



## Solution of the Ulam Stability Problem for Euler-Lagrange-Jensen $k$ -Cubic Mappings

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**Abstract.** The “oldest cubic” functional equation was introduced and solved by the second author of this paper (see: Glas. Mat. Ser. III 36(56) (2001), no. 1, 63-72). which is of the form:

$$f(x + 2y) = 3f(x + y) + f(x - y) - 3f(x) + 6f(y).$$

For further research in various normed spaces, we are introducing new cubic functional equations, and establish fundamental formulas for the general solution of such functional equations and for the “Ulam stability” of pertinent cubic functional inequalities.

### 1. Introduction

In 1940, Ulam [39] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following famous “stability Ulam question”:

We are given a group  $G$  and a metric group  $G'$  with metric  $\rho(.,.)$ . Given  $\varepsilon > 0$ , does there exist a  $\delta > 0$  such that if  $f : G \rightarrow G'$  satisfies  $\rho(f(xy), f(x)f(y)) < \delta$  for all  $x, y$  in  $G$ , then a homomorphism  $h : G \rightarrow G'$  exists with  $\rho(f(x), h(x)) < \varepsilon$  for all  $x \in G$ ?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an  $f : G \rightarrow G'$  an approximate homomorphism.

In 1941, Hyers [8] considered the case of approximately additive mappings  $f : E \rightarrow E'$ , where  $E$  and  $E'$  are Banach spaces and  $f$  satisfies the following Hyers' inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon,$$

for all  $x, y \in E$ . It was shown that the limit

$$L(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$$

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exists for all  $x \in E$  and that  $L : E \rightarrow E'$  is the unique additive mapping satisfying  $\|f(x) - L(x)\| \leq \varepsilon$ .

No continuity conditions are required for this result, but if  $f(tx)$  is continuous in the real variable  $t$  for each fixed  $x$ , then  $L$  is linear, and if  $f$  is continuous at a single point of  $E$  then  $L : E \rightarrow E'$  is also continuous.

In 1982-1994, a generalization of this result was proved by the author J. M. Rassias [26–28, 30, 33], as follows. He introduced the following weaker condition (or weaker inequality or the generalized Cauchy inequality)

$$\|f(x+y) - [f(x) + f(y)]\| \leq \theta \|x\|^p \|y\|^q,$$

for all  $x, y \in E$  controlled by (or involving) a product of different powers of norms, where  $\theta \geq 0$  and real  $p, q : r = p + q \neq 1$ , and retained the condition of continuity of  $f(tx)$  in  $t$  for fixed  $x$ . Besides he investigated that it is possible to replace  $\varepsilon$  in the above Hyers' inequality, by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. In all the cases investigated in these results, the approach to the existence question was to prove asymptotic type formulas:

$$L(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x); \quad L(x) = \lim_{n \rightarrow \infty} 2^n f(2^{-n} x).$$

**Theorem.** (J. M. Rassias: 1982-1994). Let  $X$  be a real normed linear space and let  $Y$  be a real complete normed linear space. Assume in addition that  $f : X \rightarrow Y$  is an approximately additive mapping for which there exist constants  $\theta \geq 0$  and  $p, q \in \mathbb{R}$  such that  $r = p + q \neq 1$  and  $f$  satisfies the "generalized Cauchy inequality"

$$\|f(x+y) - [f(x) + f(y)]\| \leq \theta \|x\|^p \|y\|^q,$$

for all  $x, y \in X$ . Then there exists a unique additive mapping  $L : X \rightarrow Y$  satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|} \|x\|^r$$

for all  $x \in X$ . If, in addition,  $f : X \rightarrow Y$  is a mapping such that the transformation  $t \rightarrow f(tx)$  is continuous in  $t \in \mathbb{R}$  for each fixed  $x \in X$ , then  $L$  is an  $\mathbb{R}$ -linear mapping.

In 1940, Ulam [39] proposed the "Ulam stability problem": When does a linear transformation near an "approximately linear" transformation exist?. Since then, many specialists on this "famous Ulam problem", have investigated interesting functional equations, for instance: Hyers [8], Aoki [3], Gajda [5], Gávruta [6], Rassias et al. [29, 31, 36], Jung [10, 11], Jun and Kim [9], Part et al. [21, 22, 24], Mirmostafae and Moslehian [12], Rätz [37], and others. Also very interesting results on additive, quadratic and cubic functional equations have been achieved by Mohiuddine et al. [1, 2, 13–18, 20]. The "oldest cubic" functional equation was introduced and solved by the second author [34] of this paper which is of the form:

$$f(x+2y) = 3f(x+y) + f(x-y) - 3f(x) + 6f(y).$$

Since then various "cubic" equations have been proposed and solved by a number of experts in the area of functional equations and inequalities (see also [4, 7, 19, 23, 25, 32, 35, 38, 40–42]). For further research in various normed spaces, we are introducing new cubic functional equations, and establish fundamental formulas for the general solution of such functional equations and for the "Ulam stability" of pertinent cubic functional inequalities.

## 2. Euler-Lagrange-Jensen $k$ -cubic Mappings

Let  $X$  be a real normed linear space and let  $Y$  be a real complete normed linear space. Let us introduce Euler-Lagrange-Jensen  $k$ -cubic mapping  $f : X \rightarrow Y$ , satisfying the following Euler-Lagrange-Jensen  $k$ -cubic functional equation

$$k[f(kx+y) + f(x+ky)] + (k-1)^3 \left[ f\left(\frac{kx-y}{k-1}\right) + f\left(\frac{x-ky}{1-k}\right) \right]$$

$$= (k^4 - 1)[f(x) + f(y)] + 8k(k^2 + 1)f\left(\frac{x+y}{2}\right), \quad (2.1)$$

where  $m = k + 1 \neq 0$ ;  $m \neq \pm 1$ .

Replacing  $x = y = 0$  in (2.1), one gets

$$2k^2(k^2 + 3k + 3)f(0) = 0, \quad (2.2)$$

or

$$2(m - 1)^2(m^2 + m + 1)f(0) = 0, \quad (2.3)$$

or

$$f(0) = 0, \quad (2.4)$$

if  $k \neq 0$ ;  $m \neq 1$ .

Replacing  $x = x$ ,  $y = x$  in (2.1), one obtains

$$2k[f((k+1)x) - (k^3 + 3k + 3k + 1)f(x)] = 0, \quad (2.5)$$

or

$$2(m - 1)[f(mx) - m^3 f(x)] = 0, \quad (2.6)$$

or

$$f(mx) = m^3 f(x). \quad (2.7)$$

Replacing  $x \rightarrow m^{-1}x$  in (2.7), we get

$$f(x) = m^3 f(m^{-1}x). \quad (2.8)$$

### 3. Approximate Euler-Lagrange-Jensen $k$ -cubic Mappings

Let  $X$  be a real normed linear space and let  $Y$  be a real complete normed linear space. Let us introduce approximate Euler-Lagrange-Jensen  $k$ -cubic mappings  $f : X \rightarrow Y$ , satisfying the following Euler-Lagrange-Jensen  $k$ -cubic functional inequality

$$\left\| k[f(kx + y) + f(x + ky)] + (k - 1)^3 \left[ f\left(\frac{kx - y}{k - 1}\right) + f\left(\frac{x - ky}{1 - k}\right) \right] - (k^4 - 1)[f(x) + f(y)] - 8k(k^2 + 1)f\left(\frac{x + y}{2}\right) \right\| \leq c, \quad (3.1)$$

where  $c (= \text{const.}) > 0$ , and

$$m = k + 1 \neq 0; m \neq \pm 1.$$

Replacing  $x = y = 0$  in the above inequality (3.1), one gets

$$2k^2(k^2 + 3k + 3)\|f(0)\| \leq c, \quad (3.2)$$

or

$$2(m - 1)^2(m^2 + m + 1)\|f(0)\| \leq c, \quad (3.3)$$

or

$$\|f(0)\| \leq \frac{c}{2(m - 1)^2(m^2 + m + 1)}, \quad (3.4)$$

if  $k \neq 0$ ;  $m \neq 1$ .

Note that

$$m^2 + m + 1 = (k + 1)^2 + (k + 1) + 1 = k^2 + 3k + 3 > 0. \quad (3.5)$$

Replacing  $x = x$ ,  $y = x$  in the inequality (3.1), one obtains

$$2|k| \left\| f((k+1)x) - (k^3 + 3k + 3k + 1)f(x) \right\| \leq c, \quad (3.6)$$

or

$$2|m-1| \left\| f(mx) - m^3 f(x) \right\| \leq c, \quad (3.7)$$

or

$$\left\| f(mx) - m^3 f(x) \right\| \leq c_2 = \frac{c}{2|m-1|}, \quad (3.8)$$

or

$$\left\| f(x) - m^{-3} f(mx) \right\| \leq c_1 = \frac{c_2}{|m|^3} = \frac{c}{2|m|^3|m-1|}, \quad (3.9)$$

if  $m \in \mathbb{R} - \{0, \pm 1\}; |m| > 1$ .

Replacing  $x \rightarrow mx$  in (3.9) and then multiplying by  $|m|^{-3}$ , we find

$$\left\| m^{-3} f(mx) - m^{-6} f(m^2 x) \right\| \leq |m|^{-3} c_1, \quad m \neq 0. \quad (3.10)$$

Then

$$\begin{aligned} \left\| f(x) - m^{-6} f(m^2 x) \right\| &\leq \left\| f(x) - m^{-3} f(mx) \right\| + \left\| m^{-3} f(mx) - m^{-6} f(m^2 x) \right\| \\ &\leq (1 + |m|^{-3}) c_1, \end{aligned} \quad (3.11)$$

if  $m \in \mathbb{R} - \{0, \pm 1\}; |m| > 1$ .

Employing (3.9)-(3.10), without induction, we obtain

$$\begin{aligned} \left\| f(x) - m^{-3n} f(m^n x) \right\| &\leq \left\| f(x) - m^{-3} f(mx) \right\| + \left\| m^{-3} f(mx) - m^{-6} f(m^2 x) \right\| \\ &\quad + \dots + \left\| m^{-3(n-1)} f(m^{n-1} x) - m^{-3n} f(m^n x) \right\| \\ &\leq (1 + |m|^{-3} + \dots + |m|^{-3(n-1)}) c_1 \end{aligned}$$

or the following general inequality;

$$\left\| f(x) - m^{-3n} f(m^n x) \right\| \leq \frac{1 - |m|^{-3n}}{1 - |m|^{-3}} c_1 = \frac{1}{|m|^3 - 1} (1 - |m|^{-3n}) c_2, \quad (3.12)$$

where

$$|m| > 1, \quad c_2 = |m|^3 c_1 = \frac{c}{2|m-1|}. \quad (3.13)$$

Similarly for  $|m| < 1$ ;  $m \neq 0$ , we get:

$$\left\| f(x) - m^3 f(m^{-1} x) \right\| \leq c'_1 = \frac{c}{2|m-1|} = c_2, \quad (3.14)$$

by replacing  $x \rightarrow m^{-1} x$  in (3.7).

Replacing  $x \rightarrow m^{-1} x$  in (3.14) and then multiplying by  $|m|^3$ , we find

$$\left\| m^3 f(m^{-1} x) - m^6 f(m^{-2} x) \right\| \leq |m|^3 c'_1, \quad m \neq 0. \quad (3.15)$$

Then

$$\begin{aligned} \left\| f(x) - m^6 f(m^{-2} x) \right\| &\leq \left\| f(x) - m^3 f(m^{-1} x) \right\| + \left\| m^3 f(m^{-1} x) - m^6 f(m^{-2} x) \right\| \\ &\leq (1 + |m|^3) c'_1 \end{aligned} \quad (3.16)$$

if  $m \in \mathbb{R} - \{0, \pm 1\}; |m| < 1$ .

Therefore, without induction, we obtain

$$\begin{aligned} \|f(x) - m^{3n}f(m^{-n}x)\| &\leq \|f(x) - m^3f(m^{-1}x)\| + \|m^3f(m^{-1}x) - m^6f(m^{-2}x)\| \\ &\quad + \dots + \|m^{3(n-1)}f(m^{-(n-1)}x) - m^{3n}f(m^{-n}x)\| \\ &\leq (1 + |m|^3 + \dots + |m|^{3(n-1)})c'_1 \end{aligned}$$

or the following general inequality:

$$\|f(x) - m^{3n}f(m^{-n}x)\| \leq \frac{1}{1 - |m|^3}(1 - |m|^{3n})c_2, \tag{3.17}$$

where

$$|m| < 1, \quad m \in \mathbb{R} - \{0, \pm 1\}; \quad c_2 = c'_1 = \frac{c}{2|m - 1|}.$$

**Note 1.** (i) Let us denote

$$C_n : C_n(x) = m^{-3n}f(m^n x). \tag{3.18}$$

In fact, assuming  $|m| > 1$ , and  $j > i > 0$ , then from inequality (3.12) we get

$$\begin{aligned} 0 \leq \|C_i(x) - C_j(x)\| &= \|m^{-3i}f(m^i x) - m^{-3j}f(m^j x)\| \\ &= |m|^{-3i} \|f(m^i x) - m^{-3(j-i)}f(m^{j-i}m^i x)\| \\ &\leq \frac{1}{|m|^3 - 1} (|m|^{-3i} - |m|^{-3j})c_2 \rightarrow 0, \end{aligned} \tag{3.19}$$

as  $i \rightarrow \infty$  (and  $j \rightarrow \infty$ ).

(ii) Let us denote

$$C_n : C_n(x) = m^{3n}f(m^{-n}x). \tag{3.20}$$

Similarly, assuming  $|m| < 1$ , from the general inequality (3.17), we get

$$\begin{aligned} 0 \leq \|C_i(x) - C_j(x)\| &= \|m^{3i}f(m^{-i}x) - m^{3j}f(m^{-j}x)\| \\ &= |m|^{3i} \|f(m^{-i}x) - m^{3(j-i)}f(m^{-(j-i)}m^{-i}x)\| \\ &\leq \frac{1}{1 - |m|^3} (|m|^{3i} - |m|^{3j})c_2 \rightarrow 0, \end{aligned}$$

as  $i \rightarrow \infty$  (and  $j \rightarrow \infty$ ). Thus the sequence  $\{C_n\}$ , is a Cauchy sequence.

**Note 2.** Claim the “well-defined cubicness” of the mapping

$$C : C(x) = \lim_{n \rightarrow \infty} C_n(x). \tag{3.21}$$

(i) Considering (3.18) and replacing in the above  $k$ -inequality (3.1)

$$x \rightarrow m^n x \quad \text{and} \quad y \rightarrow m^n y, \tag{3.22}$$

then multiplying by  $|m|^{-3n}$ ,  $|m| < 1$ , and also taking limits  $n \rightarrow \infty$ , we obtain

$$\begin{aligned} 0 \leq &\left\| k[C(kx + y) + C(x + ky)] + (k - 1)^3 \left[ C\left(\frac{kx - y}{k - 1}\right) + C\left(\frac{x - ky}{1 - k}\right) \right] \right. \\ &\left. - (k^4 - 1)[C(x) + C(y)] - 8k(k^2 + 1)C\left(\frac{x + y}{2}\right) \right\| \leq |m|^{-3n}c \rightarrow 0, \quad n \rightarrow \infty \end{aligned} \tag{3.23}$$

leading to the original  $k$ -cubic functional equation, and thus the cubicness of  $C : |m| > 1$ , satisfying the  $k$ -cubic functional equation (2.1).

(ii) Considering (3.20) and replacing in the above  $k$ -inequality (3.1)

$$x \rightarrow m^{-n}x \quad \text{and} \quad y \rightarrow m^{-n}y, \quad (3.24)$$

then multiplying by  $|m|^{3n}$ ,  $|m| < 1$ ;  $m \neq 0$ , and also taking limits  $n \rightarrow \infty$ , we obtain

$$0 \leq \left\| k[C(kx + y) + C(x + ky)] + (k-1)^3 \left[ C\left(\frac{kx-y}{k-1}\right) + C\left(\frac{x-ky}{1-k}\right) \right] \right. \\ \left. - (k^4 - 1)[C(x) + C(y)] - 8k(k^2 + 1)C\left(\frac{x+y}{2}\right) \right\| \leq |m|^{3n}c \rightarrow 0, \quad n \rightarrow \infty \quad (3.25)$$

leading to the original  $k$ -cubic functional equation, and thus the well-defined cubicness of  $C : |m| < 1$ ;  $m \neq 0$ , satisfying the  $k$ -cubic functional equation (2.1).

Therefore the proof for the existence of the  $k$ -cubic mapping  $C$ , given by (3.21), is complete.

**Note 3.** Claim the “uniqueness” of the mapping (3.21).

(i) Considering (3.18) and assuming  $|m| > 1$ , and two cubic mappings  $C : X \rightarrow Y$ ;  $C' : X \rightarrow Y$  satisfying (2.7) and (3.21), we get by induction on  $n$ :

$$C(m^n x) = m^{3n}C(x); \quad C'(m^n x) = m^{3n}C'(x), \quad (3.26)$$

and

$$\|f(x) - C(x)\| \leq \frac{c_2}{|m|^3 - 1}; \quad \|f(x) - C'(x)\| \leq \frac{c_2}{|m|^3 - 1}. \quad (3.27)$$

Also we find:

$$0 \leq \|C(x) - C'(x)\| \leq \|C(x) - f(x)\| + \|f(x) - C'(x)\|. \quad (3.28)$$

Thus, one proves

$$0 \leq \|C(x) - C'(x)\| = |m|^{-3n} \|C(m^n x) - C'(m^n x)\| \\ \leq |m|^{-3n} \left\{ \|C(m^n x) - f(m^n x)\| + \|f(m^n x) - C'(m^n x)\| \right\}, \\ \leq |m|^{-3n} \left\{ \frac{2c_2}{|m|^3 - 1} \right\} \rightarrow 0, \quad n \rightarrow \infty, \quad (3.29)$$

leading to

$$C(x) \equiv C'(x), \quad (3.30)$$

establishing the uniqueness of the mapping  $C : X \rightarrow Y : |m| > 1$ .

(ii) Considering (3.20), and assuming  $|m| < 1$ ;  $m \neq 0$ , and two cubic mappings  $C : X \rightarrow Y$ ;  $C' : X \rightarrow Y$  satisfying (3.21) and (2.8) we get by induction on  $n$  :

$$C(m^{-n}x) = m^{-3n}C(x); \quad C'(m^{-n}x) = m^{-3n}C'(x) \quad (3.31)$$

and

$$\|f(x) - C(x)\| \leq \frac{c_2}{1 - |m|^3}; \quad \|f(x) - C'(x)\| \leq \frac{c_2}{1 - |m|^3}. \quad (3.32)$$

Thus from (3.28) and (3.32), one proves

$$0 \leq \|C(x) - C'(x)\| = |m|^{3n} \|C(m^{-n}x) - C'(m^{-n}x)\| \\ \leq |m|^{3n} \left\{ \|C(m^{-n}x) - f(m^{-n}x)\| + \|f(m^{-n}x) - C'(m^{-n}x)\| \right\}, \\ \leq |m|^{3n} \left\{ \frac{2c_2}{1 - |m|^3} \right\} \rightarrow 0, \quad n \rightarrow \infty, \quad (3.33)$$

leading to

$$C(x) \equiv C'(x), \quad (3.34)$$

establishing the uniqueness of the mapping  $C : X \rightarrow Y : |m| < 1; m \neq 0$ .

Therefore the following Theorem 3.1 holds:

**Theorem 3.1.** Let  $X$  be a real normed linear space and let  $Y$  be a real complete normed linear space. If approximate Euler-Lagrange-Jensen  $k$ -cubic mappings  $f : X \rightarrow Y$ , satisfy the Euler-Lagrange-Jensen  $k$ -cubic functional inequality (3.1), then there exists a unique Euler-Lagrange-Jensen  $k$ -cubic mapping  $C : X \rightarrow Y$ , satisfying the following inequality

$$\|f(x) - C(x)\| \leq \frac{c_2}{|m|^3 - 1}; \quad \forall m(=k+1) \in \mathbb{R} - \{0, \pm 1\}, \quad (3.35)$$

$m$  different also from 2.

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