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A Certain Subclass of Analytic Functions Defined by Means of Differential Subordination

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Abstract. For $\alpha \in (\pi, \pi]$, let $\mathcal{R}_\alpha(\phi)$ denote the class of all normalized analytic functions in the open unit disk $\mathbb{U}$ satisfying the following differential subordination:

$$f'(z) + \frac{1}{2}\left(1 + e^{i\alpha}\right) z f''(z) \prec \phi(z) \qquad (z \in \mathbb{U}),$$

where the function $\phi(z)$ is analytic in the open unit disk $\mathbb{U}$ such that $\phi(0) = 1$. In this paper, various integral and convolution characterizations, coefficient estimates and differential subordination results for functions belonging to the class $\mathcal{R}_\alpha(\phi)$ are investigated. The Fekete-Szegö coefficient functional associated with the kth root transform $[f(z^k)]^{1/k}$ of functions in $\mathcal{R}_\alpha(\phi)$ is obtained. A similar problem for a corresponding class $\mathcal{R}_{\Sigma;\alpha}(\phi)$ of bi-univalent functions is also considered. Connections with previous known results are pointed out.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ and by C the familiar subclass of $\mathcal{S}$ whose members are convex functions in $\mathbb{U}$.

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Let $\mathcal{M}$ be the class of analytic functions $\phi(z)$ in $\mathbb{U}$, normalized by $\phi(0) = 1$. Also let $\mathcal{N}$ be the subclass of $\mathcal{M}$ consisting of all univalent functions ϕ for which $\phi(\mathbb{U})$ is a convex domain.

We denote by $\mathcal{P}$ the well-known class of analytic functions $p(z)$ with

$$p(0) = 1 \qquad \text{and} \qquad \Re\big(p(z)\big) > 0 \quad (z \in \mathbb{U}).$$

We also denote by $\mathcal{B}$ the class of analytic functions $\omega(z)$ in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Suppose that the functions f and g are analytic in $\mathbb{U}$. Then the function f is said to be subordinate to the function g, denoted by $f \prec g$, if there exists a function $\omega \in \mathcal{B}$ such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

For functions f given by (1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (z \in \mathbb{U}),$$

the Hadamard product (or convolution), denoted by $f * g$, is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z) \qquad (z \in \mathbb{U}).$$

Recently, Silverman and Silvia [25] considered the following classes of functions:

$$\mathcal{L}_\alpha = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) \right) > 0 \quad (z \in \mathbb{U}) \right\} \tag{2}$$

and

$$\mathcal{L}_\alpha(b) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \left| f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) - b \right| < b \quad (z \in \mathbb{U}) \right\}, \tag{3}$$

where $\alpha \in (-\pi, \pi]$ and $b > \frac{1}{2}$. Clearly, if $b \to \infty$, then $\mathcal{L}_\alpha(b) \to \mathcal{L}_\alpha$. For each of these two classes of functions, they obtained extreme points, coefficient estimates and convolution characterizations. Trojnar-Spelina [31], on the other hand, studied the function class $\mathcal{LP}_\alpha$ given by

$$\mathcal{LP}_\alpha = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) \prec Q(z) \quad (z \in \mathbb{U}) \right\}, \tag{4}$$

where $\alpha \in (-\pi, \pi]$. The function $Q(z)$ defined by

$$Q(0) = 1 \qquad \text{and} \qquad Q(z) = 1 + \frac{2}{\pi^2} \left[\log\left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right]^2 \quad (z \in \mathbb{U}) \tag{5}$$

maps $\mathbb{U}$ onto the domain given by

$$\Omega = \{ w : w \in \mathbb{C} \quad \text{and} \quad |w - 1| < \Re(w) \}.$$

Motivated by some of the ideas explored in the aforecited investigations [25] and [31], here we define a new class of analytic functions.

Definition 1. Let $\alpha \in (-\pi, \pi]$ and let $\phi \in \mathcal{M}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_\alpha(\phi)$ if the following differential subordination is satisfied:

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) \prec \phi(z) \qquad (z \in \mathbb{U}). \tag{6}$$

Consider the following two functions:

$$\phi_0(z) = \frac{1+z}{1-z} \qquad (z \in \mathbb{U}) \tag{7}$$

and

$$\phi_b(z) = \frac{1+z}{1-(1-\frac{1}{b})z} \qquad \left(z \in \mathbb{U};\ b > \frac{1}{2}\right). \tag{8}$$

Then it is easy to observe that the corresponding classes $\mathcal{R}_\alpha(\phi_0)$ and $\mathcal{R}_\alpha(\phi_b)$ reduce to the classes $\mathcal{L}_\alpha$ and $\mathcal{L}_\alpha(b)$, respectively. We note also that the class $\mathcal{R}_\alpha(Q)$, where the function Q is defined by (5), reduces to function class $\mathcal{LP}_\alpha$.

We now recall that the function class $\mathcal{R}$ given by

$$\mathcal{R} = \mathcal{R}_0(\phi_0) = \Big\{f : f \in \mathcal{A} \quad \text{and} \quad \Re\Big(f'(z) + zf''(z)\Big) > 0 \quad (z \in \mathbb{U})\Big\} \tag{9}$$

was investigated by Chichra [7] and also by Singh and Singh [26]. Another function class $\mathcal{R}_\beta$ given by

$$\mathcal{R}_\beta = \Big\{f : f \in \mathcal{A} \quad \text{and} \quad \Re\Big(f'(z) + zf''(z)\Big) > \beta \quad (z \in \mathbb{U})\Big\}, \tag{10}$$

which was considered by Silverman [24], can also be obtained from $\mathcal{R}_\alpha(\phi)$ upon setting

$$\alpha = 0 \qquad \text{and} \qquad \phi = \phi_\beta,$$

where

$$\phi_\beta(z) = \frac{1+(1-2\beta)z}{1-z} \qquad (z \in \mathbb{U};\ 0 \leqq \beta < 1). \tag{11}$$

In its special case when $\beta = 0$, the function class $\mathcal{R}_\beta$ reduces to the function class $\mathcal{R}$ considered by Silverman [24].

In this paper, we investigate various convolution and integral characterizations, coefficient estimates and subordination results for the general function class $\mathcal{R}_\alpha(\phi)$ which we have introduced here by Definition 1 above. In particular, in Section 6, we derive the Fekete-Szegö coefficient functional associated with the kth root transform $[f(z^k)]^{1/k}$ of functions in the class $\mathcal{R}_\alpha(\phi)$. A similar problem for a corresponding class $\mathcal{R}_{\Sigma;\alpha}(\phi)$ of bi-univalent functions is also considered in the last section (Section 7) of this paper.

2. Convolution Characterization

In this section we obtain a membership characterization of the class $\mathcal{R}_\alpha(\phi)$ in terms of convolution.

Theorem 1. *Let $\alpha \in (-\pi, \pi]$ and let $\phi \in \mathcal{M}$. A necessary and sufficient condition for a function $f \in \mathcal{A}$ to be in the class $\mathcal{R}_\alpha(\phi)$ is given by*

$$\frac{1}{z}\left(f(z) * \frac{z+z^2 e^{i\alpha}}{(1-z)^3}\right) \neq \phi(e^{i\theta}) \qquad \Big(z \in \mathbb{U};\ \theta \in [0, 2\pi)\Big).$$

Proof. We have $f \in \mathcal{R}_\alpha(\phi)$ if and only if

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) \prec \phi(z) \qquad (z \in \mathbb{U}).$$

It follows that $f \in \mathcal{R}_\alpha(\phi)$ if and only if

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) \neq \phi(e^{i\theta}) \qquad \big(z \in \mathbb{U};\ \theta \in [0, 2\pi)\big).$$

Since

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) = \frac{1-e^{i\alpha}}{2} f'(z) + \frac{1+e^{i\alpha}}{2}\big(zf'(z)\big)' \tag{12}$$

and

$$zf'(z) = f(z) * \frac{z}{(1-z)^2} \qquad \text{and} \qquad f(z) = f(z) * \frac{z}{1-z} \quad (z \in \mathbb{U}),$$

we have

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) = \left(f(z) * \left(\frac{1-e^{i\alpha}}{2}\frac{z}{1-z} + \frac{1+e^{i\alpha}}{2}\frac{z}{(1-z)^2}\right)\right)' \neq \phi(e^{i\theta})$$

or, equivalently,

$$\left(f(z) * \frac{z - \frac{1-e^{i\alpha}}{2} z^2}{(1-z)^2}\right)' = \frac{1}{z}\left(f(z) * \frac{z + z^2 e^{i\alpha}}{(1-z)^3}\right) \neq \phi(e^{i\theta}).$$

The convolution characterization asserted by Theorem 1 is thus proved. □

3. Integral Representation

In this section an integral representation for functions in the class $\mathcal{R}_\alpha(\phi)$ is provided.

Theorem 2. *Let $\alpha \in (-\pi, \pi)$ and let $\phi \in \mathcal{M}$. Suppose also that*

$$\gamma := \frac{2}{1+e^{i\alpha}}.$$

Then $f \in \mathcal{R}_\alpha(\phi)$ if and only if there exists $\omega \in \mathcal{B}$ such that the following equality:

$$f(z) = \int_0^z \frac{\gamma}{\eta^\gamma}\left(\int_0^\eta \zeta^{\gamma-1}\phi\big(\omega(\zeta)\big)d\zeta\right)d\eta \tag{13}$$

holds true for all $z \in \mathbb{U}$.

Proof. It follows from Definition 1 of the function class $\mathcal{R}_\alpha(\phi)$ that $f \in \mathcal{R}_\alpha(\phi)$ if and only if there exists $\omega \in \mathcal{B}$ such that

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) = \phi\big(\omega(z)\big) \qquad (z \in \mathbb{U}). \tag{14}$$

Making use of (12) in the above equality (14), we obtain

$$\frac{1-e^{i\alpha}}{2} f'(z) + \frac{1+e^{i\alpha}}{2}\big(zf'(z)\big)' = \phi\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

It follows that

$$\left(\frac{1-e^{i\alpha}}{1+e^{i\alpha}}\right) f'(z) + \big(zf'(z)\big)' = \frac{2}{1+e^{i\alpha}}\phi\big(\omega(z)\big) \qquad (z \in \mathbb{U}),$$

which is equivalent to

$$(\gamma - 1)z^{\gamma-1} f'(z) + z^{\gamma-1}\big(zf'(z)\big)' = \gamma z^{\gamma-1}\phi\big(\omega(z)\big) \qquad (z \in \mathbb{U}),$$

where

$$\gamma = \frac{2}{1+e^{i\alpha}}, \ \alpha \neq \pi.$$

We thus find that

$$\left(z^{\gamma-1}\left(zf'(z)\right)\right)' = \gamma z^{\gamma-1}\phi\left(\omega(z)\right),$$

which readily yields

$$z^{\gamma} f'(z) = \gamma \int_0^z \zeta^{\gamma-1}\phi\left(\omega(\zeta)\right)d\zeta. \tag{15}$$

Integrating once more the equality (15), we get (13). The proof of Theorem 2 is thus completed. □

Remark 1. If $\alpha \to \pi$, then the equality (14) reduces to

$$f'(z) = \phi\left(\omega(z)\right) \qquad (z \in \mathbb{U}).$$

It follows that $f \in \mathcal{R}_\pi(\phi)$ if and only if

$$f(z) = \int_0^z \phi\left(\omega(\zeta)\right)d\zeta.$$

For $\theta \in [0, 2\pi)$ and $\tau \in [0, 1]$, we now define the function $f(z, \theta, \tau)$ by

$$f(z,\theta,\tau) = \int_0^z \frac{\gamma}{\eta^{\gamma}}\left[\int_0^{\eta} \zeta^{\gamma-1}\phi\left(\frac{e^{i\theta}\zeta(\zeta+\tau)}{1+\zeta\tau}\right)d\zeta\right]d\tau \qquad (z \in \mathbb{U}). \tag{16}$$

By virtue of Theorem 2, the function $f(z, \theta, \tau)$ belongs to the class $\mathcal{R}_\alpha(\phi)$.

4. Coefficient Estimates

In this section we obtain coefficient estimates for functions belonging to the class $\mathcal{R}_\alpha(\phi)$.

Theorem 3. *Let $\alpha \in (-\pi, \pi]$ and let the function $\phi(z)$ given by*

$$\phi(z) = 1 + A_1 z + A_2 z^2 + \cdots$$

be in the class $\mathcal{N}$. If a function f of the form (1) *belongs to the class $\mathcal{R}_\alpha(\phi)$, then*

$$|a_n| \leqq \frac{\sqrt{2}|A_1|}{n\sqrt{n^2+1+(n^2-1)\cos\alpha}} \qquad (n \geqq 2).$$

Proof. Since $f \in \mathcal{R}_\alpha(\phi)$, we have

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) = p(z) \qquad (z \in \mathbb{U}), \tag{17}$$

where

$$p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n \prec \phi(z).$$

Equating the coefficients of z^n on both sides of (17), we find the following relation between the coefficients:

$$\frac{n}{2}[2 + (1+e^{i\alpha})(n-1)]a_n = p_{n-1} \qquad (n \geqq 2). \tag{18}$$

Since the function ϕ is univalent in $\mathbb{U}$ and $\phi(\mathbb{U})$ is a convex domain, we can apply Rogosinski's lemma (see [21]). We thus find that

$$|p_n| \leqq |A_1|, \ \ n \geqq 1.$$

Making use of (18), we get

$$|a_n| \leqq \frac{|A_1|}{\frac{n}{2}|2+(1+e^{i\alpha})(n-1)|} = \frac{\sqrt{2}|A_1|}{n\sqrt{n^2+1+(n^2-1)\cos\alpha}},$$

which completes the proof of Theorem 3. □

Remark 2.
(i) Let $\phi(z) = \phi_0(z)$ defined by (7). If f of the form (1) is in the class $\mathcal{R}_\alpha(\phi_0) = \mathcal{L}_\alpha$, then by Theorem 3, we obtain the coefficient estimates found in [25], namely

$$|a_n| \leqq \frac{2\sqrt{2}}{n\sqrt{n^2+1+(n^2-1)\cos\alpha}} \qquad (n \geqq 2).$$

If, in the above inequality, we set $\alpha \to \pi$, then we get

$$|a_n| \leqq \frac{2}{n} \qquad (n \geqq 2),$$

which is the well-known coefficient estimates for the class $\mathcal{R}$ (see [10] and [9]).
(ii) Let $\phi(z) = Q(z)$ defined by (5) and let f of the form (1) be in the class $\mathcal{R}_\alpha(Q) = \mathcal{LP}_\alpha$. Since

$$Q(z) = 1 + \frac{8}{\pi^2}z + \cdots,$$

it follows from Theorem 3, that

$$|a_n| \leqq \frac{8\sqrt{2}}{n\pi^2\sqrt{n^2+1+(n^2-1)\cos\alpha}} = \frac{8}{n\pi^2|1+\frac{n-1}{2}(1+e^{i\alpha})|}, \; n \geqq 2$$

which is the same with the inequality found in [31].

5. Results Involving Differential Subordination

In order to prove our main results of this section, we need the following lemma due to Hallenbeck and Ruscheweyh [11].

Lemma 1. (see [11]) *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ with $\Re\gamma \geqq 0$. If the function $p(z)$ given by*

$$p(z) = a + p_n z^n + p_{n+1}z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$ and

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z) \qquad (z \in \mathbb{U}), \tag{19}$$

then

$$p(z) \prec q(z) \prec h(z) \qquad (z \in \mathbb{U}), \tag{20}$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(\zeta)\zeta^{\gamma/n-1}d\zeta. \tag{21}$$

The result is sharp.

Theorem 4. *Let $\alpha \in (-\pi, \pi)$ and let $\phi \in \mathcal{N}$. If $f \in \mathcal{R}_\alpha(\phi)$, then*

$$f'(z) \prec \int_0^1 \phi(zt^{1/\gamma})dt \prec \phi(z) \qquad (z \in \mathbb{U}) \tag{22}$$

and

$$\frac{f(z)}{z} \prec \int_0^1 \int_0^1 \phi(zrt^{1/\gamma})dr\, dt \qquad (z \in \mathbb{U}), \tag{23}$$

where

$$\gamma = \frac{2}{1 + e^{i\alpha}}.$$

The results are sharp.

Proof. Assume that $f \in \mathcal{R}_\alpha(\phi)$. Then, from Definition 1, it follows that the differential subordination (6) holds true. Let $p(z) = f'(z)$. Also let

$$\gamma = \frac{2}{1 + e^{i\alpha}}.$$

Then

$$p(z) + \frac{1}{\gamma} zp'(z) = f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Since $\phi \in \mathcal{N}$ and $\Re(\gamma) \geqq 0$ for $\alpha \in (-\pi, \pi)$, in view of Lemma 1, we have

$$p(z) \prec \frac{\gamma}{z^\gamma} \int_0^z \zeta^{\gamma-1}\phi(\zeta)d\zeta \prec \phi(z) \qquad (z \in \mathbb{U}). \tag{24}$$

With the substitution $\zeta = zt^{1/\gamma}$ in the integral in (24) and, by taking into account the fact that $p(z) = f'(z)$, the differential chain (24) yields

$$f'(z) \prec \int_0^1 \phi(zt^{1/\gamma})dt \prec \phi(z).$$

The first condition (22) of Theorem 4 is thus proved.

In order to obtain the differential subordination (23), we show that the function $h(z)$ given by

$$h(z) = \int_0^1 \phi(zt^{1/\gamma})dt \qquad (z \in \mathbb{U}) \tag{25}$$

belongs to the class $\mathcal{N}$. To prove this, we employ the same technique as in [1]. We first define

$$\Phi_\gamma(z) = \int_0^1 \frac{1}{1 - zt^{1/\gamma}} dt = \sum_{n=0}^{\infty} \frac{\gamma}{n + \gamma} z^n. \tag{26}$$

For $\Re(\gamma) > 0$, the function $\Phi_\gamma(z)$ is convex in $\mathbb{U}$ (see [23]). From (26) we obtain

$$\phi(z) * \Phi_\gamma(z) = \int_0^1 \frac{1}{1 - zt^{1/\gamma}} dt * \phi(z) = \int_0^1 \phi(zt^{1/\gamma})dt = h(z).$$

It was proved in [22] that the convolution of two convex functions is also convex. Therefore, the function $h(z)$ defined by (25) is convex in $\mathbb{U}$. Moreover, since $h(0) = 1$, it follows that $h \in \mathcal{N}$.

We now let

$$p(z) = \frac{f(z)}{z}.$$

Then, by making use of (22) and (25), we have

$$p(z) + zp'(z) = f'(z) \prec \int_0^1 \phi(zt^{1/\gamma})dt = h(z) \qquad (z \in \mathbb{U}).$$

By applying Lemma 1 once more with $\gamma = 1$, we obtain

$$p(z) \prec \frac{1}{z}\int_0^z h(\zeta)d\zeta \prec h(z) \qquad (z \in \mathbb{U}). \tag{27}$$

With the substitution $\zeta = rz$ in the integral in (27), if we take into account (25) and also that

$$p(z) = \frac{f(z)}{z},$$

the first differential subordination in (27) implies that

$$\frac{f(z)}{z} \prec \int_0^1 \int_0^1 \phi\left(zrt^{1/\gamma}\right) dr\, dt.$$

The differential subordination (23) is thus proved.

Since the result in Lemma 1 is sharp, it follows that the differential subordinations in (22) and (23) are also sharp. Consequently, the proof of Theorem 4 is completed. □

The next result is an immediate consequence of Theorem 4.

Corollary 1. *Let f be in the class $\mathcal{R}_\beta$ $(0 \leqq \beta < 1)$ defined by (10). Then*

$$f'(z) \prec 2\beta - 1 - \frac{2(1-\beta)}{z}\log(1-z) \qquad (z \in \mathbb{U})$$

and

$$\frac{f(z)}{z} \prec 2\beta - 1 - \frac{2(1-\beta)}{z}\int_0^1 \frac{1}{r}\log(1-rz)dr \qquad (z \in \mathbb{U}).$$

Consider the function

$$\phi_M(z) = 1 + Mz \qquad (M > 0)$$

and the corresponding function class $\mathcal{R}_\alpha(\phi_M)$ given by

$$\mathcal{R}_\alpha(\phi_M) = \left\{f : f \in \mathbb{A} \quad \text{and} \quad |f'(z) + \frac{1+e^{i\alpha}}{2}zf''(z) - 1| \leqq M \quad (z \in \mathbb{U};\ M > 0)\right\}.$$

Variations of the class $\mathcal{R}_0(\phi_M)$ have been investigated in several works (see, for example, [36] and [12]).

The following result is another consequence of Theorem 4.

Corollary 2. *Let the function f be in the class $\mathcal{R}_\alpha(\phi_M)$. Then*

$$|f'(z) - 1| \leqq \frac{M\sqrt{2}}{\sqrt{5+3\cos\alpha}} \qquad (z \in \mathbb{U})$$

and

$$\left|\frac{f(z)}{z} - 1\right| \leqq \frac{M\sqrt{2}}{2\sqrt{5+3\cos\alpha}} \qquad (z \in \mathbb{U};\ -\pi < \alpha < \pi).$$

6. The Fekete-Szegö Problem for the Function Class $\mathcal{R}_\alpha(\phi)$

The problem of finding sharp upper bounds for the coefficient functional $|a_3 - \mu a_2^2|$ for different subclasses of the normalized analytic function class $\mathcal{A}$ is known as the Fekete-Szegö problem. Over the years, this problem has been investigated by many works including (for example) [8], [13], [16], [18], [19], [28] and [29],.

In this section and in the next one, it will be assumed that the function $\phi(z)$ is a member of the class $\mathcal{M}$ and has positive real part in $\mathbb{U}$. Since $\phi \in \mathcal{M}$, its Taylor-Maclaurin series expansion is of the form:

$$\phi(z) = 1 + A_1 z + A_2 z^2 + \cdots \qquad (z \in \mathbb{U}). \tag{28}$$

Remark 3. In view of (22), if $f \in \mathcal{R}_\alpha(\phi)$ and $\alpha \in (-\pi, \pi)$, then $f'(z) \prec \phi(z)$, which when combined with $\Re\big(\phi(z)\big) > 0$ implies that $\Re\big(f'(z)\big) > 0$. When $\alpha \to \pi$, the class $\mathcal{R}_\pi(\phi)$ consists of all functions f satisfying the same subordination $f'(z) \prec \phi(z)$.

The well-known Noshiro-Warschawski theorem (see [9] and [10]) states that a function $f \in \mathcal{A}$ with $\Re\big(f'(z)\big) > 0$ is univalent in $\mathbb{U}$. Therefore, for all $\alpha \in (-\pi, \pi]$, $\mathcal{R}_\alpha(\phi)$ is a class of univalent functions, that is, $\mathcal{R}_\alpha(\phi)$ is a subclass of the normalized univalent function class $\mathcal{S}$.

Recently, Ali *et al.* [2] considered the Fekete-Szegö functional associated with the kth root transform for several subclasses of univalent functions. We recall here that, for a univalent function $f(z)$ of the form (1), the kth root transform is defined by

$$F(z) = \left[f(z^k)\right]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1} \qquad (z \in \mathbb{U}). \tag{29}$$

In view of Remark 3, the functions in the class $\mathcal{R}_\alpha(\phi)$ are univalent. Therefore, following the same method as in [2], we consider the problem of finding sharp upper bounds for the Fekete-Szegö coefficient functional associated with the kth root transform for functions in the class $\mathcal{R}_\alpha(\phi)$.

Lemma 2 below is needed to prove our main result.

Lemma 2 (see [14]). *Let the function $p(z)$ given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be in the class $\mathcal{P}$. Then, for any complex number s,

$$|p_2 - s p_1^2| \leqq 2 \max\{1, |2s - 1|\}. \tag{30}$$

The result is sharp for the function $p(z)$ given by

$$p(z) = \frac{1+z}{1-z} \qquad \text{or} \qquad p(z) = \frac{1+z^2}{1-z^2}.$$

Theorem 5. *Let $\alpha \in (-\pi, \pi]$ and let $\phi \in \mathcal{M}$ be given by* (28)*. Suppose also that the function f of the form* (1) *is a member of the class $\mathcal{R}_\alpha(\phi)$ and the function F is the kth root transform of f defined by* (29)*. Then, for any complex number μ,*

$$|b_{2k+1} - \mu b_{k+1}^2| \leqq \frac{|A_1|}{3k\sqrt{5 + 4\cos\alpha}} \max\left\{1, \left|\frac{A_2}{A_1} - (2\mu + k - 1)\frac{3(2 + e^{i\alpha})A_1}{2k(3 + e^{i\alpha})^2}\right|\right\}. \tag{31}$$

The result is sharp.

Proof. Let $f \in \mathcal{R}_\alpha(\phi)$. Then, clearly, there exists $\omega \in \mathcal{B}$ such that

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) = \phi\Big(\omega(z)\Big). \tag{32}$$

We now define

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + \cdots . \tag{33}$$

Since $\omega \in \mathcal{B}$, it follows that $p \in \mathcal{P}$. We thus find from (33) that

$$\omega(z) = \frac{1}{2} p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right) z^2 + \cdots . \tag{34}$$

Combining (28) and (34), we have

$$\phi\Big(\omega(z)\Big) = 1 + \frac{1}{2}A_1 p_1 z + \left(\frac{1}{4}A_2 p_1^2 + \frac{1}{2}A_1 (p_2 - \frac{1}{2}p_1^2)\right) z^2 + \cdots .$$

Equating the coefficients of z and z^2 on both sides of (32), we get

$$a_2 = \frac{A_1 p_1}{2(3+e^{i\alpha})} \tag{35}$$

and

$$a_3 = \frac{1}{3(2+e^{i\alpha})}\left(\frac{1}{4}A_2 p_1^2 + \frac{1}{2}A_1(p_2 - \frac{1}{2}p_1^2)\right). \tag{36}$$

For f given by (1), a computation shows that

$$F(z) = [f(z^k)]^{1/k} = z + \frac{1}{k}a_2 z^{k+1} + \left(\frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2\right) z^{2k+1} + \cdots . \tag{37}$$

The equations (29) and (37) lead us to

$$b_{k+1} = \frac{1}{k}a_2 \quad \text{and} \quad b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{38}$$

Substituting from (35) and (36) into (38), we obtain

$$b_{k+1} = \frac{A_1 p_1}{2k(3+e^{i\alpha})}$$

and

$$b_{2k+1} = \frac{1}{3k(2+e^{i\alpha})}\left(\frac{1}{4}A_2 p_1^2 + \frac{1}{2}A_1(p_2 - \frac{1}{2}p_1^2)\right) - \frac{(k-1)A_1^2 p_1^2}{8k^2(3+e^{i\alpha})^2},$$

so that

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{A_1}{6k(2+e^{i\alpha})}\left[p_2 - \frac{1}{2}\left(1 - \frac{A_2}{A_1} + (2\mu + k - 1)\frac{3(2+e^{i\alpha})A_1}{2k(3+e^{i\alpha})^2}\right)p_1^2\right].$$

Let

$$s = \frac{1}{2}\left(1 - \frac{A_2}{A_1} + (2\mu + k - 1)\frac{3(2+e^{i\alpha})A_1}{2k(3+e^{i\alpha})^2}\right).$$

The inequality (31) now follows as an application of Lemma 2.

It is easy to check that the result is sharp for the kth root transforms of the functions $f(z,\theta,1)$ and $f(z,\theta,0)$ defined by (16) with $\tau = 1$ and $\tau = 0$, respectively. This evidently completes our proof of Theorem 5. □

For $k = 1$, the kth root transform of f reduces to the function f itself. Corollary 3 below is an immediate consequence of Theorem 5.

Corollary 3. *Let $\alpha \in (-\pi, \pi]$ and let the function $\phi \in \mathcal{M}$ be given by* (28). *Suppose also that the function f of the form* (1) *is in the class $\mathcal{R}_\alpha(\phi)$. Then, for any complex number μ,*

$$|a_3 - \mu a_2^2| \leqq \frac{|A_1|}{3\sqrt{5+4\cos\alpha}} \max\left\{1, \left|\frac{A_2}{A_1} - \mu \frac{3(2+e^{i\alpha})A_1}{(3+e^{i\alpha})^2}\right|\right\}.$$

The result is sharp.

7. The Fekete-Szegö Problem for the Bi-Univalent Function Class $\mathcal{R}_{\Sigma;\alpha}(\phi)$

The famous Koebe one-quarter theorem (see [9]) ensures that the image of the open unit disk $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Consequently, every univalent function f has an inverse f^{-1} satisfying the following relationships:

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\;\; r_0(f) \geqq \frac{1}{4}\right).$$

In some cases, the inverse function f^{-1} can be extended to the whole disk $\mathbb{U}$, in which case f^{-1} is also univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$.

It is easy to check that a bi-univalent function f given by (1) has the inverse f^{-1} with the series expansion of the form:

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{39}$$

Lewin [15] considered the class Σ of bi-univalent functions and obtained the bound for the second coefficient. Netanyahu [17] and Brannan *et al.* (see [6] and [5]) subsequently studied similar problems in this direction.

The paper of Srivastava *et al.* [30] has revived the study of bi-univalent functions in recent years. It was followed by a great number of papers on this topic (see, for example, [3], [4], [20], [27], [33], [32] and [34]).

In view of Remark 3, the functions in the class $\mathcal{R}_\alpha(\phi)$ are univalent. This motivates the next definition of the class $\mathcal{R}_{\Sigma;\alpha}(\phi)$.

Definition 2. A function $f \in \Sigma$ is said to be in the class $\mathcal{R}_{\Sigma;\alpha}(\phi)$ if the following subordination relationships hold true:

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) \prec \phi(z) \qquad \text{and} \qquad g'(w) + \frac{1+e^{i\alpha}}{2} w g''(w) \prec \phi(w), \tag{40}$$

where

$$g(w) = f^{-1}(w).$$

We find from Definition 2 that, if $f \in \mathcal{R}_{\Sigma;\alpha}(\phi)$, then both f and $g = f^{-1}$ are univalent in $\mathbb{U}$. For this reason we can consider their corresponding kth root transforms

$$F(z) = [f(z^k)]^{1/k}$$

given by (29) and

$$G(w) = [g(w^k)]^{1/k} = w + \sum_{n=1}^{\infty} d_{kn+1} w^{kn+1} \qquad \Big(g(w) = f^{-1}(w)\Big). \tag{41}$$

In this section we derive upper bounds for the Fekete-Szegö functional associated with the kth root transform of functions in the class $\mathcal{R}_{\Sigma;\alpha}(\phi)$.

Theorem 6. *Let $\alpha \in (-\pi, \pi]$ and let $\phi \in \mathcal{M}$ be given by* (28). *Suppose also that the function f of the form (1) is in the class $\mathcal{R}_{\Sigma;\alpha}(\phi)$ and F is the kth root transform of f defined by* (29). *Then, for any real number μ,*

$$\begin{aligned} &|b_{2k+1} - \mu b_{k+1}^2| \\ &\leqq \begin{cases} \dfrac{|A_1|}{3k\sqrt{5+4\cos\alpha}} & \left(\left|\frac{k+1}{2} - \mu\right| \leqq k\left|1 + \frac{A_1 - A_2}{3A_1^2}\frac{(3+e^{i\alpha})^2}{2+e^{i\alpha}}\right|\right) \\[2ex] \dfrac{|A_1|^3 \left|\frac{k+1}{2} - \mu\right|}{k^2 |3(2+e^{i\alpha})A_1^2 + (3+e^{i\alpha})^2(A_1 - A_2)|} & \left(\left|\frac{k+1}{2} - \mu\right| \geqq k\left|1 + \frac{A_1 - A_2}{3A_1^2}\frac{(3+e^{i\alpha})^2}{2+e^{i\alpha}}\right|\right) \end{cases} \end{aligned} \tag{42}$$

Proof. Let $f \in \mathcal{R}_{\Sigma;\alpha}(\phi)$. Then, in view of (40), we obtain

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) = \phi\big(u(z)\big) \tag{43}$$

and

$$g'(w) + \frac{1+e^{i\alpha}}{2} w g''(w) = \phi\big(v(w)\big), \tag{44}$$

where $u, v \in \mathcal{B}$. Suppose that

$$g(w) = w + \sum_{n=2}^{\infty} c_n w^n. \tag{45}$$

Define

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U})$$

and

$$q(z) = \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

As in the proof of Theorem 5, we have $p, q \in \mathcal{P}$ and

$$\phi\big(u(z)\big) = 1 + \frac{1}{2}A_1 p_1 z + \left(\frac{1}{4}A_2 p_1^2 + \frac{1}{2}A_1(p_2 - \frac{1}{2}p_1^2)\right) z^2 + \cdots$$

and

$$\phi\big(v(z)\big) = 1 + \frac{1}{2}A_1 q_1 z + \left(\frac{1}{4}A_2 q_1^2 + \frac{1}{2}A_1(q_2 - \frac{1}{2}q_1^2)\right) z^2 + \cdots .$$

It follows from (43) that

$$a_2 = \frac{A_1 p_1}{2(3+e^{i\alpha})} \qquad \text{and} \qquad a_3 = \frac{1}{3(2+e^{i\alpha})}\left(\frac{1}{4}A_2 p_1^2 + \frac{1}{2}A_1(p_2 - \frac{1}{2}p_1^2)\right). \tag{46}$$

Moreover, the equality (44) in conjunction with (45) yields

$$c_2 = \frac{A_1 q_1}{2(3+e^{i\alpha})} \qquad \text{and} \qquad c_3 = \frac{1}{3(2+e^{i\alpha})}\left(\frac{1}{4}A_2 q_1^2 + \frac{1}{2}A_1(q_2 - \frac{1}{2}q_1^2)\right). \tag{47}$$

Since

$$G(w) = [g(w^k)]^{1/k} = w + \frac{1}{k}c_2 w^{k+1} + \left(\frac{1}{k}c_3 - \frac{1}{2}\frac{k-1}{k^2}c_2^2\right)w^{2k+1} + \cdots \tag{48}$$

it follows from (41) and (45) that

$$d_{k+1} = \frac{1}{k}c_2 \qquad \text{and} \qquad d_{2k+1} = \frac{1}{k}c_3 - \frac{1}{2}\frac{k-1}{k^2}c_2^2. \tag{49}$$

On the other hand, from (39) and (45) we get

$$c_2 = -a_2 \qquad \text{and} \qquad c_3 = 2a_2^2 - a_3. \tag{50}$$

The equalities (49) and (50) give

$$d_{k+1} = -\frac{1}{k}a_2 \qquad \text{and} \qquad d_{2k+1} = \frac{1}{k}(2a_2^2 - a_3) - \frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{51}$$

Furthermore, from (38) and (51), we have

$$d_{k+1} = -b_{k+1} \qquad \text{and} \qquad d_{2k+1} = (k+1)b_{k+1}^2 - b_{2k+1}. \tag{52}$$

Combining the equalities (38), (46), (47), (49) and (52), and after some simple calculations, we obtain

$$(3+e^{i\alpha})kb_{k+1} = \frac{1}{2}A_1 p_1, \tag{53}$$

$$3(2+e^{i\alpha})kb_{2k+1} = \frac{1}{4}A_2 p_1^2 + \frac{1}{2}A_1(p_2 - \frac{1}{2}p_1^2) - \frac{k-1}{8k}\frac{3(2+e^{i\alpha})}{(3+e^{i\alpha})^2}A_1^2 p_1^2, \tag{54}$$

$$-(3+e^{i\alpha})kb_{k+1} = \frac{1}{2}A_1 q_1 \tag{55}$$

and

$$3(2+e^{i\alpha})k[(k+1)b_{k+1}^2 - b_{2k+1}] = \frac{1}{4}A_2 q_1^2 + \frac{1}{2}A_1(q_2 - \frac{1}{2}q_1^2) - \frac{k-1}{8k}\frac{3(2+e^{i\alpha})}{(3+e^{i\alpha})^2}A_1^2 q_1^2. \tag{56}$$

Now, in order to prove the inequality (42), we apply the same technique as in [35]. Indeed, from (53) and (55), we get

$$p_1 = -q_1. \tag{57}$$

Substracting (56) from (54) and using (57), we have

$$b_{2k+1} = \frac{k+1}{2}b_{k+1}^2 + \frac{A_1(p_2 - q_2)}{12k(2+e^{i\alpha})}. \tag{58}$$

Moreover, the sum between (54) and (56) gives

$$3(2+e^{i\alpha})k(k+1)b_{k+1}^2 = \frac{1}{2}A_1(p_2+q_2) + \frac{1}{2}(A_2 - A_1)p_1^2 - \frac{k-1}{4k}\frac{3(2+e^{i\alpha})}{(3+e^{i\alpha})^2}A_1^2 p_1^2,$$

which, in conjunction with (53), yields

$$b_{k+1}^2 = \frac{A_1^3(p_2+q_2)}{4k^2[3(2+e^{i\alpha})A_1^2+(3+e^{i\alpha})^2(A_1-A_2)]}. \tag{59}$$

On the other hand, from (58) and (59), we obtain

$$b_{2k+1}-\mu b_{k+1}^2 = \frac{A_1}{12k(2+e^{i\alpha})}\left[p_2(h(t)+1)+q_2(h(t)-1)\right],$$

where

$$h(t) = \frac{3A_1^2(2+e^{i\alpha})\left(\frac{k+1}{2}-\mu\right)}{k[3(2+e^{i\alpha})A_1^2+(3+e^{i\alpha})^2(A_1-A_2)]}.$$

Since the functions p and q are in the class $\mathcal{P}$, it follows that (see [9])

$$|p_2| \leqq 2 \qquad \text{and} \qquad |q_2| \leqq 2.$$

Therefore, we have

$$|b_{2k+1}-\mu b_{k+1}^2| \leqq \begin{cases} \dfrac{|A_1|}{3k|2+e^{i\alpha}|} & (|h(t)| \leqq 1) \\[2ex] \dfrac{|A_1||h(t)|}{3k|2+e^{i\alpha}|} & (|h(t)| \geqq 1), \end{cases}$$

which completes the proof of Theorem 6. □

Since, for $k=1$, the kth root transform reduces to the function itself, the next result is an immediate consequence of Theorem 6.

Corollary 4. *Let $\alpha \in (-\pi,\pi]$ and let $\phi \in \mathcal{M}$ be given by* (28). *Suppose also that the function f of the form* (1) *belongs to the class $\mathcal{R}_{\Sigma;\alpha}(\phi)$. Then, for any real number μ,*

$$|a_3-\mu a_2^2| \leqq \begin{cases} \dfrac{|A_1|}{3\sqrt{5+4\cos\alpha}} & \left(|1-\mu| \leqq \left|1+\frac{A_1-A_2}{3A_1^2}\,\frac{(3+e^{i\alpha})^2}{2+e^{i\alpha}}\right|\right) \\[2ex] \dfrac{|A_1|^3|1-\mu|}{|3(2+e^{i\alpha})A_1^2+(3+e^{i\alpha})^2(A_1-A_2)|} & \left(|1-\mu| \geqq \left|1+\frac{A_1-A_2}{3A_1^2}\,\frac{(3+e^{i\alpha})^2}{2+e^{i\alpha}}\right|\right) \end{cases} \tag{60}$$

Finally, when $\alpha \to \pi$, the inequality (60) reduces to a result obtained by Zaprawa [35].

8. Concluding Remarks and Observations

In our present investigation, we have successfully applied the principle of differential subordination between analytic functions. Indeed, for $\alpha \in (\pi,\pi]$, we have considered a certain function class $\mathcal{R}_\alpha(\phi)$ of all *normalized* analytic functions in the open unit disk $\mathbb{U}$, which satisfy the following differential subordination:

$$f'(z)+\frac{1}{2}\left(1+e^{i\alpha}\right)zf''(z) \prec \phi(z) \qquad (z\in\mathbb{U}),$$

where the function $\phi(z)$ is analytic in $\mathbb{U}$ such that $\phi(0)=1$. In particular, we have investigated various integral and convolution characterizations, coefficient estimates and differential subordination results for functions belonging to the class $\mathcal{R}_\alpha(\phi)$. We have also derived the Fekete-Szegö coefficient functional associated with the kth root transform $[f(z^k)]^{1/k}$ of functions in $\mathcal{R}_\alpha(\phi)$. Furthermore, we have considered a similar problem for a corresponding class $\mathcal{R}_{\Sigma;\alpha}(\phi)$ of bi-univalent functions. We have pointed out relevant connections of the results presented here with previous known results.

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