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Generalized Weighted Composition Operators from H^∞ to the Logarithmic Bloch Space

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Abstract. In this paper, we give three different characterizations for the boundedness and compactness of generalized weighted composition operators from the space of bounded analytic function to the logarithmic Bloch space.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the class of all functions analytic on $\mathbb{D}$, and by $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$, with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space $\mathcal{B}$ if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty.$$

The logarithmic Bloch space, denoted by $\mathcal{LB}$, consists of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\log} = \sup_{z\in\mathbb{D}} (1-|z|^2) \log \frac{e}{1-|z|^2} |f'(z)| < \infty.$$

Ir is easy to check that $\mathcal{LB}$ is a Banach space with the norm $\|f\|_{\mathcal{LB}} = |f(0)| + \|f\|_{\log}$. It is well known that $\mathcal{LB} \bigcap H^\infty$ is the space of multipliers of the Bloch space $\mathcal{B}$ (see [2, 31]). The space $\mathcal{LB}$ also arises in the study of Hankel operators on the Bergman space. In [1], Attele showed that the Hankel operator H_f is bounded on the Bergman space A^1 if and only if $f \in \mathcal{LB}$, where $H_f g = (I-P)(\overline{f}g)$, I is the identity operator and P is the Bergman projection from L^1 into A^1. See, for example, [3, 6, 11, 17, 26, 27, 29] for some results on logarithmic spaces and operators on them.

The differentiation operator D is defined by $Df = f'$, $f \in H(\mathbb{D})$. For a nonnegative integer n, we define

$$(D^0 f)(z) = f(z), \ \ (D^n f)(z) = f^{(n)}(z), \ \ n \geq 1, \ f \in H(\mathbb{D}).$$

Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and let n be a nonnegative integer. The linear operator $D^n_{\varphi,u}$, called the generalized weighted composition operator, is defined by (see [32–34])

$$(D^n_{\varphi,u} f)(z) = u(z) \cdot (D^n f)(\varphi(z)), \ \ f \in H(\mathbb{D}), \ \ z \in \mathbb{D}.$$

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When $n = 0$ and $u(z) = 1$, $D^n_{\varphi,u}$ is the composition operator C_φ, which is defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. A basic problem concerning composition operators on various Banach function spaces is to relate the operator theoretic properties of C_φ to the function theoretic properties of the symbol φ, which attracted a lot of attention recently, the reader can refer to [4]. If $n = 0$, then $D^n_{\varphi,u}$ is the weighted composition operator uC_φ, which is defined as follows

$$uC_\varphi f = u(f \circ \varphi), \quad f \in H(\mathbb{D}).$$

If $n = 1$, $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$. When $u(z) = 1$, $D^n_{\varphi,u} = C_\varphi D^n$. DC_φ and $C_\varphi D^n$ were studied in [5, 8–10, 18, 23, 25] and the referees therein. See, for example, [7, 11, 19–21, 28, 32–34] for the study of the generalized weighted composition operator on various function spaces.

It is well known that the composition operator is bounded on the Bloch space by Schwarz-Pick Lemma. Composition operators and weighted composition operators on Bloch-type spaces were studied, for example, in [12–16, 22, 24, 30]. In [24], Wulan, Zheng and Zhu obtained a characterization for the compactness of the composition operators acting on the Bloch space as follows:

Theorem A. *Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if*

$$\lim_{j\to\infty} \|\varphi^j\|_{\mathcal{B}} = 0.$$

Motivated by [24], Colonna and Li characterized the boundedness and compactness of the operator $uC_\varphi : H^\infty \to \mathcal{LB}$ in [3]. The result about the boundedness is stated as follows.

Theorem B. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) The operator $uC_\varphi : H^\infty \to \mathcal{LB}$ is bounded.

(b) $\sup_{j\in\mathbb{N}\cup 0} \|uC_\varphi I^j\|_{\mathcal{LB}} < \infty$, *where* $I^j(z) = z^j$.

(c) $u \in \mathcal{LB}$ *and*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)} < \infty.$$

In [23], Wu and Wulan obtained two characterizations for the compactness of the product of differentiation and composition operators acting on the Bloch space as follows:

Theorem C. *Let φ be an analytic self-map of $\mathbb{D}$, $n \in \mathbb{N}$. Then the following statements are equivalent.*

(a) $C_\varphi D^n : \mathcal{B} \to \mathcal{B}$ *is compact.*

(b) $\lim_{j\to\infty} \|C_\varphi D^n I^j\|_{\mathcal{B}} = 0$, *where* $I^j(z) = z^j$.

(c) $\lim_{|a|\to 1} \|C_\varphi D^n \sigma_a\|_{\mathcal{B}} = 0$, *where* $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ *is the Möbius map on* $\mathbb{D}$.

Motivated by these observations, in this work we show that $D^n_{\varphi,u}$ from H^∞ to the logarithmic Bloch space is bounded (respectively, compact) if and only if the sequence $(\|D^n_{\varphi,u} I^j\|_{\mathcal{LB}})^\infty_{j=n}$ is bounded (respectively, converges to 0 as $j \to \infty$), where $I^j(z) = z^j$. Moreover, we use two families of functions to characterize the boundedness and compactness of the operators $D^n_{\varphi,u}$.

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next.

2. Main Results and Proofs

In this section, we give our main results and proofs. First we characterize the boundedness of the operator $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$. We now introduce two families of functions which will be used to characterize the boundedness and compactness of the operators $D^n_{\varphi,u}$. For $a \in \mathbb{D}$, we define

$$f_a(z) = \frac{1-|a|^2}{1-\bar{a}z} \quad \text{and} \quad h_a(z) = \frac{(1-|a|^2)^2}{(1-\bar{a}z)^2}, \quad z \in \mathbb{D}.$$

Theorem 1. *Let n be a nonnegative integer, $u \in H(\mathbb{D})$ and let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ *is bounded.*

(b) $\sup_{j\geq n} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}} < \infty$, *where* $I^j(z) = z^j$.

(c) $u \in \mathcal{LB}$, $\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)| < \infty$ *and*

$$\sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} f_a\|_{\mathcal{LB}} < \infty, \quad \sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} h_a\|_{\mathcal{LB}} < \infty.$$

(d)

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} < \infty \quad \textit{and} \quad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)|}{(1-|\varphi(z)|^2)^{n}} < \infty.$$

Proof. $(a) \Rightarrow (b)$ This implication is obvious, since for $j \in \mathbb{N}$, the function I^j is bounded in H^∞ and $\|I^j\|_\infty = 1$.

$(b) \Rightarrow (c)$ Assume that (b) holds and let $Q := \sup_{j\geq n} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}}$. From the definition of f_a and h_a, it is easy to see that f_a and h_a have bounded norms in H^∞. Since

$$f_a(z) = (1-|a|^2)\sum_{j=0}^{\infty} \bar{a}^j z^j, \qquad h_a(z) = (1-|a|^2)^2 \sum_{j=0}^{\infty}(j+1)\bar{a}^j z^j,$$

using linearity we get

$$\begin{aligned}
\|D^n_{\varphi,u} f_a\|_{\mathcal{LB}} &\leq (1-|a|^2)\sum_{j=0}^{\infty}|a|^j \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}} \leq 2Q \text{ and} \\
\|D^n_{\varphi,u} h_a\|_{\mathcal{LB}} &\leq (1-|a|^2)^2\sum_{j=0}^{\infty}(j+1)|a|^j \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}} \leq 4Q.
\end{aligned}$$

Applying the operator $D^n_{\varphi,u}$ to I^j with $j = n, n+1$, we obtain

$$\begin{aligned}
(D^n_{\varphi,u} I^n)'(z) &= u'(z)n! \quad \text{and} \\
(D^n_{\varphi,u} I^{n+1})'(z) &= u'(z)(n+1)!\varphi(z) + u(z)(n+1)!\varphi'(z),
\end{aligned}$$

while for $j < n$, $(D^n_{\varphi,u} I^j)'(z) = 0$. Thus, using the boundedness of the function φ, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)| \leq \frac{1}{n!}\|D^n_{\varphi,u} I^n\|_{\mathcal{LB}} \leq \frac{Q}{n!},$$

i.e., $u \in \mathcal{LB}$ and

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)| \\
\leq\ & \frac{1}{(n+1)!}\|D^n_{\varphi,u} I^{n+1}\|_{\mathcal{LB}} + \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)| \leq \frac{(n+2)Q}{(n+1)!}.
\end{aligned}$$

$(c) \Rightarrow (d)$ Assume that (c) holds. Let

$$C_1 := \sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} f_a\|_{\mathcal{LB}} \qquad \text{and} \qquad C_2 := \sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} h_a\|_{\mathcal{LB}}.$$

For $a \in \mathbb{D}$, set

$$g_a(z) = \frac{1-|a|^2}{1-\bar{a}z} - \frac{1}{1+n}\frac{(1-|a|^2)^2}{(1-\bar{a}z)^2}, \quad z \in \mathbb{D}.$$

It is easy to check that $g_a \in H^\infty$ and $\sup_{a\in\mathbb{D}} \|g_a\|_\infty < \infty$. Therefore, from the assumption we see that

$$\sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} g_a\|_{\mathcal{LB}} \le C_1 + \frac{1}{1+n} C_2 < C_1 + C_2 < \infty. \tag{1}$$

For $\lambda \in \mathbb{D}$, we notice that

$$g^{(n)}_{\varphi(\lambda)}(\varphi(\lambda)) = 0, \ \ |g^{(n+1)}_{\varphi(\lambda)}(\varphi(\lambda))| = \frac{n!|\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}}. \tag{2}$$

Hence by (1) and (2) we get that

$$C_1 + C_2 > \|D^n_{\varphi,u} g_{\varphi(\lambda)}\|_{\mathcal{LB}} \ge \frac{n!(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}}, \tag{3}$$

for $\lambda \in \mathbb{D}$. For any fixed $r \in (0,1)$, from (3), we have

$$\begin{aligned} & \sup_{|\varphi(\lambda)|>r} \frac{(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{n+1}} \\ \le \ & \sup_{|\varphi(\lambda)|>r} \frac{1}{r^{n+1}} \frac{(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}} \le \frac{C_1+C_2}{r^{n+1}n!} < \infty. \end{aligned} \tag{4}$$

By the assumption that $\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)| < \infty$, we get

$$\begin{aligned} & \sup_{|\varphi(\lambda)|\le r} \frac{(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{n+1}} \\ \le \ & \sup_{|\varphi(\lambda)|\le r} \frac{1}{(1-r^2)^{n+1}}(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u(\lambda)||\varphi'(\lambda)| < \infty. \end{aligned} \tag{5}$$

Therefore, (4) and (5) yield the first inequality of (d).

Next, note that

$$\begin{aligned} C_1 \ & \ge \ \|D^n_{\varphi,u} f_{\varphi(\lambda)}\|_{\mathcal{LB}} \\ & \ge \ \frac{n!(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u'(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^n} - \frac{(n+1)!(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{1+n}}. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u'(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^n} \\ \le \ & \frac{C_1}{n!} + \frac{(n+1)(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}}. \end{aligned} \tag{6}$$

From (3) and (6), we get

$$\sup_{\lambda\in\mathbb{D}} \frac{(1-|\lambda|^2)\log\frac{e}{1-|\lambda|^2}|u'(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^n} < \infty. \tag{7}$$

Combining (7) with $u \in \mathcal{LB}$ and arguing as above, we get the second inequality of (d).

$(d) \Rightarrow (a)$ Assume that (d) holds. By Theorem 5.1.5 of [31], if $f \in \mathcal{B}$ and $m \in \mathbb{N}$, then

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{m+1}|f^{(m+1)}(z)| \le C_m\|f\|_{\mathcal{B}},$$

where C_m is a constant depending only on m. Since $H^\infty \subset \mathcal{B}$ and $\|f\|_{\mathcal{B}} \le \|f\|_\infty$, for all $f \in H^\infty$, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{m+1}|f^{(m+1)}(z)| \le C_m\|f\|_\infty.$$

Therefore, for any $f \in H^\infty$, we have

$$\begin{aligned}
&(1-|z|^2)\log\frac{e}{1-|z|^2}|(D^n_{\varphi,u}f)'(z)| = (1-|z|^2)\log\frac{e}{1-|z|^2}|(f^{(n)}(\varphi)u)'(z)|\\
\le\ & (1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)||f^{(n+1)}(\varphi(z))| + (1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)||f^{(n)}(\varphi(z))|\\
\le\ & C\frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\|f\|_\infty + C\frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)|}{(1-|\varphi(z)|^2)^{n}}\|f\|_\infty.
\end{aligned}$$

Moreover,

$$|(D^n_{\varphi,u}f)(0)| = |f^{(n)}(\varphi(0))u(0)| \le \frac{C|u(0)|}{(1-|\varphi(0)|^2)^n}\|f\|_\infty.$$

From (d) we see that

$$\|D^n_{\varphi,u}f\|_{\mathcal{LB}} = |(D^n_{\varphi,u}f)(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{e}{1-|z|^2}|(D^n_{\varphi,u}f)'(z)| \le C\|f\|_\infty.$$

Therefore the operator $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$ is bounded, as desired. □

To study the compactness of $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$, we need the following lemma, which can be proved in a standard way, see, for example Proposition 3.11 in [4].

Lemma 2. *Let n be a nonnegative integer, $u \in H(\mathbb{D})$ and let φ be an analytic self-map of $\mathbb{D}$. Then $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$ is compact if and only if $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$ is bounded and for any bounded sequence $(f_j)_{j\in\mathbb{N}}$ in H^∞ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|D^n_{\varphi,u}f_j\|_{\mathcal{LB}} \to 0$ as $j \to \infty$.*

Theorem 3. *Let n be a nonnegative integer, $u \in H(\mathbb{D})$ and let φ an analytic self-map of $\mathbb{D}$ such that $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$ is bounded. Then the following statements are equivalent.*

(a) $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$ is compact.

(b) $\lim_{j\to\infty}\|D^n_{\varphi,u}I^j\|_{\mathcal{LB}} = 0$, where $I^j(z) = z^j$.

(c) $\lim_{|\varphi(a)|\to1}\|D^n_{\varphi,u}f_{\varphi(a)}\|_{\mathcal{LB}} = 0$ and $\lim_{|\varphi(a)|\to1}\|D^n_{\varphi,u}h_{\varphi(a)}\|_{\mathcal{LB}} = 0$.

(d)

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} = 0 \ \text{and}\ \lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)|}{(1-|\varphi(z)|^2)^{n}} = 0.$$

Proof. $(a) \Rightarrow (b)$ Assume $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$ is compact. Since the sequence $\{I^j\}$ is bounded in H^∞ and converges to 0 uniformly on compact subsets, by Lemma 2 it follows that $\|D^n_{\varphi,u}I^j\|_{\mathcal{LB}} \to 0$ as $j \to \infty$.

$(b) \Rightarrow (c)$ Suppose (b) holds. Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\|D^n_{\varphi,u}I^j\|_{\mathcal{LB}} < \varepsilon$ for all $j \ge N$. Let $z_k \in \mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Arguing as in Theorem 1, we have

$$\begin{aligned}
&\|D^n_{\varphi,u}f_{\varphi(z_k)}\|_{\mathcal{LB}} \le (1-|\varphi(z_k)|^2)\sum_{j=0}^{\infty}|\varphi(z_k)|^j\|D^n_{\varphi,u}I^j\|_{\mathcal{LB}}\\
=\ & (1-|\varphi(z_k)|^2)\sum_{j=0}^{N-1}|\varphi(z_k)|^j\|D^n_{\varphi,u}I^j\|_{\mathcal{LB}} + (1-|\varphi(z_k)|^2)\sum_{j=N}^{\infty}|\varphi(z_k)|^j\|D^n_{\varphi,u}I^j\|_{\mathcal{LB}}\\
\le\ & 2Q(1-|\varphi(z_k)|^N) + 2\varepsilon.
\end{aligned}$$

Since $|\varphi(z_k)| \to 1$ as $k \to \infty$, by the arbitrary of ε, we get $\lim_{k\to\infty} \|D^n_{\varphi,u} f_{\varphi(z_k)}\|_{\mathcal{LB}} = 0$, i.e., we obtain $\lim_{|\varphi(a)|\to 1} \|D^n_{\varphi,u} f_{\varphi(a)}\|_{\mathcal{LB}} = 0$.

Notice that

$$\sum_{j=0}^{N-1}(j+1)r^j = \frac{1-r^N-Nr^N(1-r)}{(1-r)^2}, \quad 0 \le r < 1,$$

arguing as Theorem 1 we get

$$\begin{aligned}
\|D^n_{\varphi,u} h_{\varphi(z_k)}\|_{\mathcal{LB}} &\le (1-|\varphi(z_k)|^2)^2 \sum_{j=0}^{\infty}(j+1)|\varphi(z_k)|^j \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}} \\
&= (1-|\varphi(z_k)|^2)^2 \sum_{j=0}^{N-1}(j+1)|\varphi(z_k)|^j \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}} + (1-|\varphi(z_k)|^2)^2 \sum_{j=N}^{\infty}(j+1)|\varphi(z_k)|^j \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}} \\
&\le 4Q\Big(1-|\varphi(z_k)|^N - N|\varphi(z_k)|^N(1-|\varphi(z_k)|)\Big) + 4\varepsilon.
\end{aligned}$$

Therefore, $\lim_{k\to\infty} \|D^n_{\varphi,u} h_{\varphi(z_k)}\|_{\mathcal{LB}} \le 4\varepsilon$. By the arbitrary of ε, we obtain $\lim_{|\varphi(a)|\to 1} \|D^n_{\varphi,u} h_{\varphi(a)}\|_{\mathcal{LB}} = 0$, as desired.

$(c) \Rightarrow (d)$ To prove (d) it only need to show that if $(z_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$, then

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)\log\frac{e}{1-|z_k|^2}|u(z_k)||\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^{n+1}} = 0, \quad \lim_{k\to\infty} \frac{(1-|z_k|^2)\log\frac{e}{1-|z_k|^2}|u'(z_k)|}{(1-|\varphi(z_k)|^2)^{n}} = 0.$$

Let $(z_k)_{k\in\mathbb{N}}$ be such a sequence such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. From the assumption and arguing as Theorem 1 we obtain

$$\lim_{k\to\infty} \|D^n_{\varphi,u} g_{\varphi(z_k)}\|_{\mathcal{LB}} \le \lim_{k\to\infty} \|D^n_{\varphi,u} f_{\varphi(z_k)}\|_{\mathcal{LB}} + \frac{1}{n+1}\lim_{k\to\infty} \|D^n_{\varphi,u} h_{\varphi(z_k)}\|_{\mathcal{LB}} = 0.$$

Hence $\lim_{k\to\infty} \|D^n_{\varphi,u} g_{\varphi(z_k)}\|_{\mathcal{LB}} = 0$. Similarly to the proof of Theorem 1, we have

$$\frac{n!(1-|z_k|^2)\log\frac{e}{1-|z_k|^2}|u(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}} \le \|D^n_{\varphi,u} g_{\varphi(z_k)}\|_{\mathcal{LB}} \to 0, \ \text{ as } \ k \to \infty,$$

which implies

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)\log\frac{e}{1-|z_k|^2}|u(z_k)||\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^{n+1}} = \lim_{k\to\infty} \frac{(1-|z_k|^2)\log\frac{e}{1-|z_k|^2}|u(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}} = 0. \tag{8}$$

In addition,

$$\begin{aligned}
&\|D^n_{\varphi,u} f_{\varphi(z_k)}\|_{\mathcal{LB}} + \frac{(n+1)!(1-|z_k|^2)\log\frac{e}{1-|z_k|^2}|u(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}} \\
\ge\ & \frac{n!(1-|z_k|^2)\log\frac{e}{1-|z_k|^2}|u'(z_k)||\varphi(z_k)|^{n}}{(1-|\varphi(z_k)|^2)^{n}}.
\end{aligned}$$

From (8) and the assumption that $\|D^n_{\varphi,u} f_{\varphi(z_k)}\|_{\mathcal{LB}} \to 0$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)\log\frac{e}{1-|z_k|^2}|u'(z_k)|}{(1-|\varphi(z_k)|^2)^{n}} = \lim_{k\to\infty} \frac{(1-|z_k|^2)\log\frac{e}{1-|z_k|^2}|u'(z_k)||\varphi(z_k)|^{n}}{(1-|\varphi(z_k)|^2)^{n}} = 0,$$

as desired.

$(d) \Rightarrow (a)$ Assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in H^∞ converging to 0 uniformly on compact subsets of $\mathbb{D}$. By the assumption, for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that

$$\frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|\varphi'(z)||u(z)|}{(1-|\varphi(z)|^2)^{n+1}} < \varepsilon \quad \text{and} \quad \frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)|}{(1-|\varphi(z)|^2)^{n}} < \varepsilon \tag{9}$$

when $\delta < |\varphi(z)| < 1$. Let $K = \{z \in \mathbb{D} : |\varphi(z)| \le \delta\}$. Since $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ is bounded, as shown in the proof of Theorem 1,

$$C_3 := \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)| < \infty \tag{10}$$

and

$$C_4 := \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)| < \infty. \tag{11}$$

By (9), (10) and (11), we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{e}{1-|z|^2}|(D^n_{\varphi,u}f_k)'(z)| \\
\le\ & \sup_{z\in K}(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)||f_k^{(n+1)}(\varphi(z))| + \sup_{z\in K}(1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)||f_k^{(n)}(\varphi(z))| \\
& + C\sup_{z\in\mathbb{D}\setminus K}\frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\|f_k\|_\infty + C\sup_{z\in\mathbb{D}\setminus K}\frac{(1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)|}{(1-|\varphi(z)|^2)^{n}}\|f_k\|_\infty \\
\le\ & C_4\sup_{z\in K}|f_k^{(n+1)}(\varphi(z))| + C_3\sup_{z\in K}|f_k^{(n)}(\varphi(z))| + C\varepsilon\|f_k\|_\infty.
\end{aligned}$$

Hence

$$\|D^n_{\varphi,u}f_k\|_{\mathcal{LB}} \le C_4\sup_{|w|\le\delta}|f_k^{(n+1)}(w)| + C_3\sup_{|w|\le\delta}|f_k^{(n)}(w)| + C\varepsilon\|f_k\|_\infty + |u(0)||f_k^{(n)}(\varphi(0))|. \tag{12}$$

Since $(f_k)_{k\in\mathbb{N}}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, by Cauchy's estimates we see that $(f_k^{(n)})$ and $(f_k^{(n+1)})$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$. From (12), letting $k \to \infty$ and using the fact that ε is an arbitrary positive number, we obtain $\|D^n_{\varphi,u}f_k\|_{\mathcal{LB}} \to 0$ as $k \to \infty$. By Lemma 2, we see that the operator $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ is compact. □

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