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Convergence Theorem, Convergence Rate and Convergence Speed for Continuous Real Functions

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Abstract. In this work, we study convergence theorem, convergence rate and convergence speed of a new three-step iterative scheme for continuous functions on an arbitrary interval. We also give numerical examples for comparing with iterations of Mann, Ishikawa, Noor and Kadioglu-Yildirim.

1. Introduction

Let E be a closed interval on the real line and let $f : E \to E$ be a continuous function. A point $p \in E$ is called a *fixed point* of f if $f(p) = p$.

One classical way to approximate a fixed point of a nonlinear mapping was introduced, in 1953, by Mann [6] as follows: a sequence $\{u_n\}_{n=1}^{\infty}$ defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n f(u_n) \tag{1}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$. Such an iteration process is known as *Mann iteration*. In 1991, D. Borwein and J. Borwein [1] proved the convergence theorem for a continuous function on the closed and bounded interval in the real line by using iteration (1).

Another classical iteration process was introduced by Ishikawa [4] as follows: a sequence $\{s_n\}_{n=1}^{\infty}$ defined by $s_1 \in E$ and

$$\begin{aligned} t_n &= (1 - b_n)s_n + b_n f(s_n) \\ s_{n+1} &= (1 - \alpha_n)s_n + \alpha_n f(t_n) \end{aligned} \tag{2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{b_n\}$ are sequences in $[0, 1]$. Such an iterative method is known as *Ishikawa iteration*. In 2006, Qing and Qihou [10] proved the convergence theorem of the sequence generated by iteration (2) for a continuous function on the closed interval in the real line (see also [11]).

In 2000, Noor [7] defined the following iterative scheme by $l_1 \in E$ and

$$\begin{aligned} m_n &= (1 - a_n)l_n + a_n f(l_n) \\ v_n &= (1 - b_n)l_n + b_n f(m_n) \\ l_{n+1} &= (1 - \alpha_n)l_n + \alpha_n f(v_n) \end{aligned} \tag{3}$$

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for all $n \geq 1$, where $\{\alpha_n\}, \{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$. Such an iterative method is known as *Noor iteration*. Phuengrattana and Suantai [8] considered the convergence of a new three-step called the SP-iteration for continuous functions on an arbitrary interval in the real line.

Recently, Kadioglu and Yildirim [5] defined the following KY-iteration process: $w_1 \in E$ and

$$\begin{aligned} r_n &= (1-a_n)w_n + a_n f(w_n) \\ q_n &= (1-b_n-c_n)w_n + b_n f(r_n) + c_n f(w_n) \\ w_{n+1} &= (1-\alpha_n-\beta_n)w_n + \alpha_n f(q_n) + \beta_n f(r_n) \end{aligned} \tag{4}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences in [0,1]. They showed that (4) converges to a fixed point of f. Moreover the rate of convergence is better than those of Mann, Ishikawa and Noor in the sense of Rhoades [13]. We denote the above iteration by $KY(w_1, a_n, b_n, c_n, \alpha_n, \beta_n, f)$.

Some interesting results concerning fixed point theory of continuous functions can be found in [2, 3, 9, 12–14].

In this paper, we propose a new three-step iteration process for solving a fixed point problem for continuous functions on an arbitrary interval in the real line. Numerical examples are also presented to compare with iterations of Mann, Ishikawa, Noor and Kadioglu-Yildirim.

2. Convergence Theorem

In this section, we study convergence theorem for the iteration process defined by the following for continuous functions on an arbitrary interval.

Theorem 2.1. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. For $x_1 \in E$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be defined by*

$$\begin{aligned} z_n &= (1-a_n)x_n + a_n f(x_n) \\ y_n &= (1-b_n-c_n)z_n + b_n f(z_n) + c_n f(x_n) \\ x_{n+1} &= (1-\alpha_n-\beta_n)y_n + \alpha_n f(y_n) + \beta_n f(z_n) \end{aligned} \tag{5}$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0,1)$ with $0 \leq b_n + c_n < 1$ and $0 \leq \alpha_n + \beta_n < 1$ satisfying the following conditions :

(i) $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if it converges to a fixed point of f.

Proof. Sufficiency is obvious. It suffices to show that if $\{x_n\}_{n=1}^{\infty}$ is bounded, then $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point. We will show that $\{x_n\}_{n=1}^{\infty}$ is convergent. Suppose that $\{x_n\}_{n=1}^{\infty}$ is divergent. Then there exist $a, b \in \mathbb{R}$, $a = \liminf_{n\to\infty} x_n$, $b = \limsup_{n\to\infty} x_n$ and $a < b$. First, we show that if $a < m < b$, then $f(m) = m$. Suppose that $f(m) \neq m$. Without loss of generality, we assume that $f(m) - m > 0$. Since f is continuous, there exists $\delta \in (0, b-a)$ such that, for $|x - m| \leq \delta$,

$$f(x) - x > 0. \tag{6}$$

By the boundedness of $\{x_n\}_{n=1}^{\infty}$ and the continuity of f, we have $\{f(x_n)\}_{n=1}^{\infty}$ is bounded. So are $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$, $\{f(y_n)\}_{n=1}^{\infty}$ and $\{f(z_n)\}_{n=1}^{\infty}$. From (5), we have $x_{n+1} - y_n = \alpha_n(f(y_n) - y_n) + \beta_n(f(z_n) - y_n)$, $y_n - z_n = b_n(f(z_n) - z_n) + c_n(f(x_n) - z_n)$ and $z_n - x_n = a_n(f(x_n) - x_n)$. By conditions (i) and (ii), we see that $|x_{n+1} - y_n| \to 0$, $|y_n - z_n| \to 0$ and $|z_n - x_n| \to 0$. Since $|x_{n+1} - x_n| \leq |x_{n+1} - y_n| + |y_n - z_n| + |z_n - x_n|$ and $|y_n - x_n| \leq |y_n - z_n| + |z_n - x_n|$, we have $|x_{n+1} - x_n| \to 0$ and $|y_n - x_n| \to 0$. Thus there exists a natural number N such that

$$|x_{n+1} - x_n| < \frac{\delta}{3}, |z_n - x_n| < \frac{\delta}{3}, |y_n - x_n| < \frac{\delta}{3} \tag{7}$$

for all $n > N$. Since $b = \limsup_{n\to\infty} x_n > m$. there exists $k_1 > N$ such that $x_{n_{k_1}} > m$. Let $k = n_{k_1}$, then $x_k > m$. For x_k, we consider the following two cases:

Case 1 : if $x_k \geq m + \frac{\delta}{3}$, then by (7), we have $x_{k+1} - x_k > -\frac{\delta}{3}$. Thus $x_{k+1} > x_k - \frac{\delta}{3} \geq m$ and $x_{k+1} > m$.

Case 2 : if $m < x_k < m + \frac{\delta}{3}$, then by (7), we have $m - \frac{\delta}{3} < y_k < m + \frac{2\delta}{3}$ and $m - \frac{\delta}{3} < z_k < m + \frac{2\delta}{3}$. So we have $|x_k - m| < \frac{\delta}{3} < \delta, |y_k - m| < \frac{2\delta}{3} < \delta$ and $|z_k - m| < \frac{2\delta}{3} < \delta$. From (6), we have

$$f(x_k) - x_k > 0, f(y_k) - y_k > 0, f(z_k) - z_k > 0. \tag{8}$$

By (5), we obtain

$$\begin{aligned} x_{k+1} &= x_k - x_k + (1 - \alpha_k - \beta_k)y_k + \alpha_k f(y_k) + \beta_k f(z_k) \\ &= x_k + (1 - \alpha_k - \beta_k)(y_k - x_k) + \alpha_k(f(y_k) - x_k) + \beta_k(f(z_k) - x_k) \\ &= x_k + (1 - \alpha_k - \beta_k)(y_k - x_k) + \alpha_k(f(y_k) - y_k) + \alpha_k(y_k - x_k) + \beta_k(f(z_k) - z_k) + \beta_k(z_k - x_k) \\ &= x_k + (1 - \beta_k)(y_k - x_k) + \alpha_k(f(y_k) - y_k) + \beta_k(f(z_k) - z_k) + \beta_k a_k(f(x_k) - x_k). \end{aligned} \tag{9}$$

Also, we have

$$\begin{aligned} y_k - x_k &= (1 - b_k - c_k)z_k + b_k f(z_k) + c_k f(x_k) - x_k \\ &= (z_k - x_k) + b_k(f(z_k) - z_k) + c_k(f(x_k) - z_k) \\ &= (z_k - x_k) + b_k(f(z_k) - z_k) + c_k(f(x_k) - x_k) + c_k(x_k - z_k) \\ &= (1 - c_k)(z_k - x_k) + b_k(f(z_k) - z_k) + c_k(f(x_k) - x_k) \\ &= (1 - c_k)a_k(f(x_k) - x_k) + b_k(f(z_k) - z_k) + c_k(f(x_k) - x_k). \end{aligned} \tag{10}$$

Substituting (10) into (9), we obtain

$$\begin{aligned} x_{k+1} &= x_k + (1 - \beta_k)\Big(a_k(1 - c_k)(f(x_k) - x_k) + b_k(f(z_k) - z_k) + c_k(f(x_k) - x_k)\Big) + \alpha_k(f(y_k) - y_k) + \beta_k(f(z_k) - z_k) \\ &\quad + \beta_k a_k(f(x_k) - x_k) \\ &= x_k + \Big((1 - \beta_k)\Big(a_k(1 - c_k) + c_k\Big) + \beta_k a_k\Big)(f(x_k) - x_k) + \Big(b_k(1 - \beta_k) + \beta_k\Big)(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k). \end{aligned}$$

From (8), we have $x_{k+1} > m$. So, by Case 1 and Case 2, we can conclude that $x_{k+1} > m$. Employing the same argument, we obtain $x_{k+2} > m, x_{k+3} > m, \ldots$ Hence, by induction, $x_n > m$ for all $n > k$. Therefore $a = \liminf_{n\to\infty} x_n \geq m$, which contradicts with $a < m$. It follows that $f(m) = m$.

For the sequence $\{x_n\}_{n=1}^{\infty}$, we consider the following two cases:

Case 1 : There exists x_m such that $a < x_m < b$, then $f(x_m) = x_m$ and

$$z_m = (1 - a_m)x_m + a_m f(x_m) = x_m.$$

which yields

$$y_m = (1 - b_m - c_m)z_m + b_m f(z_m) + c_m f(x_m) = (1 - b_m - c_m)x_m + b_m f(x_m) + c_m f(x_m) = x_m.$$

So we have

$$x_{m+1} = (1 - \alpha_m - \beta_m)y_m + \alpha_m f(y_m) + \beta_m f(z_m) = (1 - \alpha_m - \beta_m)x_m + \alpha_m f(x_m) + \beta_m f(x_m) = x_m.$$

By induction, we obtain $x_m = x_{m+1} = x_{m+2} = x_{m+3} = \ldots$, so that $x_n \to x_m$. This shows that $x_m = a$ and $x_n \to a$, which contradicts to the divergence of $\{x_n\}_{n=1}^{\infty}$.

Case 2 : For all n, $x_n \leq a$ or $x_n \geq b$, since $b - a > 0$ and $|x_{n+1} - x_n| \to 0$, there exists N_0 such that $|x_{n+1} - x_n| < \frac{b-a}{3}$ for all $n > N_0$. If $x_n \leq a$ for $n > N_0$, then $b = \limsup_{n\to\infty} x_n \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > N_0$, then $a = \liminf_{n\to\infty} x_n \geq b$, which is also a contradiction with $a < b$. Hence $\{x_n\}_{n=1}^{\infty}$ is convergent.

Finally, we show that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f. Let $x_n \to p$ and suppose that $f(p) \neq p$. Since $z_n = (1 - a_n)x_n + a_n f(x_n)$ and $a_n \to 0$, we obtain $z_n \to p$. From $y_n = (1 - b_n - c_n)z_n + b_n f(z_n) + c_n f(x_n)$, $b_n \to 0$ and $c_n \to 0$, it follows that $y_n \to p$. Let $h_k = f(x_k) - x_k, r_k = f(y_k) - y_k$ and $s_k = f(z_k) - z_k$. By the continuity of f, we see that

$\lim_{k\to\infty} h_k = \lim_{k\to\infty}(f(x_k) - x_k) = f(p) - p \neq 0,$

$\lim_{k\to\infty} r_k = \lim_{k\to\infty}(f(y_k) - y_k) = f(p) - p \neq 0,$
$\lim_{k\to\infty} s_k = \lim_{k\to\infty}(f(z_k) - z_k) = f(p) - p \neq 0.$
Put $w = f(p) - p$. From (5) we get

$$x_{n+1} = x_n + \Big((1-\beta_n)\Big(a_n(1-c_n) + c_n\Big) + \beta_n a_n\Big)(f(x_n) - x_n) + \Big(b_n(1-\beta_n) + \beta_n\Big)(f(z_n) - z_n) + \alpha_n(f(y_n) - y_n).$$

It follows that

$$x_n = x_1 + \sum_{k=1}^{n}\Big(a_k(1-\beta_k)(1-c_k) + c_k(1-\beta_k) + \beta_k a_k\Big)h_k + \sum_{k=1}^{n}\Big(b_k(1-\beta_k) + \beta_k\Big)s_k + \sum_{k=1}^{n}\alpha_k r_k.$$

From $h_k \to w, r_k \to w, s_k \to w$ and conditions (i), (ii), we can easily check that $\{x_n\}_{n=1}^{\infty}$ is divergent. Thus $f(p) = p$ and we complete the proof. □

Corollary 2.2. *Let $f : [a,b] \to [a,b]$ be a continuous function. For $x_1 \in [a,b]$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be defined by (5), where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1) with $0 \le b_n + c_n < 1$ and $0 \le \alpha_n + \beta_n < 1$ satisfying the following conditions:*
(i) $\sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$.
Then $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Remark 2.3. *If we take $c_n = \beta_n = 0$, we then obtain Theorem 2.1 of Phuengrattana and Suantai [8].*

3. Rate of Convergence

In this section, we compare the convergence rate of (5) with the KY-iteration proposed in [5].
To this end, we use the concept introduced by Rhoades [13] as follows:

Definition 3.1. *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous function. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are two iterations which converge to the fixed point p of f. Then $\{x_n\}_{n=1}^{\infty}$ is said to converge faster than $\{w_n\}_{n=1}^{\infty}$ if*

$$|x_n - p| \le |w_n - p|$$

for all $n \ge 1$.

We next prove some crucial lemmas which will be used in the sequel.

Lemma 3.2. *[5] Let E be a closed interval on the real line and let $f : E \to E$ be a continuous and nondecreasing function. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1)$ with $0 \le b_n + c_n < 1$ and $0 \le \alpha_n + \beta_n < 1$. Let $\{w_n\}_{n=1}^{\infty}$ be defined by the KY-iteration. Then the following hold:*
(i) If $f(w_1) < w_1$, then $f(w_n) < w_n$ for all $n \ge 1$ and $\{w_n\}_{n=1}^{\infty}$ is nonincreasing.
(ii) If $f(w_1) > w_1$, then $f(w_n) > w_n$ for all $n \ge 1$ and $\{w_n\}_{n=1}^{\infty}$ is nondecreasing.

Lemma 3.3. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1)$ with $0 \le b_n + c_n < 1$ and $0 \le \alpha_n + \beta_n < 1$. Let $\{x_n\}_{n=1}^{\infty}$ be defined by (5). Then the following hold:*
(i) If $f(x_1) < x_1$, then $f(x_n) < x_n$ for all $n \ge 1$ and $\{x_n\}_{n=1}^{\infty}$ is nonincreasing.
(ii) If $f(x_1) > x_1$, then $f(x_n) > x_n$ for all $n \ge 1$ and $\{x_n\}_{n=1}^{\infty}$ is nondecreasing.

Proof. (i) Let $f(x_1) < x_1$. Then $f(x_1) < z_1 \le x_1$. Since f is nondecreasing, $f(z_1) \le f(x_1) < z_1 \le x_1$. This implies $f(z_1) < y_1 \le z_1$. Thus $f(y_1) \le f(z_1) \le f(x_1) < z_1 \le x_1$. For y_1, we consider the following cases:

Case 1: If $f(z_1) < y_1 \le z_1$, then $f(y_1) \le f(z_1) < y_1 < x_1$. It follows that if $f(y_1) < x_2 \le y_1$, then $f(x_2) \le f(y_1) < x_2$, if $y_1 < x_2 \le z_1$, then $f(x_2) \le f(z_1) < y_1 < x_2$ and if $z_1 < x_2 \le x_1$, then $f(x_2) \le f(x_1) < z_1 < x_2$. So, we have $f(x_2) < x_2$.

Case 2: If $z_1 < y_1 \le x_1$, then $f(y_1) \le f(x_1) < z_1 \le x_1$. This implies that $f(y_1) < x_2 \le x_1$ and $f(x_2) \le f(x_1) < z_1 < y_1 < x_2$. We thus have $f(x_2) < x_2$.

From Case 1 and Case 2, we have $f(x_2) < x_2$. So we can show that $f(x_n) < x_n$ for all $n \ge 1$. So $z_n \le x_n$ for all $n \ge 1$. Since f is nondecreasing, we have $f(z_n) \le f(x_n) < x_n$ for all $n \ge 1$. Thus $y_n \le x_n$ for all $n \ge 1$, and $f(y_n) \le f(x_n) < x_n$ for all $n \ge 1$. Hence, we have $x_{n+1} \le x_n$ for all $n \ge 1$, and thus $\{x_n\}_{n=1}^{\infty}$ is nonincreasing.

(ii) Following the proof line as in (i), we obtain the desired result. □

Lemma 3.4. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0, 1)$ with $0 \le b_n + c_n < 1$ and $0 \le \alpha_n + \beta_n < 1$. For $w_1 = x_1 \in E$, let $\{w_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be sequences defined by the KY-iteration and (5), respectively. Then the following are satisfied:*

(i) If $f(w_1) < w_1$, then $x_n < w_n$ for all $n \ge 1$.

(ii) If $f(w_1) > w_1$, then $x_n > w_n$ for all $n \ge 1$.

Proof. (i) Let $f(w_1) < w_1$. Then $f(x_1) < x_1$ since $w_1 = x_1$. From (5), we get $f(x_1) < z_1 \le x_1$. Since f is nondecreasing, we obtain $f(z_1) \le f(x_1) < z_1 \le x_1$. Hence $f(z_1) < y_1 \le z_1$.

Using the KY-iteration and (5), we obtain the following estimation:

$$z_1 - r_1 = (1 - a_1)(x_1 - w_1) + a_1(f(x_1) - f(w_1)) = 0.$$

So, $z_1 = r_1$, and also

$$y_1 - q_1 = (1 - b_1 - c_1)(z_1 - w_1) + b_1(f(z_1) - f(r_1)) + c_1(f(x_1) - f(w_1)) \le 0.$$

Since f is nondecreasing, we have $f(y_1) \le f(q_1)$. We next obtain

$$x_2 - w_2 = (1 - \alpha_1 - \beta_1)(y_1 - w_1) + \alpha_1(f(y_1) - f(q_1)) + \beta_1(f(z_1) - f(r_1)) \le 0,$$

so, $x_2 \le w_2$. Assume that $x_k \le w_k$. Thus $f(x_k) \le f(w_k)$.

From Lemma 3.2 (i), we get $f(w_k) < w_k$ and $f(x_k) < x_k$. It follows that $f(x_k) < z_k \le x_k$ and $f(z_k) \le f(x_k) < z_k$. Hence

$$z_k - r_k = (1 - a_k)(x_k - w_k) + a_k(f(x_k) - f(w_k)) \le 0.$$

So, $z_k \le r_k$. Since $f(z_k) \le f(r_k)$,

$$y_k - q_k = (1 - b_k - c_k)(z_k - w_k) + b_k(f(z_k) - f(r_k)) + c_k(f(x_k) - f(w_k)) \le 0,$$

so $y_k \le q_k$, which yields $f(y_k) \le f(q_k)$. This shows that

$$x_{k+1} - w_{k+1} = (1 - \alpha_k - \beta_k)(y_k - w_k) + \alpha_k(f(y_k) - f(q_k)) + \beta_k(f(z_k) - f(r_k)) \le 0,$$

which gives, $x_{k+1} \le w_{k+1}$. By induction, we conclude that $x_n \le w_n$ for all $n \ge 1$.

(ii) From Lemma 3.2 (ii) and the same proof as in (i), we can show that $x_n \ge w_n$ for all $n \ge 1$. □

For convenience, we write algorithm (5) by $BC(x_1, a_n, b_n, c_n, \alpha_n, \beta_n, f)$.

Proposition 3.5. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in E : p = f(p)\}$. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0, 1)$ with $0 \le b_n + c_n < 1$ and $0 \le \alpha_n + \beta_n < 1$. If $f(x_1) > x_1$, then $\{x_n\}_{n=1}^{\infty}$ defined by $KY(x_1, a_n, b_n, c_n, \alpha_n, \beta_n, f)$ and $BC(x_1, a_n, b_n, c_n, \alpha_1, \beta_n, f)$ do not converge to a fixed point of f.*

Proof. From Lemma 3.3 (ii), we know that $\{x_n\}_{n=1}^{\infty}$ is nondecreasing. Since the initial point $x_1 > \sup\{p \in E : p = f(p)\}$, it follows that $\{x_n\}_{n=1}^{\infty}$ does not converge to a fixed point of f. □

Proposition 3.6. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in E : p = f(p)\}$. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1)$ with $0 \le b_n + c_n < 1$ and $0 \le \alpha_n + \beta_n < 1$. If $f(x_1) < x_1$, then $\{x_n\}_{n=1}^{\infty}$ defined by $KY(x_1, a_n, b_n, c_n, \alpha_n, \beta_n, f)$ and $BC(x_1, a_n, b_n, c_n, \alpha_1, \beta_n, f)$ do not converge to a fixed point of f.*

Proof. *From Lemma 3.3 (i), we know that $\{x_n\}_{n=1}^{\infty}$ is nonincreasing. Since the initial point $x_1 < \inf\{p \in E : p = f(p)\}$, it follows that $\{x_n\}_{n=1}^{\infty}$ does not converge to a fixed point of f.* □

We are now in position to prove the main results of this paper.

Theorem 3.7. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1)$ with $0 \le b_n + c_n < 1$ and $0 \le \alpha_n + \beta_n < 1$. For $w_1 = x_1 \in E$, let $\{w_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be sequences defined by the KY-iteration and the BC-iteration, respectively. Then the BC-iteration $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if the KY-iteration $\{w_n\}_{n=1}^{\infty}$ converges to p. Moreover, the BC-iteration $\{x_n\}_{n=1}^{\infty}$ converges faster than the KY-iteration.*

Proof. Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$.

(⇒) Let the BC-iteration $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$. From Theorem 3.7 (iii) in [8] and Theorem 3 in [5], we get the convergence of the KY-iteration.

(⇐) Suppose that the KY-iteration $\{w_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$. We split the proof into three cases as follows:

Case 1: $w_1 = x_1 > U$, Case 2: $w_1 = x_1 < L$, Case 3: $L \le w_1 = x_1 \le U$.

Case 1: $w_1 = x_1 > U$. By Proposition 3.5, we get $f(w_1) < w_1$ and $f(x_1) < x_1$. So, by Lemma 3.4 (i), we have $x_n \le w_n$ for all $n \ge 1$. By induction, we can show that $U \le x_n$ for all $n \ge 1$. Then, we have $0 \le x_n - p \le w_n - p$, which yields $|x_n - p| \le |w_n - p|$ for all $n \ge 1$. This shows that $x_n \to p$. By Definition 3.1, we conclude that the BC-iteration $\{x_n\}_{n=1}^{\infty}$ converges faster than the KY-iteration $\{w_n\}_{n=1}^{\infty}$.

Case 2: $w_1 = x_1 < L$. By Proposition 3.6, we get $f(w_1) > w_1$ and $f(x_1) > x_1$. This implies, by Lemma 3.4 (ii), that $x_n \ge w_n$ for all $n \ge 1$. So, by induction, we can show that $x_n \le L$ for all $n \ge 1$. Then, we have $|x_n - p| \le |w_n - p|$ for all $n \ge 1$. It follows that $x_n \to p$ and the BC-iteration $\{x_n\}_{n=1}^{\infty}$ converges faster than the KY-iteration $\{w_n\}_{n=1}^{\infty}$.

Case 3: $L \le w_1 = x_1 \le U$. Suppose that $f(w_1) \ne w_1$. If $f(w_1) < w_1$, we have, by Lemma 3.2 (i), that $\{w_n\}_{n=1}^{\infty}$ is nonincreasing with limit p. Lemma 3.4 (i) gives $p \le x_n \le w_n$ for all $n \ge 1$. It follows that $|x_n - p| \le |w_n - p|$ for all $n \ge 1$. Therefore $x_n \to p$ and the result follows. If $f(w_1) > w_1$, by Lemma 3.2 (ii) and Lemma 3.4 (ii), then we can also show that the result holds. □

Remark 3.8. *We note that, by Theorem 2 in [5] and Theorem 3.7 in [8], the convergence of Mann, Ishikawa, Noor and the KY-iteration are all equivalent. Hence, by Theorem 3.7, the BC-iteration converges faster than Mann, Ishikawa and Noor iterations.*

4. Speed of Convergence

In this section, we study the convergence speed of our algorithm defined in this paper.

Theorem 4.1. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{a'_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$, $\{c'_n\}_{n=1}^{\infty}$, $\{\alpha'_n\}_{n=1}^{\infty}$ and $\{\beta'_n\}_{n=1}^{\infty}$ be sequences in $[0,1)$ with $0 \le b_n + c_n < 1$, $0 \le \alpha_n + \beta_n < 1$, $0 \le a'_n + b'_n < 1$ and $0 \le \alpha'_n + \beta'_n < 1$ such that $a_n \le a'_n$, $b_n \le b'_n$, $c_n \le c'_n$, $\alpha_n \le \alpha'_n$ and $\beta_n \le \beta'_n$ for all $n \ge 1$. For $x'_1 = x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ and $\{x'_n\}_{n=1}^{\infty}$ be defined by $BC(x_1, a_n, b_n, c_n, \alpha_n, \beta_n, f)$ and $BC(x'_1, a'_n, b'_n, c'_n, \alpha'_n, \beta'_n, f)$, respectively. If $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$, then $\{x'_n\}_{n=1}^{\infty}$ converges to p. Moreover, $\{x'_n\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$.*

Proof. Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$. Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$. we divide our proof into the following three cases:

Case 1: $x_1 = x_1' > U$. By Proposition 3.5, we have $f(x_1) < x_1$ and $f(x_1) < z_1 \le x_1$. By Lemma 3.3 (i), we obtain that $f(x_n) < x_n$ for all $n \ge 1$. Moreover, we can show that $f(z_n) < z_n$ and $f(y_n) < y_n$ for all $n \ge 1$. From the BC-iteration, we have

$$\begin{aligned} z_1' - z_1 &= (1-a_1')x_1' + a_1' f(x_1') - (1-a_1)x_1 - a_1 f(x_1) \\ &= (x_1' - x_1) + a_1'(f(x_1') - x_1') + a_1(x_1 - f(x_1)) \\ &= (a_1' - a_1)(f(x_1) - x_1) \le 0, \end{aligned}$$

that is $z_1' \le z_1$. Since f is nondecreasing, $f(z_1') \le f(z_1)$. So we get

$$\begin{aligned} y_1' - y_1 &= (1-b_1'-c_1')z_1' + b_1' f(z_1') + c_1' f(x_1') - (1-b_1-c_1)z_1 - b_1 f(z_1) - c_1 f(x_1) \\ &= (z_1' - z_1) + b_1'(f(z_1') - z_1') - b_1'(f(z_1) - z_1) + b_1'(f(z_1) - z_1) + c_1'(f(x_1') - z_1') - c_1'(f(x_1) - z_1) \\ &\quad + c_1'(f(x_1) - z_1) + b_1(z_1 - f(z_1)) + c_1(z_1 - f(x_1)) \\ &= (z_1' - z_1) + b_1'(f(z_1') - f(z_1)) + b_1'(z_1 - z_1') + (b_1' - b_1)(f(z_1) - z_1) + c_1'(f(x_1') - f(x_1)) + c_1'(z_1 - z_1') \\ &\quad + (c_1' - c_1)(f(x_1) - z_1) \\ &= (1-b_1'-c_1')(z_1' - z_1) + b_1'(f(z_1') - f(z_1)) + (b_1' - b_1)(f(z_1) - z_1) + c_1'(f(x_1') - f(x_1)) \\ &\quad + (c_1' - c_1)(f(x_1) - z_1) \le 0, \end{aligned}$$

which implies $y_1' \le y_1$ and $f(y_1') \le f(y_1)$. Noting $y_1 - f(y_1) > 0$ and $f(z_1) < y_1$, we have

$$\begin{aligned} x_2' - x_2 &= (1-\alpha_1'-\beta_1')y_1' + \alpha_1' f(y_1') + \beta_1' f(z_1') - (1-\alpha_1-\beta_1)y_1 - \alpha_1 f(y_1) - \beta_1 f(z_1) \\ &= (y_1' - y_1) + \alpha_1'(f(y_1') - y_1') + \beta_1'(f(z_1') - y_1') + \alpha_1(y_1 - f(y_1)) + \beta_1(y_1 - f(z_1)) \\ &= (y_1' - y_1) + \alpha_1'(f(y_1') - y_1') - \alpha_1'(f(y_1) - y_1) + \alpha_1'(f(y_1) - y_1) + \beta_1'(f(z_1') - y_1') - \beta_1'(f(z_1) - y_1) \\ &\quad + \beta_1'(f(z_1) - y_1) + \alpha_1(y_1 - f(y_1)) + \beta_1(y_1 - f(z_1)) \\ &= (y_1' - y_1) + \alpha_1'(f(y_1') - f(y_1)) + \alpha_1'(y_1 - y_1') + (\alpha_1' - \alpha_1)(f(y_1) - y_1) + \beta_1'(f(z_1') - f(z_1)) \\ &\quad + \beta_1'(y_1 - y_1') + (\beta_1' - \beta_1)(f(z_1) - y_1) \\ &= (1-\alpha_1'-\beta_1')(y_1' - y_1) + \alpha_1'(f(y_1') - f(y_1)) + (\alpha_1' - \alpha_1)(f(y_1) - y_1) + \beta_1'(f(z_1') - f(z_1)) \\ &\quad + (\beta_1' - \beta_1)(f(z_1) - y_1) \le 0, \end{aligned}$$

which also implies $x_2' \le x_2$. Assume that $x_k' \le x_k$. Since $f(x_k') \le f(x_k) < x_k$, we have $z_k' - z_k \le (1-a_k')(x_k' - x_k) + a_k'(f(x_k') - f(x_k)) \le 0$, that is $z_k' \le z_k$. Since $f(z_k') \le f(z_k) < z_k$, we have $y_k' - y_k = (1-b_k'-c_k')(z_k' - z_k) + b_k'(f(z_k') - f(z_k)) + (b_k' - b_k)(f(z_k) - z_k) + c_k'(f(x_k') - f(x_k)) + (c_k' - c_k)(f(x_k) - z_k) \le 0$. So $y_k' \le y_k$, and $f(y_k') \le f(y_k) < y_k$. We then obtain

$$\begin{aligned} x_{k+1}' - x_{k+1} &= (y_k' - y_k) + \alpha_k'(f(y_k') - f(y_k)) + \alpha_k'(y_k - y_k') + (\alpha_k' - \alpha_k)(f(y_k) - y_k) + \beta_k'(f(z_k') - f(z_k)) \\ &\quad + \beta_k'(y_k - y_k') + (\beta_k' - \beta_k)(f(z_k) - y_k) \\ &= (1-\alpha_k'-\beta_k')(y_k' - y_k) + \alpha_k'(f(y_k') - f(y_k)) + (\alpha_k' - \alpha_k)(f(y_k) - y_k) + \beta_k'(f(z_k') - f(z_k)) \\ &\quad + (\beta_k' - \beta_k)(f(z_k) - y_k) \le 0, \end{aligned}$$

which yields $x_{k+1}' \le x_{k+1}$. By mathematical induction, we have $x_n' \le x_n$ for all $n \ge 1$. We note that $U < x_1'$. By induction, we can show that $U \le x_n'$ for all $n \ge 1$. Hence, we have $|x_n' - p| \le |x_n - p|$ for all $n \ge 1$. Therefore $x_n' \to p$ and $\{x_n'\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$.

Case 2: $x_1 = x_1' < L$. From Proposition 3.6, we get $f(x_1) > x_1$. In the same way as Case 1, we can show that $x_n' \ge x_n$ for all $n \ge 1$. We note that $x_1' < L$. By induction, we can show that $x_n' \le L$ for all $n \ge 1$. So $|x_n' - p| \le |x_n - p|$ for all $n \ge 1$. Hence $x_n' \to p$ and $\{x_n'\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$.

Case 3: $L \le x_1 = x_1' \le U$. Suppose that $f(x_1) \ne x_1$. If $f(x_1) < x_1$, then we have, by Lemma 3.3 (i), that $\{x_n\}_{n=1}^{\infty}$ is nonincreasing with limit p. We also have $p \le x_n'$ for all $n \ge 1$. By using the same argument as in Case 1, we can show that $x_n' \le x_n$ for all $n \ge 1$, so $p \le x_n' \le x_n$ for all $n \ge 1$. It follows that $|x_n' - p| \le |x_n - p|$ for all $n \ge 1$. Hence we have $x_n' \to p$ and $\{x_n'\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$. If $f(x_1) > x_1$, then we have, by Lemma 3.3 (ii), that $\{x_n\}_{n=1}^{\infty}$ is nondecreasing with limit p. We also have $p \ge x_n'$ for all $n \ge 1$. By using the same argument as in Case 2, we can show that $x'_n \geq x_n$ for all $n \geq 1$, so $p \geq x'_n \geq x_n$ for all $n \geq 1$. It follows that $|x'_n - p| \leq |x_n - p|$ for all $n \geq 1$. Hence we obtain that $x'_n \to p$ and $\{x'_n\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$. □

Corollary 4.2. *Let $f : [a,b] \to [a,b]$ be a continuous function. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{a'_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$, $\{c'_n\}_{n=1}^{\infty}$, $\{\alpha'_n\}_{n=1}^{\infty}$ and $\{\beta'_n\}_{n=1}^{\infty}$ be sequences in $[0,1)$ with $0 \leq b_n + c_n < 1$, $0 \leq \alpha_n + \beta_n < 1$, $0 \leq a'_n + b'_n < 1$ and $0 \leq \alpha'_n + \beta'_n < 1$ such that $a_n \leq a'_n$, $b_n \leq b'_n$, $c_n \leq c'_n$, $\alpha_n \leq \alpha'_n$ and $\beta_n \leq \beta'_n$ for all $n \geq 1$. For $x'_1 = x_1 \in [a,b]$, let $\{x_n\}_{n=1}^{\infty}$ and $\{x'_n\}_{n=1}^{\infty}$ be defined by $BC(x_1, a_n, b_n, c_n, \alpha_n, \beta_n, f)$ and $BC(x'_1, a'_n, b'_n, c'_n, \alpha'_n, \beta'_n, f)$, respectively. If $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$, then $\{x'_n\}_{n=1}^{\infty}$ converges to p. Moreover, $\{x'_n\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$.*

5. Numerical Examples

In this section, we demonstrate numerical examples to support our main results.

Example 5.1. *Let $f : [0,2] \to [0,2]$ be defined by $f(x) = (2x^3 - x^2 + \sin\frac{x}{2})/10$. Then f is continuous and nondecreasing. Use the initial point $u_1 = s_1 = l_1 = w_1 = x_1 = 1$ and the control conditions $a_n = \frac{1}{(n+1)^{1.5}}$, $b_n = \frac{1}{(n+1)^3}$, $c_n = \frac{1}{(n+1)^2}$, $\alpha_n = \frac{1}{(n+1)^{0.5}}$ and $\beta_n = \frac{1}{(n+1)^2}$.*

	Mann	Ishikawa	Noor	KY-iteration	BC-iteration	
n	u_n	s_n	l_n	w_n	x_n	$\lvert f(x_n) - x_n \rvert$
1	1.000000	1.000000	1.000000	1.000000	1.000000	0.852057
5	0.052708	0.048564	0.048185	0.007517	0.002162	0.002054
10	0.006520	0.006008	0.005962	0.000820	0.000171	0.000163
15	0.001399	0.001290	0.001280	0.000169	0.000031	0.000029
20	0.000392	0.000362	0.000359	0.000047	0.000008	0.000007
25	0.000130	0.000120	0.000119	0.000015	0.000002	0.000002
30	0.000048	0.000044	0.000044	0.000006	0.000001	0.000001
35	0.000020	0.000018	0.000018	0.000002	0.000000	0.000000
40	0.000008	0.000008	0.000008	0.000001	0.000000	0.000000
45	0.000004	0.000004	0.000004	0.000000	0.000000	0.000000
50	0.000002	0.000002	0.000002	0.000000	0.000000	0.000000
55	0.000001	0.000001	0.000001	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 1 Comparison of the convergence rate between Mann, Ishikawa, Noor, KY-iteration and BC-iteration

Remark 5.2. *From Table 1, we see that the BC-iteration converges significantly to a fixed point $p = 0$ of f faster than Mann, Ishikawa, Noor and KY-iteration.*

We end this section by giving numerical examples for the convergence speed of our algorithm.

Example 5.3. *Let $f : [-1,2] \to [-1,2]$ be defined by $f(x) = (\sqrt{x^5 + 1})/5$. Use the initial point $x_1 = x'_1 = 2$ and the control conditions $a'_n = \frac{1}{(n+1)^{1.2}}$, $b'_n = \frac{1}{(n+1)^3}$, $c'_n = \frac{1}{(n+1)^2}$, $\alpha'_n = \frac{1}{(n+1)^{0.25}}$, $\beta'_n = \frac{1}{(n+1)^2}$, $a_n = \frac{1}{(n+1)^{1.5}}$, $b_n = \frac{1}{(n+1)^4}$, $c_n = \frac{1}{(n+1)^3}$, $\alpha_n = \frac{1}{(n+1)^{0.5}}$ and $\beta_n = \frac{1}{(n+1)^3}$.*

	BC-iteration		BC′-iteration	
n	x_n	$\lvert f(x_n) - x_n\rvert$	x'_n	$\lvert f(x'_n) - x'_n\rvert$
1	2.000000	0.851087	2.000000	0.851087
5	0.240367	0.040287	0.200957	0.000924
10	0.203369	0.003335	0.200036	0.000004
12	0.201602	0.001569	0.200033	0.000001
13	0.201141	0.001108	0.200032	0.000000
15	0.200613	0.000580	0.200032	0.000000
20	0.200173	0.000141	0.200032	0.000000
25	0.200074	0.000042	0.200032	0.000000
30	0.200046	0.000014	0.200032	0.000000
35	0.200037	0.000005	0.200032	0.000000
40	0.200034	0.000002	0.200032	0.000000
45	0.200033	0.000001	0.200032	0.000000
50	0.200032	0.000000	0.200032	0.000000

Table 2 Comparison of the convergence speed

Remark 5.4. *From Table 2, we see that the BC′-iteration converges to a fixed point $p \approx 0.200032$ faster than the BC-iteration.*

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