Fixed Point Theorems of Soft Contractive Mappings

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Abstract. The first aim of this paper is to examine some important properties of soft metric spaces. Second is to introduce soft continuous mappings and investigate properties of soft continuous mappings. Third, to prove some fixed point theorems of soft contractive mappings on soft metric spaces.

1. Introduction

In the year 1999, Molodtsov ([13]) initiated a novel concept of soft set theory as a new mathematical tool for dealing with uncertainties. A soft set is a collection of approximate descriptions of an object. Soft systems provide a very general framework with the involvement of parameters. Since soft set theory has a rich potential, applications of soft set theory in other disciplines and real life problems are progressing rapidly.

Maji et al. ([10, 11]) worked on soft set theory and presented an application of soft sets in decision making problems. Chen ([3]) introduced a new definition of soft set parametrization reduction and a comparison of it with attribute reduction in rough set theory, Ali et all.([1]) gave some new operations in soft set theory.

Shabir and Naz ([14]) presented soft topological spaces and investigated some properties of soft topological spaces. Later, many researches about soft topological spaces were studied in ([7, 8, 12]). In these studies, the concept of soft point is expressed by different approaches. In the study we use the concept of soft point which was given in ([2, 5]).

It is known that there are many generalizations of metric spaces: Menger spaces, fuzzy metric spaces, generalized metric spaces, abstract (cone) metric spaces or K-metric and K-normed spaces etc. Recently Das and Samanta ([4, 5]) introduced a different notion of soft metric space by using a different concept of soft point and investigated some important properties of these spaces.

A number of authors have defined contractive type mapping on a complete metric space which are generalizations of the well-known Banach contraction, and which have the property that each such mapping has a unique fixed point ([9, 15]). The fixed point can always be found by using Picard iteration, beginning with some initial choice .

In the present study, we first give, as preliminaries, some well-known results in soft set theory. Firstly, we examine some important properties of soft metric spaces defined in ([5]). Secondly, we investigate

properties of soft continuous mappings on soft metric spaces. Finally, we introduced soft contractive mappings on soft metric spaces and prove some fixed point theorems of soft contractive mappings.

2. Preliminaries

Definition 2.1. ([13]) Let X be an initial universe set and E be a set of parameters. A pair (F, E) is called a soft set over X if and only if F is a mapping from E into the set of all subsets of the set X, i.e. , $F : E \to P(X)$ where $P(X)$ is the power set of X.

Definition 2.2. ([10]) The intersection of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cap B$ and $\forall e \in C$, $H(e) = F(e) \cap G(e)$. This is denoted by $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Definition 2.3. ([10]) The union of two soft sets (F, A) and (G, B) over X is the soft set, where $C = A \cup B$ and $\forall e \in C$,

$$H(e) = \begin{cases} F(e) & \text{,if } e \in A - B \\ G(e) & \text{,if } e \in B - A \\ F(e) \cup G(e) & \text{,if } e \in A \cap B \end{cases}$$

This relationship is denoted by $(F, A) \tilde{\cup} (G, B) = (H, C)$.

Definition 2.4. ([10]) A soft set (F, A) over X is said to be a null soft set denoted by $\emptyset$ if for all $e \in A$, $F(e) = \emptyset$ (null set).

Definition 2.5. ([10]) A soft set (F, A) over X is said to be an absolute soft set denoted by $\tilde{X}$ if for all $e \in A$, $F(e) = X$.

Definition 2.6. ([10]) The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) \backslash (G, E)$, is defined as $H(e) = F(e) \backslash G(e)$ for all $e \in E$.

Definition 2.7. ([10]) The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to P(X)$ is a mapping given by $F^c(e) = X - F(e)$,for all $e \in A$.

Definition 2.8. ([6]) Let $\mathbb{R}$ be the set of real numbers and $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. If a real soft set is a singleton soft set, it will be called a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ etc. $\tilde{0}$ and $\tilde{1}$ are the soft real numbers where $\tilde{0}(e) = 0$, $\tilde{1}(e) = 1$for all $e \in E$ respectively.

Definition 2.9. ([6]) Let $\tilde{r}, \tilde{s}$ be two soft real numbers. Then the following statements hold:

i. $\tilde{r} \tilde{\leq} \tilde{s}$ if $\tilde{r}(e) \tilde{\leq} \tilde{s}(e)$ for all $e \in E$;
ii. $\tilde{r} \tilde{\geq} \tilde{s}$ if $\tilde{r}(e) \tilde{\geq} \tilde{s}(e)$ for all $e \in E$;
iii. $\tilde{r} \tilde{<} \tilde{s}$ if $\tilde{r}(e) \tilde{<} \tilde{s}(e)$ for all $e \in E$;
iv. $\tilde{r} \tilde{>} \tilde{s}$ if $\tilde{r}(e) \tilde{>} \tilde{s}(e)$ for all $e \in E$.

Definition 2.10. ([2, 5]) A soft set (F, E) over X is said to be a soft point, denoted by $\tilde{x}_e$, if for the element $e \in E$, $F(e) = \{x\}$ and $F(\acute{e}) = \emptyset$ for all $\acute{e} \in E/\{e\}$.

Definition 2.11. ([2, 5]) Two soft points $\tilde{x}_e, \tilde{y}_{\acute{e}}$ are said to be equal if $e = \acute{e}$ and $x = y$. Thus $\tilde{x}_e \neq \tilde{y}_{\acute{e}} \Leftrightarrow x \neq y$ or $e \neq \acute{e}$.

Proposition 2.12. ([2]) *Every soft set can be expressed as a union of all soft points belonging to it as* $(F, E) = \underset{\tilde{x}_e \in (F,E)}{\cup} \tilde{x}_e$. *Conversely, any set of soft points can be considered as a soft set.*

Definition 2.13. ([14]) Let τ be a collection of soft sets over X. Then τ is said to be a soft topology on X if

(1) $\emptyset, \tilde{X}$ belong to τ
(2) The union of any number of soft sets in τ belongs to τ
(3) The intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X.

Definition 2.14. ([8]) Let (X, τ, E) be a soft topological space over X. Then soft interior of (F, E), denoted by $(F, E)^\circ$, is defined as the union of all soft open sets contained in (F, E).

Definition 2.15. ([8]) Let (X, τ, E) be a soft topological space over X. Then soft closure of (F, E), denoted by $\overline{(F, E)}$, is defined as the intersection of all soft closed super sets of (F, E).

Definition 2.16. ([8]) Let (X, τ, E) be a soft topological space over X. Then soft boundary of soft set (F, E) over X ,denoted by $\partial(F, E)$, is defined as $\partial(F, E) = \overline{(F, E)} \tilde{\cap} \overline{(F, E)^c}$.

Definition 2.17. ([7]) Let (X, τ, E) and $(Y, \acute{\tau}, E)$ be two soft topological spaces, $f : (X, \tau, E) \rightarrow (Y, \acute{\tau}, E)$ be a mapping. For each soft neighborhood (H, E) of $(\widetilde{f(x_e)}, E)$, if there exists a soft neighborhood (F, E) of $(\tilde{x}_e, E)$ such that $f((F, E)) \subset (H, E)$, then f is said to be soft continuous mapping at $(\tilde{x}_e, E)$.

If f is soft continuous mapping for all $(\tilde{x}_e, E)$, then f is called soft continuous mapping.

Let $SP(\tilde{X})$ be the collection of all soft points of X and $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers.

Definition 2.18. ([5]) A mapping $\tilde{d} : SP(\tilde{X}) \times SP(\tilde{X}) \rightarrow \mathbb{R}(E)^*$ is said to be a soft metric on the soft set $\tilde{X}$ if $\tilde{d}$ satisfies the following conditions:

(M1) $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) \tilde{\geq} \bar{0}$ for all $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} \tilde{X}$,
(M2) $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = \bar{0}$ if and only if $\tilde{x}_e = \tilde{y}_{e'}$,
(M3) $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = \tilde{d}(\tilde{y}_{e'}, \tilde{x}_e)$ for all $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} \tilde{X}$,
(M4) For all $\tilde{x}_e, \tilde{y}_{e'}, \tilde{z}_{e''} \tilde{\in} \tilde{X}$, $\tilde{d}(\tilde{x}_e, \tilde{z}_{e''}) \tilde{\leq} \tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) + \tilde{d}(\tilde{y}_{e'}, \tilde{z}_{e''})$

The soft set $\tilde{X}$ with a soft metric $\tilde{d}$ on $\tilde{X}$ is called a soft metric space and denoted by $(\tilde{X}, \tilde{d}, E)$.

Definition 2.19. ([5]) Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space and $\tilde{\varepsilon}$ be a non-negative soft real number. $B(\tilde{x}_e, \tilde{\varepsilon}) = \left\{ \tilde{y}_{e'} \tilde{\in} \tilde{X} : \tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) \tilde{<} \tilde{\varepsilon} \right\} \subset SP(\tilde{X})$ is called the soft open ball with center $\tilde{x}_e$ and radius $\tilde{\varepsilon}$ and $B[\tilde{x}_e, \tilde{\varepsilon}] = \left\{ \tilde{y}_{e'} \tilde{\in} \tilde{X} : \tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) \tilde{\leq} \tilde{\varepsilon} \right\} \subset SP(\tilde{X})$ is called the soft closed ball with center $\tilde{x}_e$ and radius $\tilde{\varepsilon}$.

Definition 2.20. ([5]) Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space and (F, E) be a non-null soft subset of $\tilde{X}$ in $(\tilde{X}, \tilde{d}, E)$. Then (F, E) is said to be a soft open set in $\tilde{X}$ with respect to $\tilde{d}$ if and only if all soft points of (F, E) is soft interior points of (F, E).

Definition 2.21. ([5]) Let $\left\{ \tilde{x}^n_{e_n} \right\}$ be a sequence of soft points in a soft metric space $(\tilde{X}, \tilde{d}, E)$. Then the sequence $\left\{ \tilde{x}^n_{e_n} \right\}$ is said to be convergent in $(\tilde{X}, \tilde{d}, E)$ if there is a soft point $\tilde{x}^0_{e_0} \tilde{\in} \tilde{X}$ such that $\tilde{d}(\tilde{x}^n_{e_n}, \tilde{x}^0_{e_0}) \rightarrow \bar{0}$ as $n \rightarrow \infty$. This means for every $\tilde{\varepsilon} \tilde{>} \tilde{0}$, chosen arbitrarily, there is a natural number $N = N(\tilde{\varepsilon})$ such that $\tilde{0} \tilde{\leq} \tilde{d}(\tilde{x}^n_{e_n}, \tilde{x}^0_{e_0}) \tilde{<} \tilde{\varepsilon}$, whenever $n > N$.

Theorem 2.22. ([5]) *Limit of a sequence in a soft metric space, if exist, is unique.*

Definition 2.23. ([5]) (Cauchy Sequence) The sequence $\left\{ \tilde{x}^n_{e_n} \right\}$ of soft points in $(\tilde{X}, \tilde{d}, E)$ is called a Cauchy sequence in $\tilde{X}$ if corresponding to every $\tilde{\varepsilon} \tilde{>} \tilde{0}$, there is a $m \in N$ such that $\tilde{d}(\tilde{x}^i_{e_i}, \tilde{y}^j_{e_j}) \tilde{\leq} \tilde{\varepsilon}$, for all $i, j \geq m$ i.e. $\tilde{d}(\tilde{x}^i_{e_i}, \tilde{y}^j_{e_j}) \rightarrow \tilde{0}$ as $i, j \rightarrow \infty$.

Definition 2.24. ([5]) (Complete Metric Space) The soft metric space $(\tilde{X}, \tilde{d}, E)$ is called complete if every Cauchy Sequence in $\tilde{X}$ converges to some point of $\tilde{X}$. The soft metric space $(\tilde{X}, \tilde{d}, E)$ is called incomplete if it is not complete.

3. Soft Topology Generated by Soft Metric

In this section, we study some important results of soft metric spaces.

Let $\tilde{X}$ be the absolute soft set and E be a parameter set and $\tilde{X}_e$ be a family of soft points i.e. $\tilde{X}_e = \{\tilde{x}_e : x \in X\}$ for $\forall e \in E$. Then there exists a bijective mapping between the soft set $\tilde{X}_e$ and the set X. If $e \neq e' \in E$, then $\tilde{X}_e \cap \tilde{X}_{e'} = \emptyset$ and $SP(\tilde{X}) = \underset{e \in E}{\cup} \tilde{X}_e$.

Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. It is clear that $(\tilde{X}_e, \tilde{d}_e, \{e\})$ is a soft metric space for $e \in E$. Then by using the soft metric $\tilde{d}_e$, we define a metric on X as $d_e(x, y) = \tilde{d}_e(\tilde{x}_e, \tilde{y}_e)$. Note that $e \neq e' \in E$, then d_e and $d_{e'}$ on X are generally different metrics.

Proposition 3.1. *Every soft metric space is a family of parameterized metric spaces.*

Proof. The proof is obvious. □

The converse of Proposition 3.1 may not be true in general. This is shown by the following example.

Example 3.2. Let $E = \mathbb{R}$ be a parameter set and (X, d) be a metric space. We define the function $\tilde{d} : SP(\tilde{X}) \times SP(\tilde{X}) \to \mathbb{R}(E)$ by $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = d(x, y)^{1+|e-e'|}$ for all $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SP(\tilde{X})$. Then for all $e \in E$, d_e is a metric on X. If $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = \tilde{0}$, then this does not always mean that $\tilde{x}_e = \tilde{y}_{e'}$, so $\tilde{d}$ is not a soft metric on $\tilde{X}$.

Proposition 3.3. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space and $\tau_{\tilde{d}}$ be a soft topology generated by the soft metric $\tilde{d}$. Then for every $e \in E$, the topology $(\tau_{\tilde{d}})_e$ on X is the topology τ_{d_e} generated by the metric d_e on X.*

Proof. The proof is obvious. □

Lemma 3.4. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. Then the following expressions are true:*

(i) $\tilde{x}_e \tilde{\in} \overline{(F, E)} \Leftrightarrow \tilde{d}(\tilde{x}_e, (F, E)) = \tilde{0}$;
(ii) $\tilde{x}_e \tilde{\in} (F, E)^\circ \Leftrightarrow \tilde{d}(\tilde{x}_e, (F, E)^c) \tilde{>} \tilde{0}$;
(iii) $\tilde{x}_e \tilde{\in} \partial(F, E) \Leftrightarrow \tilde{d}(\tilde{x}_e, (F, E)) = \tilde{d}(\tilde{x}_e, (F, E)^c) = \tilde{0}$.

Proof. The proof is clear. □

Note that if (F, E) is a soft closed set in the soft metric space $(\tilde{X}, \tilde{d}, E)$ and $\tilde{x}_e \tilde{\notin} (F, E)$, then there exists a soft open ball $B(\tilde{x}_e, \tilde{\varepsilon})$ such that $B(\tilde{x}_e, \tilde{\varepsilon}) \tilde{\cap} (F, E) = \emptyset$.

Theorem 3.5. *Every soft metric space is a soft normal space.*

Proof. Let (F_1, E) and (F_2, E) be two disjoint soft closed sets in the soft metric space $(\tilde{X}, \tilde{d}, E)$. For every soft points $\tilde{x}_e \tilde{\in} (F_1, E)$ and $\tilde{y}_{e'} \tilde{\in} (F_2, E)$, we choose soft open balls $B(\tilde{x}_e, \tilde{\varepsilon}_{\tilde{x}_e})$ and $B(\tilde{y}_{e'}, \tilde{\delta}_{\tilde{y}_{e'}})$ such that $B(\tilde{x}_e, \tilde{\varepsilon}_{\tilde{x}_e}) \tilde{\cap} (F_2, E) = \emptyset$ and $B(\tilde{y}_{e'}, \tilde{\delta}_{\tilde{y}_{e'}}) \tilde{\cap} (F_1, E) = \emptyset$. Thus, we have $(F_1, E) \tilde{\subseteq} \tilde{\cup} B(\tilde{x}_e, (\tilde{\varepsilon}/3)_{\tilde{x}_e}) = (U, E)$ and $(F_2, E) \tilde{\subseteq} \tilde{\cup} B(\tilde{y}_{e'}, (\tilde{\delta}/3)_{\tilde{y}_{e'}}) = (V, E)$. We want to show that $(U, E) \tilde{\cap} (V, E) = \emptyset$.

Assume that $(U, E) \tilde{\cap} (V, E) \neq \emptyset$. Then there exists a soft point $\tilde{z}_{e''}$ such that $\tilde{z}_{e''} \tilde{\in} (U, E) \tilde{\cap} (V, E)$. Therefore, there exist soft open balls $B(\tilde{x}_e, (\tilde{\varepsilon}/3)_{\tilde{x}_e})$ and $B(\tilde{y}_{e'}, (\tilde{\delta}/3)_{\tilde{y}_{e'}})$ such that $\tilde{z}_{e''} \tilde{\in} B(\tilde{x}_e, (\tilde{\varepsilon}/3)_{\tilde{x}_e})$ and $\tilde{z}_{e''} \tilde{\in} B(\tilde{y}_{e'}, (\tilde{\delta}/3)_{\tilde{y}_{e'}})$.

Here, we have $\tilde{d}(\tilde{x}_e, \tilde{z}_{e''}) \tilde{<} (\tilde{\varepsilon}/3)_{\tilde{x}_e}$ and $\tilde{d}(\tilde{y}_{e'}, \tilde{z}_{e''}) \tilde{<} (\tilde{\delta}/3)_{\tilde{y}_{e'}}$. If we get $\max \left\{ (\tilde{\varepsilon}/3)_{\tilde{x}_e}, (\tilde{\delta}/3)_{\tilde{y}_{e'}} \right\} = (\tilde{\varepsilon}/3)_{\tilde{x}_e}$, then we have $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) \tilde{\leq} \tilde{d}(\tilde{x}_e, \tilde{z}_{e''}) + \tilde{d}(\tilde{z}_{e''}, \tilde{y}_{e'}) \tilde{<} (\tilde{\varepsilon}/3)_{\tilde{x}_e} + (\tilde{\delta}/3)_{\tilde{y}_{e'}} \tilde{\leq} \tilde{\varepsilon}_{\tilde{x}_e}$ and so $\tilde{y}_{e'} \tilde{\in} B(\tilde{x}_e, \tilde{\varepsilon}_{\tilde{x}_e})$ and which contradicts with our assumption. Therefore, $(U, E) \tilde{\cap} (V, E) = \emptyset$. □

4. Soft Contractive Mappings

In this section we shall prove some fixed point theorems of soft contractive mappings.

Let $(\tilde{X}, \tilde{d}, E)$ and $(\tilde{Y}, \tilde{\rho}, \acute{E})$ be two soft metric spaces. The mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \rightarrow (\tilde{Y}, \tilde{\rho}, \acute{E})$ is a soft mapping, where $f : X \rightarrow Y, \varphi : E \rightarrow \acute{E}$ are two mappings.

Proposition 4.1. *For each soft point $\tilde{x}_e \tilde{\in} SP(\tilde{X})$, $(f, \varphi)(\tilde{x}_e)$ is a soft point in $\tilde{Y}$.*

Proof. Let $\tilde{x}_e \tilde{\in} SP(\tilde{X})$ be a soft point. Then

$$(f, \varphi)(\tilde{x}_e)(e') = \underset{e \in \varphi^{-1}(e')}{\cup} f(x_e(e)) = (f(x))_{\varphi(e)}.$$

$\square$

Definition 4.2. Let $(\tilde{X}, \tilde{d}, E)$ and $(\tilde{Y}, \tilde{\rho}, E')$ be two soft metric spaces and $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \rightarrow (\tilde{Y}, \tilde{\rho}, E')$ be a soft mapping. The mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \rightarrow (\tilde{Y}, \tilde{\rho}, E')$ is a soft continuous mapping at the soft point $\tilde{x}_e \tilde{\in} SP(\tilde{X})$ if for every soft open ball $B((f, \varphi)(\tilde{x}_e), \tilde{\varepsilon})$ of $(\tilde{Y}, \tilde{\rho}, \acute{E})$, there exists a soft open ball $B(\tilde{x}_e, \tilde{\delta})$ of $(\tilde{X}, \tilde{d}, E)$ such that $f(B(\tilde{x}_e, \tilde{\delta})) \subseteq B((f, \varphi)(\tilde{x}_e), \tilde{\varepsilon})$.

If (f, φ) is a soft continuous mapping at every soft point $\tilde{x}_e$ of $(\tilde{X}, \tilde{d}, E)$, then it is said to be soft continuous mapping on $(\tilde{X}, \tilde{d}, E)$.

Now, this definition can be expressed using $\tilde{\varepsilon} - \tilde{\delta}$ as follows:

The mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \rightarrow (\tilde{Y}, \tilde{\rho}, E')$ is said to be a soft continuous mapping at the soft point $\tilde{x}_e \tilde{\in} SP(\tilde{X})$ if for every $\tilde{\varepsilon} \tilde{>} 0$ there exists a $\tilde{\delta} \tilde{>} 0$ such that $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) \tilde{<} \tilde{\delta}$ implies that $\tilde{\rho}((f, \varphi)(\tilde{x}_e), (f, \varphi)(\tilde{y}_{e'})) \tilde{<} \tilde{\varepsilon}$.

Theorem 4.3. *Let $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \rightarrow (\tilde{Y}, \tilde{\rho}, E')$ be a soft mapping. Then the following conditions are equivalent:*

(1) $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \rightarrow (\tilde{Y}, \tilde{\rho}, E')$ is a soft continuous mapping,

(2) For each soft open set (G, E') over $\tilde{Y}$, $(f, \varphi)^{-1}((G, E'))$ is a soft open set over $\tilde{X}$,

(3) For each soft closed set (H, E') over $\tilde{Y}$, $(f, \varphi)^{-1}((H, E'))$ is a soft closed set over $\tilde{X}$,

(4) For each soft set (F, E) over $\tilde{X}$, $(f, \varphi)(\overline{(F, E)}) \subset \overline{((f, \varphi)((F, E)))}$ is a soft closed set over $\tilde{X}$,

(5) For each soft set over (G, E') over $\tilde{Y}$, $\overline{((f, \varphi)^{-1}(G, E'))} \subset (f, \varphi)^{-1}(\overline{(G, E')})$,

(6) For each soft set over (G, E') over $\tilde{Y}$, $(f, \varphi)^{-1}((G, E')^{\circ}) \subset ((f, \varphi)^{-1}(G, E'))^{\circ}$.

Proof. $(1) \Rightarrow (2)$ Let (f, φ) be a soft continuous mapping and (G, E') be a soft open set on $\tilde{Y}$. Consider the soft set $(f, \varphi)^{-1}(G, E')$. If $(f, \varphi)^{-1}(G, E') = \emptyset$, then the proof is completed. Let $(f, \varphi)^{-1}(G, E') \neq \emptyset$. In this case there exists at least one soft point $\tilde{x}_e \tilde{\in} (f, \varphi)^{-1}(G, E')$. Then we have $(f, \varphi)(\tilde{x}_e) \tilde{\in} (G, E')$. Since (G, E') is a soft open set, there exists a soft open ball $B((f, \varphi)(\tilde{x}_e), \tilde{\varepsilon})$ such that $B((f, \varphi)(\tilde{x}_e), \tilde{\varepsilon}) \subset (G, E')$ holds. Also since (f, φ) is a soft continuous mapping, there exists a soft open ball $B(\tilde{x}_e, \tilde{\delta})$ such that $(f, \varphi)(B(\tilde{x}_e, \tilde{\delta})) \subset B((f, \varphi)(\tilde{x}_e), \tilde{\varepsilon})$. Thus,

$$B(\tilde{x}_e, \tilde{\delta}) \subset (f, \varphi)^{-1}(f, \varphi)(B(\tilde{x}_e, \tilde{\delta})) \subset (f, \varphi)^{-1}B((f, \varphi)(\tilde{x}_e), \tilde{\varepsilon}) \subset (f, \varphi)^{-1}(G, E')$$

Consequently, $(f, \varphi)^{-1}(G, E')$ is a soft open set.

$(2) \Rightarrow (3)$ Let (H, E') be any soft closed set over $\tilde{Y}$. Then $(H, E')^c$ is a soft open set. From (2), we have $((f, \varphi)^{-1}((H, E')))^c$ is a soft open set over $\tilde{X}$. Thus $(f, \varphi)^{-1}((H, E'))$ is a soft closed set.

$(3) \Rightarrow (4)$ Let (F, E) be a soft set over $\tilde{X}$. Since

$$(F, E) \subset (f, \varphi)^{-1}((f, \varphi)((F, E))) \text{ and } (f, \varphi)((F, E)) \subset \overline{((f, \varphi)((F, E)))},$$

we have $(F, E) \subset (f, \varphi)^{-1}((f, \varphi)(F, E)) \subset (f, \varphi)^{-1}(\overline{(f, \varphi)(F, E)})$. By part (3), since $(f, \varphi)^{-1}(\overline{(f, \varphi)(F, E)})$ is a soft closed set over $\tilde{X}$, $\overline{(F, E)} \subset (f, \varphi)^{-1}(\overline{(f, \varphi)(F, E)})$. Thus $(f, \varphi)(\overline{(F, E)}) \subset (f, \varphi)((f, \varphi)^{-1}(\overline{(f, \varphi)(F, E)}))$ is obtained.

(4) $\Rightarrow$ (5) Let (G, E') be a soft set over $\tilde{Y}$ and $(f, \varphi)^{-1}(G, E') = (F, E)$. By part (4), we have $(f, \varphi)\left(\overline{(F, E)}\right) = (f, \varphi)\left((f, \varphi)^{-1}(G, E')\right) \subset \overline{(f, \varphi)\left((f, \varphi)^{-1}(G, E')\right)} \subset \overline{(G, E')}$. Then $\overline{\left((f, \varphi)^{-1}(G, E')\right)} = \overline{(F, E)} \subset (f, \varphi)^{-1}\left((f, \varphi)\overline{(F, E)}\right) \subset (f, \varphi)^{-1}\left(\overline{(G, E')}\right)$.

(5) $\Rightarrow$ (6) Let (G, E') be a soft set over $\tilde{Y}$. Substituting $(G, E')^c$ for condition in (5). Then $\overline{(f, \varphi)^{-1}\left((G, E')^c\right)} \subset (f, \varphi)^{-1}\left(\overline{(G, E')^c}\right)$. Since $(G, E')^\circ = \left(\overline{(G, E')^c}\right)^c$, then we have

$$
\begin{aligned}
(f, \varphi)^{-1}\left((G, E')^\circ\right) &= (f, \varphi)^{-1}\left(\left(\overline{(G, E')^c}\right)^c\right) \\
&= \left((f, \varphi)^{-1}\left(\overline{(G, E')^c}\right)\right)^c \subset \left(\overline{(f, \varphi)^{-1}\left((G, E')^c\right)}\right)^c \\
&= \left(\overline{\left((f, \varphi)^{-1}(G, E')\right)^c}\right)^c \\
&= \left((f, \varphi)^{-1}(G, E')\right)^\circ .
\end{aligned}
$$

(6) $\Rightarrow$ (1) Let (G, E') be a soft set over $\tilde{Y}$. Then since

$$
\left((f, \varphi)^{-1}(G, E')\right)^\circ \subset (f, \varphi)^{-1}(G, E') = (f, \varphi)^{-1}\left((G, E')^\circ\right) \subset \left((f, \varphi)^{-1}(G, E')\right)^\circ,
$$

$\left((f, \varphi)^{-1}(G, E')\right)^\circ = (f, \varphi)^{-1}(G, E')$ is obtained. This implies that $(f, \varphi)^{-1}(G, E')$ is a soft open set. $\square$

Definition 4.4. The soft mapping $(f, \varphi) : (\check{X}, \tilde{d}, E) \to (\check{Y}, \tilde{\rho}, E')$ is said to be soft sequentially continuous at the soft point $\tilde{x}_e \tilde{\in} SP(\check{X})$ iff for every sequence of soft points $\left\{\tilde{x}^n_{e_n}\right\}$ converging to the soft point $\tilde{x}_e$ in the metric space $(\check{X}, \tilde{d}, E)$, the sequence $(f, \varphi)\left(\left\{\tilde{x}^n_{e_n}\right\}\right)$ in $(\check{Y}, \tilde{\rho}, E')$ converges to a soft point $(f, \varphi)(\tilde{x}_e) \tilde{\in} SP(\check{Y})$.

Theorem 4.5. *Soft continuity is equivalent to soft sequential continuity in soft metric spaces.*

Proof. Let $(f, \varphi) : (\check{X}, \tilde{d}, E) \to (\check{Y}, \tilde{\rho}, E')$ be a soft continuous mapping and $\left\{\tilde{x}^n_{e_n}\right\}$ be any sequence of soft points converging to the soft point $\tilde{x}_e \tilde{\in} SP(\check{X})$. Let $B\left((f, \varphi)(\tilde{x}_e), \tilde{\varepsilon}\right)$ be a soft open ball in $(\check{Y}, \tilde{\rho}, E')$. By continuity of (f, φ) choose a soft open ball $B(\tilde{x}_e, \tilde{\delta})$ containing $\tilde{x}_e$ such that $(f, \varphi)\left(B(\tilde{x}_e, \tilde{\delta})\right) \tilde{\subseteq} B\left((f, \varphi)(\tilde{x}_e), \tilde{\varepsilon}\right)$. Since $\left\{\tilde{x}^n_{e_n}\right\}$ converges to $\tilde{x}_e$ there exists $n_0 \in \mathbb{N}$ such that $\left\{\tilde{x}^n_{e_n}\right\} \tilde{\in} B(\tilde{x}_e, \tilde{\delta})$ for all $n \geq n_0$. Therefore for all $n \geq n_0$ we have $(f, \varphi)\left(\left\{\tilde{x}^n_{e_n}\right\}\right) \tilde{\in} (f, \varphi)\left(B(\tilde{x}_e, \tilde{\delta})\right) \tilde{\subseteq} B\left((f, \varphi)(\tilde{x}_e), \tilde{\varepsilon}\right)$, as required.

Conversely, assume for contradiction that $(f, \varphi) : (\check{X}, \tilde{d}, E) \to (\check{Y}, \tilde{\rho}, E')$ is soft sequential continuous but not soft continuous mapping. Since (f, φ) is not soft continuous at the soft point $\tilde{x}_e$, there exists such that $\tilde{\varepsilon} \tilde{>} \tilde{0}$ for all $\tilde{\delta} \tilde{>} \tilde{0}$ there exists $\tilde{y}_{e'} \tilde{\in} SP(\check{X})$ such that $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) \tilde{<} \tilde{\delta}$ and $\tilde{\rho}((f, \varphi)(\tilde{x}_e), (f, \varphi)(\tilde{y}_{e'})) \tilde{>} \tilde{\varepsilon}_0$. For $n \geq 1 (n \in \mathbb{N})$, define $\tilde{\delta}_n = \frac{1}{n}$. For $n \geq 1$ we may choose $\left\{\tilde{y}^n_{e_n}\right\}$ in $(\check{X}, \tilde{d}, E)$ such that $\tilde{d}(\tilde{x}^n_{e_n}, \tilde{y}^n_{e_n}) \tilde{<} \tilde{\delta}_n$ and $\tilde{\rho}((f, \varphi)(\tilde{x}_e), (f, \varphi)(\tilde{y}_{e'})) \tilde{>} \tilde{\varepsilon}_0$. Therefore, by definition the sequence $\left\{\tilde{y}^n_{e_n}\right\} (n \geq 1)$ converges to $\tilde{x}_e$. However, by definition the sequence $\left\{(f, \varphi)\left(\tilde{y}^n_{e_n}\right)\right\} (n \geq 1)$ does not converge to $(f, \varphi)(\tilde{x}_e)$. That is, (f, φ) is not soft sequentially continuous at $\tilde{x}_e$. $\square$

Definition 4.6. Let $(\check{X}, \tilde{d}, E)$ be a soft metric space. A function $(f, \varphi) : (\check{X}, \tilde{d}, E) \to (\check{X}, \tilde{d}, E)$ is called a soft contraction mapping if there exists a soft real number $\tilde{\alpha} \in \mathbb{R}(E)$, $\tilde{0} \tilde{\leq} \tilde{\alpha} \tilde{<} \tilde{1}$ ($\mathbb{R}(E)$ denotes the soft real numbers set) such that for every soft points $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SP(\check{X})$ we have $\tilde{d}((f, \varphi)(\tilde{x}_e), (f, \varphi)(\tilde{y}_{e'})) \tilde{\leq} \tilde{\alpha} \tilde{d}(\tilde{x}_e, \tilde{y}_{e'})$.

Proposition 4.7. *Every soft contraction mapping is a soft continuous mapping.*

Proof. Let $\tilde{x}_e \tilde{\in} SP(\check{X})$ be any soft point and $\tilde{\varepsilon} \tilde{>} \tilde{0}$ be arbitrary. If we choose $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) \tilde{<} \tilde{\delta} \tilde{<} \tilde{\varepsilon}$, then since (f, φ) is a soft contraction mapping we have

$$
\tilde{d}((f, \varphi)(\tilde{x}_e), (f, \varphi)(\tilde{y}_{e'})) \tilde{\leq} \tilde{\alpha} \tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) \tilde{<} \tilde{\alpha} \tilde{\delta} \tilde{<} \tilde{\varepsilon}
$$

and so (f, φ) is a soft continuous mapping. $\square$

Theorem 4.8. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft complete metric space. If the mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ is a soft contraction mapping on a complete soft metric space, then there exists a unique soft point $\tilde{x}_e \tilde{\in} SP(\tilde{X})$ such that $(f, \varphi)(\tilde{x}_e) = \tilde{x}_e$.*

Proof. Let $\tilde{x}_e^0$ be any soft point in $SP(\tilde{X})$. Set

$$\tilde{x}_{e_1}^1 = (f, \varphi)(\tilde{x}_e^0) = \left(f(\tilde{x}_e^0)\right)_{\varphi(e)}, \tilde{x}_{e_2}^2 = (f, \varphi)(\tilde{x}_{e_1}^1) = \left(f^2(\tilde{x}_e^0)\right)_{\varphi^2(e)}, ..., \tilde{x}_{e_{n+1}}^{n+1} = \left((f, \varphi)(\tilde{x}_{e_n}^n)\right) = \left(f^{n+1}(\tilde{x}_e^0)\right)_{\varphi^{n+1}(e)}, ...$$

We have

$$\begin{aligned} \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_{e_n}^n) &= \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), (\tilde{x}_{e_{n-1}}^{n-1})\right) \\ &\tilde{\leq} \tilde{\alpha}.\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1}) \tilde{\leq} \tilde{\alpha}^2.\tilde{d}(\tilde{x}_{e_{n-1}}^{n-1}, \tilde{x}_{e_{n-2}}^{n-2}) \\ &\tilde{\leq} ...\tilde{\leq} \tilde{\alpha}^n.\tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0). \end{aligned}$$

So for $n > m$

$$\begin{aligned} \tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m) &\tilde{\leq} \tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1}) + \tilde{d}(\tilde{x}_{e_{n-1}}^{n-1}, \tilde{x}_{e_{n-2}}^{n-2}) + \cdots + \tilde{d}(\tilde{x}_{e_{m+1}}^{m+1}, \tilde{x}_{e_m}^m) \\ &\tilde{\leq} \left(\tilde{\alpha}^{n-1} + \tilde{\alpha}^{n-2} + \cdots + \tilde{\alpha}^m\right) \tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0) \\ &\tilde{\leq} \frac{\tilde{\alpha}^m}{\tilde{1} - \tilde{\alpha}} \tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0) \end{aligned}$$

we get $\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m) \tilde{\leq} \frac{\tilde{\alpha}^m}{\tilde{1}-\tilde{\alpha}} \tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0)$. This implies $\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m) \to \tilde{0}$ as $(n, m \to \infty)$. Hence $\left\{\tilde{x}_{e_n}^n\right\}$ is a soft Cauchy sequence, by the completeness of $\tilde{X}$, there is a soft point $\tilde{x}_e^* \tilde{\in} SP(\tilde{X})$ such that $\tilde{x}_{e_n}^n \to \tilde{x}_e^*$ as $(n \to \infty)$. Since

$$\begin{aligned} \tilde{d}\left((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*\right) &\tilde{\leq} \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), (f, \varphi)(\tilde{x}_e^*)\right) + \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), \tilde{x}_e^*\right) \\ &\tilde{\leq} \tilde{\alpha}.\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_e^*) + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*), \end{aligned}$$

$$\tilde{d}\left((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*\right) \tilde{\leq} \tilde{\alpha} \left(\tilde{\alpha}\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_e^*) + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*)\right) \to \tilde{0}$$

Hence $\tilde{d}\left((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*\right) \to \tilde{0}$. This implies $(f, \varphi)(\tilde{x}_e^*) = \tilde{x}_e^*$. So the soft point $\tilde{x}_e^*$ is a fixed soft point of the mapping (f, φ).

Now, if $\tilde{y}_{e'}^*$ is another fixed soft point of (f, φ), then

$$\tilde{d}(\tilde{x}_e^*, \tilde{y}_{e'}^*) = \tilde{d}\left((f, \varphi)(\tilde{x}_e^*), (f, \varphi)(\tilde{y}_{e'}^*)\right) \tilde{\leq} \tilde{\alpha}.\tilde{d}(\tilde{x}_e^*, \tilde{y}_{e'}^*).$$

Hence for $\tilde{\alpha} \tilde{<} \tilde{1}$, $\tilde{d}(\tilde{x}_e^*, \tilde{y}_{e'}^*) = \tilde{0} \Rightarrow \tilde{x}_e^* = \tilde{y}_{e'}^*$. Therefore the fixed soft point of (f, φ) is unique. $\quad\square$

Theorem 4.9. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft complete metric space. Suppose the soft mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ satisfies the soft contractive condition*

$$\tilde{d}\left((f, \varphi)(\tilde{x}_e), (f, \varphi)(\tilde{y}_{e'})\right) \tilde{\leq} \tilde{\alpha} \left[\tilde{d}\left((f, \varphi)(\tilde{x}_e), \tilde{x}_e\right) + \tilde{d}\left((f, \varphi)(\tilde{y}_{e'}), \tilde{y}_{e'}\right)\right],$$

for all $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SP(\tilde{X})$, where $\tilde{\alpha} \tilde{\in} \left[\tilde{0}, \frac{\tilde{1}}{2}\right)$ is a soft constant. Then (f, φ) has a unique fixed soft point in $SP(\tilde{X})$.

Proof. Choose $\tilde{x}_e^0$ be any soft point in $SP(\tilde{X})$. Set

$$\tilde{x}_{e_1}^1 = (f, \varphi)(\tilde{x}_e^0) = \left(f(\tilde{x}_e^0)\right)_{\varphi(e)}, \tilde{x}_{e_2}^2 = (f, \varphi)(\tilde{x}_{e_1}^1) = \left(f^2(\tilde{x}_e^0)\right)_{\varphi^2(e)}, ..., \tilde{x}_{e_{n+1}}^{n+1} = \left((f, \varphi)(\tilde{x}_{e_n}^n)\right) = \left(f^{n+1}(\tilde{x}_e^0)\right)_{\varphi^{n+1}(e)}, ...$$

We have

$$\begin{aligned} \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_{e_n}^n) &= \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), (\tilde{x}_{e_{n-1}}^{n-1})\right) \\ &\tilde{\leq} \tilde{\alpha} \left[\tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), \tilde{x}_{e_n}^n\right) + \tilde{d}\left((f, \varphi)(\tilde{x}_{e_{n-1}}^{n-1}), \tilde{x}_{e_{n-1}}^{n-1}\right)\right] \\ &= \tilde{\alpha} \left[\tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_{e_n}^n) + \tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1})\right]. \end{aligned}$$

$$\tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_{e_n}^n) \tilde{\leq} \frac{\tilde{\alpha}}{\tilde{1} - \tilde{\alpha}} \tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1}) = \tilde{h}\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1}),$$

where $\tilde{h} = \frac{\tilde{\alpha}}{\tilde{1}-\tilde{\alpha}}$. For $n > m$,

$$
\begin{aligned}
\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m) \quad &\tilde{\leq} \quad \tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1}) + \tilde{d}(\tilde{x}_{e_{n-1}}^{n-1}, \tilde{x}_{e_{n-2}}^{n-2}) + \cdots + \tilde{d}(\tilde{x}_{e_{m+1}}^{m+1}, \tilde{x}_{e_m}^m) \\
&\tilde{\leq} \quad \left(\tilde{h}^{n-1} + \tilde{h}^{n-2} + \cdots + \tilde{h}^m \right) \tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0) \\
&\tilde{\leq} \quad \frac{\tilde{h}^m}{\tilde{1} - \tilde{h}} \tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0).
\end{aligned}
$$

we get $\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m) \tilde{\leq} \frac{\tilde{h}^m}{\tilde{1}-\tilde{h}} \tilde{\alpha}\tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0)$. This implies $\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m) \to \tilde{0}$ as $(n, m \to \infty)$. Hence $\left\{ \tilde{x}_{e_n}^n \right\}$ is a soft Cauchy sequence, by the completeness of $\tilde{X}$, there is a soft point $\tilde{x}_e^* \tilde{\in} SP(\tilde{X})$ such that $\tilde{x}_{e_n}^n \to \tilde{x}_e^*$ as $(n \to \infty)$. Since

$$
\begin{aligned}
\tilde{d}((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*) \quad &\tilde{\leq} \quad \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), (f, \varphi)(\tilde{x}_e^*)\right) + \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), \tilde{x}_e^*\right) \\
&\tilde{\leq} \quad \tilde{\alpha}.\left[\tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), \tilde{x}_{e_n}^n\right) + \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), \tilde{x}_e^*\right)\right] + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*) \\
&\tilde{\leq} \quad \frac{\tilde{1}}{\tilde{1} - \tilde{\alpha}}\left[\tilde{\alpha}.\tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), \tilde{x}_{e_n}^n\right) + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*)\right],
\end{aligned}
$$

$$\tilde{d}((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*) \tilde{\leq} \tilde{\alpha}.\frac{\tilde{1}}{\tilde{1} - \tilde{\alpha}}\left(\tilde{\alpha}\tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_{e_n}^n) + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*)\right) \to \tilde{0}$$

Hence $\tilde{d}((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*) \to \tilde{0}$. This implies $(f, \varphi)(\tilde{x}_e^*) = \tilde{x}_e^*$. So the soft point $\tilde{x}_e^*$ is a fixed soft point of the mapping (f, φ).

Now, if $\tilde{y}_{e'}^*$ is another fixed soft point of (f, φ), then

$$\tilde{d}(\tilde{x}_e^*, \tilde{y}_{e'}^*) = \tilde{d}((f, \varphi)(\tilde{x}_e^*), (f, \varphi)(\tilde{y}_{e'}^*)) \tilde{\leq} \tilde{\alpha}.\left[\tilde{d}((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*) + \tilde{d}((f, \varphi)(\tilde{y}_{e'}^*), \tilde{y}_{e'}^*)\right] = \tilde{0}.$$

Hence for $\tilde{\alpha} \tilde{<} \tilde{1}$, $\tilde{d}(\tilde{x}_e^*, \tilde{y}_{e'}^*) = \tilde{0} \Rightarrow \tilde{x}_e^* = \tilde{y}_{e'}^*$. Therefore the fixed soft point of (f, φ) is unique. $\quad\square$

Theorem 4.10. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft complete metric space. Suppose the soft mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ satisfies the soft contractive condition*

$$\tilde{d}((f, \varphi)(\tilde{x}_e), (f, \varphi)(\tilde{y}_{e'})) \tilde{\leq} \tilde{\alpha}\left[\tilde{d}((f, \varphi)(\tilde{x}_e), \tilde{y}_{e'}) + \tilde{d}((f, \varphi)(\tilde{y}_{e'}), \tilde{x}_e)\right]$$

for all $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SP(\tilde{X})$, where $\tilde{\alpha} \tilde{\in} \left[\tilde{0}, \frac{\tilde{1}}{2}\right)$ is a soft constant. Then (f, φ) has a unique fixed soft point in $SP(\tilde{X})$.

Proof. Let $\tilde{x}_e^0$ be any soft point in $SP(\tilde{X})$. Set

$$\tilde{x}_{e_1}^1 = (f, \varphi)(\tilde{x}_e^0) = \left(f(\tilde{x}_e^0)\right)_{\varphi(e)}, \tilde{x}_{e_2}^2 = (f, \varphi)(\tilde{x}_{e_1}^1) = \left(f^2(\tilde{x}_e^0)\right)_{\varphi^2(e)}, ..., \tilde{x}_{e_{n+1}}^{n+1} = \left((f, \varphi)(\tilde{x}_{e_n}^n)\right) = \left(f^{n+1}(\tilde{x}_e^0)\right)_{\varphi^{n+1}(e)}, ...$$

We have

$$
\begin{aligned}
\tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_{e_n}^n) \quad &= \quad \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), (f, \varphi)(\tilde{x}_{e_{n-1}}^{n-1})\right) \\
&\tilde{\leq} \quad \tilde{\alpha}.\left[\tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), \tilde{x}_{e_{n-1}}^{n-1}\right) + \tilde{d}\left((f, \varphi)(\tilde{x}_{e_{n-1}}^{n-1}), \tilde{x}_{e_n}^n\right)\right] \\
&\tilde{\leq} \quad \tilde{\alpha}.\left[\tilde{d}\left((\tilde{x}_{e_n+1}^{n+1}), \tilde{x}_{e_n}^n\right) + \tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1})\right].
\end{aligned}
$$

So,

$$\tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_{e_n}^n) \tilde{\leq} \frac{\tilde{\alpha}}{\tilde{1} - \tilde{\alpha}} \tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1}) = \tilde{h}.\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1}),$$

where $\tilde{h} = \frac{\tilde{\alpha}}{\tilde{1}-\tilde{\alpha}}$. For $n > m$,

$$
\begin{aligned}
\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m) &\tilde{\leq} \tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_{n-1}}^{n-1}) + \tilde{d}(\tilde{x}_{e_{n-1}}^{n-1}, \tilde{x}_{e_{n-2}}^{n-2}) + \cdots + \tilde{d}(\tilde{x}_{e_{m+1}}^{m+1}, \tilde{x}_{e_m}^m) \\
&\tilde{\leq} \left(\tilde{h}^{n-1} + \tilde{h}^{n-2} + \cdots + \tilde{h}^m\right)\tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0) \\
&\tilde{\leq} \frac{\tilde{h}^m}{\tilde{1}-\tilde{h}}\tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0).
\end{aligned}
$$

we get $\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m)\tilde{\leq}\frac{\tilde{h}^m}{\tilde{1}-\tilde{h}}\tilde{\alpha}\tilde{d}(\tilde{x}_{e_1}^1, \tilde{x}_e^0)$. This implies $\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m) \to \tilde{0}$ as $(n, m \to \infty)$. Hence $\left\{\tilde{x}_{e_n}^n\right\}$ is a soft Cauchy sequence, by the completeness of $\tilde{X}$, there is a soft point $\tilde{x}_e^* \tilde{\in} SP(\tilde{X})$ such that $\tilde{x}_{e_n}^n \to \tilde{x}_e^*$ as $(n \to \infty)$. Since

$$
\begin{aligned}
\tilde{d}((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*) &\tilde{\leq} \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), (f, \varphi)(\tilde{x}_e^*)\right) + \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), \tilde{x}_e^*\right) \\
&\tilde{\leq} \tilde{\alpha}.\left[\tilde{d}\left((f, \varphi)(\tilde{x}_e^*), \tilde{x}_{e_n}^n\right) + \tilde{d}\left((f, \varphi)(\tilde{x}_{e_n}^n), \tilde{x}_e^*\right)\right] + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*) \\
&\tilde{\leq} \tilde{\alpha}.\left[\tilde{d}((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*) + \tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_e^*) + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*)\right] + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*) \\
&\tilde{\leq} \frac{\tilde{1}}{\tilde{1}-\tilde{\alpha}}\left[\tilde{\alpha}.\left(\tilde{d}\left(\tilde{x}_{e_n}^n, \tilde{x}_e^*\right) + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*)\right) + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*)\right],
\end{aligned}
$$

$$
\tilde{d}((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*)\tilde{\leq}\tilde{\alpha}.\frac{\tilde{1}}{\tilde{1}-\tilde{\alpha}}\left(\tilde{\alpha}.\left(\tilde{d}\left(\tilde{x}_{e_n}^n, \tilde{x}_e^*\right) + \tilde{d}(\tilde{x}_{e_{n+1}}^{n+1}, \tilde{x}_e^*)\right)\right) \to \tilde{0}
$$

Hence $\tilde{d}((f, \varphi)(\tilde{x}_e^*), \tilde{x}_e^*) \to \tilde{0}$. This implies $(f, \varphi)(\tilde{x}_e^*) = \tilde{x}_e^*$. So the soft point $\tilde{x}_e^*$ is a fixed soft point of the mapping (f, φ).

Now, if $\tilde{y}_{e'}^*$ is another fixed soft point of (f, φ), then

$$
\tilde{d}(\tilde{x}_e^*, \tilde{y}_{e'}^*) = \tilde{d}\left((f, \varphi)(\tilde{x}_e^*), (f, \varphi)(\tilde{y}_{e'}^*)\right)\tilde{\leq}\tilde{\alpha}.\left[\tilde{d}\left((f, \varphi)(\tilde{x}_e^*), \tilde{y}_{e'}^*\right) + \tilde{d}\left((f, \varphi)(\tilde{y}_{e'}^*), \tilde{x}_e^*\right)\right] = \tilde{0}.
$$

Hence for $\tilde{\alpha}\tilde{<}\tilde{1}$, $\tilde{d}(\tilde{x}_e^*, \tilde{y}_{e'}^*) = \tilde{0} \Rightarrow \tilde{x}_e^* = \tilde{y}_{e'}^*$. Therefore the fixed soft point of (f, φ) is unique. $\quad\square$

Proposition 4.11. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. If $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ is a soft contraction mapping, then the mapping $f_e : (X, d_e) \to (X, d_{\varphi(e)})$ is a contraction mapping for all $e \in E$.*

The following example shows that converse of Proposition 4.11 does not hold.

Example 4.12. Let $E = \mathbb{R}$ be a parameter set and $X = \mathbb{R}^2$. Consider usual metrics on this sets and define soft metric on $\tilde{X}$ by $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = |e - e'| + d(x, y)$. Then if we define the soft mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ as follows

$$
(f, \varphi)(\tilde{x}_e) = \left(\frac{1}{2}x\right)_{3e},
$$

then

$$
\tilde{d}\left((f, \varphi)(\widetilde{0, 1})_2, (f, \varphi)(\widetilde{1, 0})_1\right) = \tilde{d}\left(\left(\widetilde{0, \frac{1}{2}}\right)_6, \left(\widetilde{\frac{1}{2}, 0}\right)_3\right) = 3 + \frac{\sqrt{2}}{2},
$$

$$
\tilde{d}\left(\left(\widetilde{0, 1}\right)_2, \left(\widetilde{1, 0}\right)_1\right) = 1 + \sqrt{2}.
$$

Since $3 + \frac{\sqrt{2}}{2} > 1 + \sqrt{2}$, we see that the soft mapping (f, φ) is not a soft contraction mapping. But the mapping $f_e : (X, d_e) \to (X, d_{3e})$ is a contraction mapping for all $e \in E$.

Corollary 4.13. *Let E be a parameter set and X be a set. By using the given metrics defined on these sets, we can form a soft metric. If (f, φ) is a soft contraction mapping on the soft space, then f or φ may not be a contraction mappings.*

Proof. The proof is clear. □

The following example justifies Corollary 4.13

Example 4.14. Let $(\mathbb{R}, \tilde{d}, E)$ be a soft metric space with the following metrics

$$d(x, y) = |x - y|, d_1(x, y) = \min\left\{|x - y|, 1\right\},$$

and

$$\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = d(x, y) + \frac{1}{2} d_1(e, e'),$$

where $E = [1, \infty)$. Let the functions $\varphi : [1, \infty) \to [1, \infty)$ and $f : \mathbb{R} \to \mathbb{R}$ are defined as $\varphi(x) = x + \frac{1}{x}$ and $f(x) = \frac{1}{5}x$ respectively.

Here, it is obvious that the conditions of contraction mapping are hold for the composite function

$$(f, \varphi) : (\mathbb{R}, \tilde{d}, E) \to (\mathbb{R}, \tilde{d}, E).$$

We want to show that (f, φ) is a soft contraction mapping, whereas the function $\varphi(x) = x + \frac{1}{x}$ is not a contraction mapping with the defined metric $d_1(x, y) = \min\left\{|x - y|, 1\right\}$.

$$
\begin{aligned}
\tilde{d}\left((f, \varphi)(\tilde{x}_e), (f, \varphi)(\tilde{y}_{e'})\right) &= \tilde{d}\left(\left(\widetilde{\frac{1}{5}x}\right)_{e + \frac{1}{e}}, \left(\widetilde{\frac{1}{5}y}\right)_{e' + \frac{1}{e'}}\right) \\
&= \frac{1}{5}|x - y| + \frac{1}{2} d_1\left(e + \frac{1}{e}, e' + \frac{1}{e'}\right) \\
&= \frac{1}{5}|x - y| + \frac{1}{2} \min\left\{\left|e + \frac{1}{e} - e' - \frac{1}{e'}\right|, 1\right\} \\
&= \frac{1}{5}|x - y| + \frac{1}{2} \min\left\{|e - e'|\left|1 - \frac{1}{ee'}\right|, 1\right\} \\
&\leq \frac{1}{5}|x - y| + \frac{1}{2} \min\left\{|e - e'|, 1\right\} \\
&\leq \frac{1}{5}d(x, y) + \frac{1}{2} d_1(e, e') \\
&\leq \frac{3}{4}\left(d(x, y) + d_1(e, e')\right)
\end{aligned}
$$

which shows that (f, φ) is a soft contraction mapping.

5. Conclusion

In the present study, we have continued to investigate properties of soft metric spaces. We also introduce soft continuous mappings. Later we prove some fixed point theorems of soft contractive mappings on soft metric spaces. We hope that the findings in this paper will help researcher enhance and promote the further study on soft metric spaces to carry out a general framework for their applications in real life.

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