrum of H. Put

$$f(H) = \sum_{k=1}^{\infty} f(\lambda_k(H))E_k$$

and

$$\gamma_p(H) = e^{2u_p(H)}.$$

Theorem 1.1 immediately implies.

Corollary 6.1. *Let conditions (1.1) and (1.2) hold. Then* $\|f(H)\| \leq \gamma_p(H)\sup_k |f(\lambda_k(H))|$.

In particular, we have

$$\|e^{-Ht}\| \leq \gamma_p(H)e^{-\beta(H)t} \quad (t \geq 0),$$

where $\beta(H) = \inf_k \ Re \ \lambda_k(H)$ and

$$\|R_\lambda(H)\| \leq \frac{\gamma_p(H)}{\rho(H,\lambda)} \quad (\lambda \notin \sigma(H)). \tag{6.1}$$

Let A and $\tilde{A}$ be linear operators. Then the quantity

$$sv_A(\tilde{A}) := \sup_{t\in\sigma(\tilde{A})} \inf_{s\in\sigma(A)} |t-s|$$

is said to be the variation of $\tilde{A}$ with respect to A.

Now let $\tilde{H}$ be a linear operator in $\mathfrak{H}$ with $Dom(H) = Dom(\tilde{H})$ and

$$\xi := \|H - \tilde{H}\| < \infty. \tag{6.2}$$

From (6.1) it follows that $\lambda \notin \sigma(\tilde{H})$, provided $\xi\gamma_p(H) < \rho(H,\lambda)$. So for any $\mu \in \sigma(\tilde{H})$ we have $\xi\gamma_p(H) \geq \rho(H,\mu)$. This inequality implies our next result.

Corollary 6.2. *Let conditions (1.1), (1.2) and (6.2) hold. Then* $sv_H(\tilde{H}) \le \xi\gamma_p(H)$.

Now consider unbounded perturbations. To this end put

$$H^{-\nu} = \sum_{k=1}^{\infty} \lambda_k^{-\nu}(H) E_k \quad (0 < \nu \le 1).$$

Similarly H^{ν} is defined. We have

$$\|H^{\nu} R_{\lambda}(H)\| \le \frac{\gamma(H)}{\phi_{\nu}(H,\lambda)} \quad (\lambda \notin \sigma(H)), \tag{6.3}$$

where

$$\phi_{\nu}(H,\lambda) = \inf_k |(\lambda - \lambda_k(H))\lambda_k^{-\nu}(H)|.$$

Now let $\tilde{H}$ be a linear operator in $\mathfrak{H}$ with $Dom(H) = Dom(\tilde{H})$ and

$$\xi_{\nu} := \|(H - \tilde{H})H^{-\nu}\| < \infty. \tag{6.4}$$

Take into account that

$$R_{\lambda}(H) - R_{\lambda}(\tilde{H}) = R_{\lambda}(H)(\tilde{H} - H)R_{\lambda}(\tilde{H}) = R_{\lambda}(\tilde{H})(\tilde{H} - H)H^{-\nu}H^{\nu}R_{\lambda}(H).$$

Thus, $\lambda \notin \sigma(\tilde{H})$, provided the conditions (6.4) and $\xi_{\nu}\gamma_p(H) < \phi_{\nu}(H,\lambda)$ hold. So for any $\mu \in \sigma(\tilde{H})$ we have

$$\xi_{\nu}\gamma(H) \ge \phi_{\nu}(H,\mu). \tag{6.5}$$

The quantity

$$\nu - \mathrm{rsv}_H(\tilde{H}) := \sup_{t\in\sigma(\tilde{H})} \inf_{s\in\sigma(H)} |(t-s)s^{-\nu}|$$

is said to be the $\nu-$ relative spectral variation of operator $\tilde{H}$ with respect to H. Now (6.5) implies.

Corollary 6.3. *Let conditions (1.1), (1.2) and (6.4) hold. Then* $\nu - \mathrm{rsv}_H(\tilde{H}) \le \xi_{\nu}\gamma_p(H)$.

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