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Normal Functions and Shared Sets

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Abstract. In this paper, we obtain some criteria for normal functions that share sets with their derivatives.

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$. A family $\mathcal{F}$ of meromorphic functions defined on D is said to be normal on D, in the sense of Montel, if for any sequence $\{f_n\}$ in $\mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$, such that $\{f_{n_j}\}$ converges spherically locally uniformly on D to a meromorphic function or ∞(see [2,8,10]). A function f meromorphic in the unit disc $\Delta = \{z : |z| < 1\}$ is called to a normal function if and only if the family $\{f(S(z))\}$, where $z' = S(z)$ denotes an arbitrary one-one conformal mapping of Δ onto itself, is normal(see [4]).

Obviously, there exists a close relation between normal families and normal functions, and it is natural to expect the criteria of normal functions corresponding to the known criteria of normal families. As we know, Montel's theorem asserts that a family of meromorphic functions $\mathcal{F}$ is normal in a domain D if there are three distinct points a, b, c in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that each $f \in \mathcal{F}$ omits a, b, c in D. The corresponding result for normal functions is due to Lehto and Virtanen [4], which states that a function f meromorphic in Δ is normal if there are three distinct points a, b, c in the extended complex plane $\widehat{\mathbb{C}}$ such that $f \neq a, b, c$ in Δ.

However, this is not always true, especially for relating to derivatives. For example, Miranda proved that the family $\mathcal{F}$ of all holomorphic functions $f(z)$, such that $f \neq 0, f' \neq 1$ in Δ forms a normal family, which is called Miranda criterion; but an example given by Hayman and Storvick [3] shows that there exists non-normal function f such that $f \neq 0$ and $f' \neq 1$ in Δ.

Let f, g be two meromorphic functions on a domain $D \subset \mathbb{C}$, a, b be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we denote it by $f(z) = a \Rightarrow g(z) = b$ in D. If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = b$ in D. the functions f and g are said to share the value a if $f(z) = a \Leftrightarrow g(z) = a$ in D. Let $S_1, S_2 \subset \mathbb{C}$. We define

$$D(f, S_1) := \bigcup_{a \in S_1} \{z \in D : f(z) = a\},$$

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$$D(g,S_2) := \bigcup_{a\in S_2}\{z \in D : g(z) = a\}.$$

If $D(f,S_1) \subset D(g,S_2)$, we denote it by $f(z) \in S_1 \Rightarrow g(z) \in S_2$ in D. Furthermore, if $D(f,S_1) \subset D(g,S_2)$ and $D(g,S_2) \subset D(f,S_1)$, that is, $D(f,S_1) = D(g,S_2)$, we write $f(z) \in S_1 \Leftrightarrow g(z) \in S_2$ in D. the functions f and g are said to share the set S in D if $f(z) \in S \Leftrightarrow g(z) \in S$ in D.

In 1992, Schwick [9] studied the relation between normal families and shared values. He proved the following result.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, and let a_1, a_2, a_3 be distinct complex numbers. If, for every function $f \in \mathcal{F}$, f and f' share a_1, a_2, a_3, then $\mathcal{F}$ is normal in D.*

Liu-Pang [5] improved the above result, by replacing 'share value a_1, a_2, a_3' by 'share the set $\{a_1, a_2, a_3\}$' in Theorem A, as follows.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, and let $S = \{a_1, a_2, a_3\}$ be a set in $\mathbb{C}$. If, for every function $f \in \mathcal{F}$, f and f' share S, then $\mathcal{F}$ is normal in D.*

Set

$$\Omega_0 = \{z \in \widehat{\mathbb{C}}; z \in \widehat{\Delta}(1,1) \text{ } or \text{ } 1/z \in \widehat{\Delta}(1,1)\},$$

where $\widehat{\Delta}(1,1) = \{z \in \mathbb{C}; |z-1| < 1 \text{ } or \text{ } (z-1)^k = 1 \text{ } for\ some\ positive\ integer\ k\}$. It is not difficult to see that nonnegative real values and ∞ are in Ω_0, while negative real values are not.

Recently, Chang and Wang [1] proved that the 3-element set in Theorem B can be reduced to a 2-element set.

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, and let $S = \{a_1, a_2\}$, where a_1, a_2 are nonzero constants such that $a_1/a_2 \in \Omega_0$,. If, for every function $f \in \mathcal{F}$, f and f' share S, then $\mathcal{F}$ is normal in D.*

In this paper, corresponding to the above results, we prove some criteria for normal functions that share sets with their derivatives.

Theorem 1.1. *Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$ be two sets in $\mathbb{C}$ such that $a_1 a_2 \neq 0$ and $b_1/b_2 \notin \mathbb{Z}^- \cup 1/\mathbb{Z}^-$. Let f be a meromorphic function in the unit disc Δ, and suppose that there exists a positive number M such that $|f'(z)| \leq M$ whenever $f(z) = 0$. If $f \in S_1 \Leftrightarrow f' \in S_2$ in Δ, then f is normal.*

Here $\mathbb{Z}^-$ denotes the set of all negative integers, and $1/\mathbb{Z}^-$ stands for the set $\{1/k; k \in \mathbb{Z}^-\}$.

Theorem 1.2. *Let $S_1 = \{a_1, a_2, a_3\}$ and $S_2 = \{b_1, b_2, b_3\}$ be two sets in $\mathbb{C}$. Let f be a meromorphic function in the unit disc Δ. If $f \in S_1 \Leftrightarrow f' \in S_2$ in Δ, then f is normal.*

If one is a two-element set and the other is a three-element set, we have the following results.

Theorem 1.3. *Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2, b_3\}$ be two sets in $\mathbb{C}$ such that $a_1 a_2 \neq 0$. Let f be a meromorphic function in the unit disc Δ. If there exists a positive number M such that $|f'(z)| \leq M$ whenever $f(z) = 0$, and $f \in S_1 \Leftrightarrow f' \in S_2$ in Δ, then f is normal.*

Theorem 1.4. *Let $S_1 = \{a_1, a_2, a_3\}$ and $S_2 = \{b_1, b_2\}$ be two sets in $\mathbb{C}$ such that $b_1/b_2 \notin \mathbb{Z}^- \cup 1/\mathbb{Z}^-$. Let f be a meromorphic function in the unit disc Δ. If $f \in S_1 \Leftrightarrow f' \in S_2$ in Δ, then f is normal.*

The following are direct consequences of Theorems 1 and 2.

Corollary 1.5. *Let $S = \{a_1, a_2\}$ be a set in $\mathbb{C}$ such that $a_1 a_2 \neq 0$ and $a_1/a_2 \notin \mathbb{Z}^- \cup 1/\mathbb{Z}^-$, and f be a meromorphic function in the unit disc Δ, and suppose that there exists a positive number M such that $|f'(z)| \leq M$ whenever $f(z) = 0$. If f and f' share S in Δ, then f is normal.*

Corollary 1.6. *Let $S = \{a_1, a_2, a_3\}$ be a set in $\mathbb{C}$, f be a meromorphic function in the unit disc Δ. If f and f' share S in Δ, then f is normal.*

2. Lemmas

To prove our results, we need some preliminaries. The next is the well-known Lohwater-Pommerenke's theorem [6].

Lemma 2.1. *Let f be a function meromorphic in the unit disc Δ. If f is not normal, then there exist a sequence of points $z_n \in \Delta$, and a sequence of positive numbers ρ_n with $\rho_n \to 0$ such that $g_n(z) = f(z_n + \rho_n z)$ converges spherically uniformly to a non-constant meromorphic function on each compact subset of the complex plane $\mathbb{C}$.*

The following is the local version of Zalcman's lemma, which is due to Pang and Zalcman [7].

Lemma 2.2. *Let k be a positive integer and let $\mathcal{F}$ be a family of meromorphic functions in a domain D, such that each function $f \in \mathcal{F}$ has only zeros with multiplicities at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0, f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each α, $0 \leq \alpha \leq k$, there exist a sequence of complex numbers $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\xi) \leq g^{\#}(0) = kA + 1$. Moreover, $g(\xi)$ has order at most 2.

Here as usual, $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ is the spherical derivative of f.

Lemma 2.3. *(see [2,8,10]) Let f be a transcendental meromorphic function. Then f or $f' - 1$ has infinitely many zeros.*

Lemma 2.4. *(see [1]) Let P be a nonconstant polynomial of degree k, and a, b two distinct nonzero finite numbers. If $P(z) = 0$ if and only if $P'(z) \in \{a, b\}$, then $k \geq 2$ and either $a + (k-1)b = 0$ or $(k-1)a + b = 0$.*

3. Proof of Theorems

Proof of Theorem 1. Assume that f is not a normal function. Then, by Lemma 1, there exist points $z_n \in \Delta$, positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = f(z_n + \rho_n \zeta) \to g(\zeta) \tag{1}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$.

If $g(\zeta') = 0$, Hurwitz's theorem (1) imply that there exist points $\zeta_n \to \zeta'$ such that $f(z_n + \rho_n \zeta_n) = 0$. Then by the the assumptions given, $|f'(z_n + \rho_n \zeta_n)| \leq M$, and thus

$$|g'(\zeta_n)| = \rho_n |f'(z_n + \rho_n \zeta_n)| \leq \rho_n M.$$

It follows that $g'(\zeta') = 0$. We know that all zeros of g are multiple.

By Nevanlinna's second fundamental theorem, either $g - a_1$ or $g - a_2$ has zeros. Without loss of generality, we assume that ζ_0 is a zero of $g - a_1$ with multiplicity k. Then there exists $\delta > 0$ such that g_n (for sufficiently large n) is holomorphic on $\Delta(\zeta_0, \delta) = \{\zeta : |\zeta - \zeta_0| < \delta\}$.

Set

$$h_n(\zeta) = \frac{g_n(\zeta) - a_1}{\rho_n}. \tag{2}$$

Clearly, $\{h_n\}$ is well defined and holomorphic on $\Delta(\zeta_0, \delta)$. We claim that $\{h_n\}$ is not normal at ζ_0. Indeed, suppose that $\{h_n\}$ is normal at ζ_0. By the definition, there exist $0 < \delta_1 < \delta$ and a subsequence of $\{h_n\}$ which (to avoid complication in notation) we still denote by $\{h_n\}$, such that $\{h_n\}$ converges uniformly in $\Delta(\zeta_0, \delta_1)$ to a holomorphic function h or ∞. Noting that $g(\zeta_0) = a_1$ and g is no constant, there exists $\zeta_0' \in \Delta(\zeta_0, \delta_1)$ such that $\zeta_0' \neq \zeta_0$ and $g(\zeta_0') \neq a_1$, and then $|g_n(\zeta_0') - a_1| > |g(\zeta_0') - a_1|/2 > 0$ for sufficiently large n. It follows that

$$|h_n(\zeta_0')| = \frac{|g_n(\zeta_0') - a_1|}{\rho_n} > \frac{|g(\zeta_0') - a_1|}{2\rho_n} \to \infty.$$

Thus $h_n \to \infty$ in $\Delta(\zeta_0, \delta_1)$. On the other hand, by Hurwitz's theorem, we may find points $\zeta_n \to \zeta_0$ such that(for sufficiently large n) $g_n(\zeta_n) - a_1 = 0$, and hence

$$h(\zeta_0) = \lim_{n\to\infty} h_n(\zeta_n) = \lim_{n\to\infty} \frac{g_n(\zeta_n) - a_1}{\rho_n} = 0,$$

a contradiction.

We also claim that $|h_n'(\zeta)| \le |b_1| + |b_2|$ whenever $h_n(\zeta) = 0$. In fact, if $h_n(\zeta) = 0$, by (1) and (2), $f(z_n + \rho_n\zeta) = a_1 \in S_1$. Since $f \in S_1 \Rightarrow f' \in S_2$, $f'(z_n + \rho_n\zeta) \in S_2$. Then $|h_n'(\zeta)| = |f'(z_n + \rho_n\zeta)| \le |b_1| + |b_2|$.

Thus applying for Lemma 2, we can extract a subsequence of $\{h_n\}$ (which, renumbering, we continue to call $\{h_n\}$), points $\zeta_n \to \zeta_0$, and positive numbers $\sigma_n \to 0$ such that

$$H_n(\xi) = \frac{h_n(\zeta_n + \sigma_n\xi)}{\sigma_n} = \frac{g_n(\zeta_n + \sigma_n\xi) - a_1}{\sigma_n\rho_n} \to H(\xi) \tag{3}$$

locally uniformly with respect to the spherical metric, where H is a nonconstant meromorphic function on $\mathbb{C}$. Moreover, $H^{\#}(\xi) \le H^{\#}(0) = |b_1| + |b_2| + 1$.

Claim: (I) H is entire; (II) H has at most k distinct zeros; (III) $H(\xi) = 0$ if and only if $H'(\xi) \in S_2$.

Since $\{h_n\}$ is is holomorphic on $\Delta(\zeta_0, \delta)$, and $\zeta_n + \sigma_n\xi \to \zeta_0$ for each $\xi \in \mathbb{C}$, we see from (3) that H is entire on $\mathbb{C}$. (I) is proved.

Suppose that H has (at least) $k+1$ distinct zeros: $\xi_1, \xi_2, \cdots, \xi_{k+1}$. By Hurwitz's theorem and (3), we can find $k+1$ distinct sequences $\{\xi_{nj}\}$ such that $\xi_{nj} \to \xi_j$ and $H_n(\xi_{nj}) = 0$ ($j = 1, 2, \ldots, k+1$). Then $g_n(\zeta_n + \sigma_n\xi_{nj}) - a_1 = 0$. Noting that $\zeta_n + \sigma_n\xi_{nj} \to \zeta_0$ and $\zeta_n + \sigma_n\xi_{ni} \neq \zeta_n + \sigma_n\xi_{nj}$ for $1 \le i < j \le k+1$, it follows from (1) that, as a zero of $g - a_1$, ζ_0 must have multiplicity at least $k+1$, we arrive at a contradiction since ζ_0 is a zero of $g - a_1$ of multiplicity k. Thus (II) is proved.

Let $H(\xi_0) = 0$. In view of $H \not\equiv 0$ and (3), Hurwitz's theorem implies that there exist points $\xi_n \to \xi_0$ such that $H_n(\xi_n) = 0$. It follows that $g_n(\zeta_n + \sigma_n\xi_n) - a_1 = 0$, and hence

$$f(z_n + \rho_n\zeta_n + \rho_n\sigma_n\xi_n) = a_1 \in S_1.$$

Since $f \in S_1 \Rightarrow f' \in S_2$, we have

$$H_n'(\zeta_n) = f'(z_n + \rho_n\zeta_n + \rho_n\sigma_n\xi_n) \in S_2.$$

In view of $H_n'(\xi_n) \to H'(\xi_0)$, we get $H'(\xi_0) \in S_2$.

Conversely, suppose that $H'(\xi_0) \in S_2$, say $H'(\xi_0) = b_1$. We have $H'(\xi) \not\equiv b_1$. For otherwise $H(\xi) = b_1\xi + c$ with $c \in \mathbb{C}$, then $H^{\#}(0) = |b_1|/(1 + |c|^2) < |b_1| + |b_2| + 1$, a contradiction. Thus by Hurwitz's theorem, there exist $\xi_n \to \xi_0$ such that $H_n'(\xi_n) = b_1$, so that

$$f'(z_n + \rho_n\zeta_n + \rho_n\sigma_n\xi_n) = b_1 \in S_2.$$

Since $f \in S_1 \Leftarrow f' \in S_2$, $f(z_n + \rho_n\zeta_n + \rho_n\sigma_n\xi_n) \in S_1$.

If there exists $N > 0$ such that $f(z_n + \rho_n\zeta_n + \rho_n\sigma_n\xi_n) = a_2$ for $n > N$, then we get from (3) that

$$\begin{aligned} H(\xi_0) &= \lim_{n\to\infty} H_n(\xi_n) \\ &= \lim_{n\to\infty} \frac{f(z_n + \rho_n\zeta_n + \rho_n\sigma_n\xi_n) - a_1}{\rho_n\sigma_n} \\ &= \lim_{n\to\infty} \frac{a_2 - a_1}{\rho_n\sigma_n} = \infty, \end{aligned}$$

violating the fact that $H'(\xi_0) = b_1$. Thus, there exists a subsequence which we continue to denote by $\{f(z_n + \rho_n\zeta_n + \rho_n\sigma_n\xi_n)\}$ such that

$$f(z_n + \rho_n\zeta_n + \rho_n\sigma_n\xi_n) = a_1.$$

Hence we have

$$H(\xi_0) = \lim_{n\to\infty} H_n(\xi_n) = \lim_{n\to\infty} \frac{f(z_n + \rho_n\zeta_n + \rho_n\sigma_n\xi_n) - a_1}{\rho_n\sigma_n} = 0.$$

This completes the proof of (III).

Now (II) and (III) imply that both $H' - b_1$ and $H' - b_2$ have only finitely many zeros. Then, we know from (I) and Lemma 3 that H is a polynomial.

If $b_1b_2 \neq 0$, Lemma 4 implies that $b_1/b_2 \in \mathbb{Z}^- \cup 1/\mathbb{Z}^-$, a contradiction. Then one of b_1, b_2 must be zero, say $b_1 = 0$, that is, $S_2 = \{0, b_2\}$. We claim that $\deg H \geq 2$. Indeed, otherwise H would be a polynomial of degree 1, and then (ii) implies that $H(\xi) = b_2\xi + c$, where c is a constant. Thus $H'(\xi) = b_2 \in S_2$ for each $\xi \in \mathbb{C}$, whereas H has only one zero at $-c/b_2$, a contradiction.

So $H'' \not\equiv 0$. Set

$$P = \frac{HH''}{H'(H' - b_2)}. \tag{4}$$

Clearly, $P(\not\equiv 0)$ is a rational function, and its poles arise only from the zeros of H' and $H' - b$. Firstly, if ξ_0 is a zero of H' of multiplicity n, then ξ_0 is a zero of H'' of multiplicity $n - 1$. By (III), ξ_0 is also a zero of H of multiplicity $n + 1$. Then $P(\xi_0) = 0$. This means that the zero of H' is also the zero of P. Secondly, if ξ_0 is a zero of $H' - b_2$ with multiplicity m, then ξ_0 is a zero of H'' of multiplicity $m - 1$. Aging by (III), ξ_0 is a simple zero of H. It follows that $P(\xi_0) \neq 0, \infty$. Therefore, P has no poles, so that P is a polynomial.

By (4), $PH'(H' - b_2) = HH''$. Comparing the degree of both side gives $\deg P = 0$, so that P is nonzero constant. It follows (from what has been proved above) that H' has no zero. But this contradicts the fact that H is a polynomial of degree $\deg H \geq 2$. Theorem 1 is thus proved. □

Proof of Theorem 2. Assume that f is not a normal function. By Lemma 1, there exist points $z_n \in \Delta$, positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = f(z_n + \rho_n\zeta) \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$.

We first prove that all zeros of $g - a_i$ (for each $a_i \in S_1$) are multiple. Suppose that $g(\zeta') - a_1 = 0$. By Hurwitz's theorem, there exist points $\zeta_n \to \zeta'$ such that $f(z_n + \rho_n\zeta_n) = a_1 \in S_1$. Since $f \in S_1 \Rightarrow f' \in S_2$, $f'(z_n + \rho_n\zeta_n) \in S_2$, and then

$$|g'(\zeta_n)| = \rho_n|f'(z_n + \rho_n\zeta_n)| \leq \rho_n(|b_1| + |b_2| + |b_3|).$$

It follows that $g'(\zeta') = 0$. We see that all zeros of $g - a_1$ are multiple. Similarly, all zeros of $g - a_i (i = 2, 3)$ are also multiple.

Then, by Nevanlinna's second fundamental theorem, at least one of $g - a_i (i = 1, 2, 3)$ must have zeros. Without loss of generality, we assume that ζ_0 is a zero of $g - a_1$ with multiplicity k.

Similarly, we set

$$h_n(\zeta) = \frac{g_n(\zeta) - a_1}{\rho_n}.$$

As in the proof of Theorem 1, we know that $\{h_n\}$ is not normal at ζ_0, and $|h_n'(\zeta)| \leq |b_1| + |b_2| + |b_3|$ whenever $h_n(\zeta) = 0$. By Lemma 2, we can extract a subsequence of $\{h_n\}$ (which, renumbering, we continue to call $\{h_n\}$), points $\zeta_n \to \zeta_0$, and positive numbers $\sigma_n \to 0$ such that

$$H_n(\xi) = \frac{h_n(\zeta_n + \sigma_n\xi)}{\sigma_n} = \frac{g_n(\zeta_n + \sigma_n\xi) - a_1}{\sigma_n\rho_n} \to H(\xi)$$

locally uniformly with respect to the spherical metric, where H is a nonconstant meromorphic function on $\mathbb{C}$ such that $H^{\#}(\xi) \le H^{\#}(0) = |b_1| + |b_2| + |b_3| + 1$.

Also as in the proof of Theorem 1, we have (I′) H is entire; (II′) H has at most k distinct zeros; (III′) $H(\xi) = 0$ if and only if $H'(\xi) \in S_2$. These and Lemma 3 imply that H is a polynomial.

Let

$$H(\xi) = c_n\xi^n + c_{n-1}\xi^{n-1} + \cdots + c_0,$$

where n is a positive integer, and $c_0, c_1, \ldots, c_n (\neq 0)$ are constants. Then, by Nevanlinna's second fundamental theorem,

$$2T(r, H') \le \sum_{i=1}^{3} N(r, \frac{1}{H' - b_i}) + S(r, H'). \tag{5}$$

By (III′), we have

$$\sum_{i=1}^{3} N(r, \frac{1}{H' - b_i}) \le N(r, \frac{1}{H}) = n \log r. \tag{6}$$

Clearly, $T(r, H') = (n - 1) \log r$ and $S(r, H') = O(1)$. Substituting this and (6) in (5) gives

$$(n - 2) \log r \le O(1), \text{ as } r \to \infty,$$

so that $n \le 2$.

If $n = 1$, then $H(\xi) = c_1\xi + c_0$ and $H'(\xi) = c_1$, where $c_1 \neq 0$. It follows from (III′) that $c_1 \in S_2$. Then $H'(\xi) \in S_2$ for each $\xi \in \mathbb{C}$, but $H(\xi)$ has only one zero, which contradicts (III′).

If $n = 2$, then $H(\xi) = c_2\xi^2 + c_1\xi + c_0 (c_2 \neq 0)$, and $H'(\xi) = 2c_2\xi + c_1$. It follows that $H'(\xi_i) \in S_2$, where $\xi_i = (b_i - c_1)/2c_2 (i = 1, 2, 3)$ are three distinct numbers. But H has at most two zeros, which also contradicts (III′). Theorem 2 is thus proved. □

Proof of Theorem 3 and Theorem 4. Theorem 3 and Theorem 4 can be proved by using the same argument as in Theorem 1 and Theorem 2. We here omit the details. □

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