



## Arc-transitive Cubic Abelian bi-Cayley Graphs and BCI-graphs

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**Abstract.** A finite simple graph is called a bi-Cayley graph over a group  $H$  if it has a semiregular automorphism group, isomorphic to  $H$ , which has two orbits on the vertex set. Cubic vertex-transitive bi-Cayley graphs over abelian groups have been characterized recently by Zhou and Feng (Europ. J. Combin. 36 (2014), 679–693). In this paper we consider the latter class of graphs and select those in the class which are also arc-transitive. Furthermore, such a graph is called 0-type when it is bipartite, and the bipartition classes are equal to the two orbits of the respective semiregular automorphism group. A 0-type graph can be represented as the graph  $\text{BiCay}(H, S)$ , where  $S$  is a subset of  $H$ , the vertex set of which consists of two copies of  $H$ , say  $H_0$  and  $H_1$ , and the edge set is  $\{\{h_0, g_1\} : h, g \in H, gh^{-1} \in S\}$ . A bi-Cayley graph  $\text{BiCay}(H, S)$  is called a BCI-graph if for any bi-Cayley graph  $\text{BiCay}(H, T)$ ,  $\text{BiCay}(H, S) \cong \text{BiCay}(H, T)$  implies that  $T = hS^\alpha$  for some  $h \in H$  and  $\alpha \in \text{Aut}(H)$ . It is also shown that every cubic connected arc-transitive 0-type bi-Cayley graph over an abelian group is a BCI-graph.

### 1. Introduction

In this paper all graphs will be simple and finite and all groups will be finite. For a graph  $\Gamma$ , we let  $V(\Gamma)$ ,  $E(\Gamma)$ ,  $A(\Gamma)$ , and  $\text{Aut}(\Gamma)$  denote the vertex set, the edge set, the arc set, and the full group of automorphisms of  $\Gamma$ , respectively. A graph  $\Gamma$  is called a *bi-Cayley graph* over a group  $H$  if it has a semiregular automorphism group, isomorphic to  $H$ , which has two orbits in the vertex set. Given such  $\Gamma$ , there exist subsets  $R, L, S$  of  $H$  such that  $R^{-1} = R$ ,  $L^{-1} = L$ ,  $1 \notin R \cup L$ , and  $\Gamma \cong \text{BiCay}(H, R, L, S)$ , where the latter graph is defined to have the union of the *right part*  $H_0 = \{h_0 : h \in H\}$  and the *left part*  $H_1 = \{h_1 : h \in H\}$  as the vertex set while the edge set consists of three sets:

$\{\{h_0, g_0\} : gh^{-1} \in R\}$  (*right edges*),

$\{\{h_1, g_1\} : gh^{-1} \in L\}$  (*left edges*),

$\{\{h_0, g_1\} : gh^{-1} \in S\}$  (*spoke edges*).

In what follows we will also refer to  $\text{BiCay}(H, R, L, S)$  as a *bi-Cayley representation* of  $\Gamma$ . Regarding bi-Cayley graphs, our notation and terms will follow [37]. For the case when  $|S| = 1$ , the bi-Cayley

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Table 1: Cubic symmetric abelian 0-type bi-Cayley graphs.

no.	$H$	$S$	$k$ -reg.	other name
1.	$\mathbb{Z}_{rm} \times \mathbb{Z}_m = \langle a, b \mid a^r = b^m = 1, b^m = a^{m(u+1)} \rangle$ , $r = 3^s p_1^{e_1} \cdots p_t^{e_t}$ , $r > 3$ and $r \geq 11$ if $m = 1$ , $s \in \{0, 1\}$ , every $p_i \equiv 1 \pmod{3}$ , and $u^2 + u + 1 \equiv 0 \pmod{r}$	$\{1, a, b\}$	1	–
2.	$\mathbb{Z}_8 = \langle a \rangle$	$\{1, a^2, a^3\}$	2	Möbius-Kantor graph
3.	$\mathbb{Z}_m^2 = \langle a, b \rangle$ , $m > 1$ , $m \neq 3$	$\{1, a, b\}$	2	–
4.	$\mathbb{Z}_{3m} \times \mathbb{Z}_m = \langle a, b \mid a^{3m} = b^{3m} = 1, a^m = b^m \rangle$ , $m > 1$	$\{1, a, b\}$	2	–
5.	$\mathbb{Z}_3 = \langle a \rangle$	$\{1, a, a^{-1}\}$	3	$K_{3,3}$
6.	$\mathbb{Z}_3^2 = \langle a, b \rangle$	$\{1, a, b\}$	3	Pappus graph
7.	$\mathbb{Z}_7 = \langle a \rangle$	$\{1, a, a^3\}$	4	Heawood graph

Table 2: Cubic symmetric abelian 2-type bi-Cayley graphs.

$H$	$R$	$L$	$S$	$k$ -trans	other name
$\langle a, b \rangle = \mathbb{Z}_2^2$	$\{a, b\}$	$\{a, b\}$	$\{1\}$	2	$GP(4, 1)$
$\langle a \rangle \times \langle b \rangle = \mathbb{Z}_2 \times \mathbb{Z}_{10}$	$\{ab^3, ab^{-3}\}$	$\{b, b^{-1}\}$	$\{1\}$	2	–
$\langle a \rangle = \mathbb{Z}_n$	$\{a\}$	$\{a^k\}$	$\{1\}$	2	$GP(n, k)$ , $(n, k) = (4, 1), (8, 3), (10, 2), (12, 5), (24, 5)$
$\langle a \rangle = \mathbb{Z}_n$	$\{a\}$	$\{a^k\}$	$\{1\}$	3	$GP(n, k)$ , $(n, k) = (5, 2), (10, 3)$

graph  $\text{BiCay}(H, R, L, S)$  is also called a *one-matching bi-Cayley graph* (see [22]). Also, if  $|R| = |L| = s$ , then  $\text{BiCay}(H, R, L, S)$  is said to be an *s-type bi-Cayley graph*, and if  $H$  is abelian, then  $\text{BiCay}(H, R, L, S)$  is simply called an *abelian bi-Cayley graph*. If  $|L| = |R| = 0$ , then  $\text{BiCay}(H, S)$  will be written for  $\text{BiCay}(H, \emptyset, \emptyset, S)$ . Bi-Cayley graphs have been studied from various aspects [5, 14, 15, 18–20, 22, 28, 34, 36, 37], they have been used by constructions of strongly regular graphs [1, 2, 24, 30–32] and semisymmetric graphs [9, 10, 27]. Bi-Cayley graphs are related with the so called quasi  $m$ -Cayley graphs, see [17]. The cubic vertex-transitive abelian bi-Cayley graphs have been classified recently by Zhou and Feng [37] (by a cubic graph we mean a regular graph of valency 3).

In this paper we turn to the class of cubic connected arc-transitive bi-Cayley graphs over abelian groups. Recall that a graph  $\Gamma$  is called *arc-transitive* when  $\text{Aut}(\Gamma)$  is transitive on  $A(\Gamma)$ . From now on we say that  $\Gamma$  is *symmetric* when it is connected and arc-transitive. In the first part of our paper we are going to determine the cubic symmetric abelian bi-Cayley graphs. Clearly, such a graph is  $s$ -type for  $s \in \{0, 1, 2\}$ ; and in fact, the classification in the case of 0-type and 2-type graphs follows from the results in [7, 12, 22, 23]. The respective graphs are listed in Tables 1 and 2.

**Remark 1.1.** *In order to derive the 0-type graphs, the key observation is that each such graph is of girth 4 or 6. Namely, if  $S = \{a, b, c\}$ , then we find in  $\text{BiCay}(H, S)$  the closed walk:*

$$(1_0, a_1, (b^{-1}a)_0, (cb^{-1}a)_1, (b^{-1}c)_0, c_1, 1_0),$$

here we use that  $cb^{-1}a = ab^{-1}c$  holds as  $H$  is abelian. It is a folklore result that the cubic symmetric graphs of girth at most 4 are  $K_4, K_{3,3}$  and  $Q_3$ , the graph of the cube. There are infinitely many cubic symmetric graphs of girth 6, but fortunately, all have been determined in [7, 12, 23]. A description of these graphs is given in Theorem 2.2, and using this theorem, it is not hard to deduce Table 1 (see Remark 2.3 for the details).

**Remark 1.2.** *The 2-type bi-Cayley graphs  $\text{BiCay}(\mathbb{Z}_n, \{1, -1\}, \{k, -k\}, \{0\})$  are also known as the generalized Petersen graphs, denoted by  $GP(n, k)$ . It was proved by Frucht et al. [13] that  $GP(n, k)$  is symmetric for exactly seven pairs:  $(4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5)$  and  $(24, 5)$ . Recall that, a bi-Cayley graph  $\text{BiCay}(H, R, L, S)$  is*

one-matching if  $|S| = 1$ . Symmetric one-matching abelian bi-Cayley graphs are classified in [22, Theorem 1.1], and since the 2-type cubic bi-Cayley graphs are one-matching, Table 2 can be read off directly from the latter theorem.

In this paper we complete the classification of cubic symmetric abelian bi-Cayley graphs by proving the following theorem:

**Theorem 1.3.** *There are exactly four cubic symmetric 1-type abelian bi-Cayley graphs:  $K_4, Q_3, GP(8, 3)$  and  $GP(12, 5)$ .*

**Remark 1.4.** *It should be noted that, the cubic 1-type abelian bi-Cayley graphs have been described in [37, Theorem 4.2], and it follows quickly from this result that every such graph is of girth at most 6. Now, one can use the characterization of symmetric cubic graphs of girth 6 in order to derive Theorem 1.3. However, our proof will follow a different approach, namely, we are going to use voltage graphs.*

In the second part of this paper we turn to the BCI-property of cubic symmetric abelian 0-type bi-Cayley graphs. Recall that, these are the graphs in the form  $\text{BiCay}(H, S)$ , where  $H$  is a finite abelian group and  $S$  is a subset of  $H$ . A bi-Cayley graph  $\text{BiCay}(H, S)$  is said to be a *BCI-graph* if for every  $\text{BiCay}(H, T)$ ,  $\text{BiCay}(H, T) \cong \text{BiCay}(H, S)$  implies that  $T = hS^\sigma$  for some  $h \in H$  and  $\sigma \in \text{Aut}(H)$ ; and the group  $H$  is called an *m-BCI-group* if every bi-Cayley graph over  $H$  of degree at most  $m$  is a BCI-graph. The study of *m-BCI-groups* was initiated in [36], where it was shown that every group is a 1-BCI-group, and a group is a 2-BCI-group if and only if it has the property that any two elements of the same order are either fused or inverse fused (these groups are described in [25]). The problem of classifying all 3-BCI-groups is still open, partial results can be found in [18–21, 35, 36]. It was proved by the authors (see [21, Theorem 1.1]) that the nilpotent 3-BCI-groups are the groups  $U \times V$ , where  $U$  is homocyclic of odd order, and  $V$  is trivial, or  $\mathbb{Z}_{2^r}$ , or  $\mathbb{Z}'_{2^r}$ , or the quaternion group  $Q_8$  (homocyclic means that it is a direct product of cyclic groups of the same order). Consequently, the class of abelian 3-BCI groups is quite restricted. As our second main result, we prove that the situation changes completely when one considers only symmetric graphs.

**Theorem 1.5.** *Every cubic symmetric abelian 0-type bi-Cayley graph is a BCI-graph.*

## 2. Preliminaries

Let  $G$  be a group acting on a finite set  $V$ . For  $g \in G$  and  $v \in V$ , the image of  $v$  under  $g$  will be written as  $v^g$ . For a subset  $U \subseteq V$ , we will denote by  $G_U$  the point-wise stabilizer of  $U$  in  $G$ , while by  $G_{\{U\}}$  the set-wise stabilizer of  $U$  in  $G$ . If  $U = \{u\}$ , then  $G_u$  will be written for  $G_{\{u\}}$ . If  $G$  is transitive on  $V$  and  $\Delta \subseteq V$  is a *block* for  $G$  (see [8, page 12]), then the partition  $\delta = \{\Delta^g : g \in G\}$  is called the *system of blocks* for  $G$  induced by  $\Delta$ . The group  $G$  acts on  $\delta$  naturally, the corresponding *kernel* will be denoted by  $G_\delta$ , i.e.,  $G_\delta = \{g \in G : \Delta'^g = \Delta' \text{ for all } \Delta' \in \delta\}$ . For further definitions and results from permutation group theory that will appear later, we refer the reader to [8].

Below we collect the main ingredients of this paper.

**2.1 Cubic symmetric graphs.** For a positive integer  $k$ , a *k-arc* of a graph  $\Gamma$  is an ordered  $(k+1)$ -tuple  $(v_0, v_1, \dots, v_k)$  of vertices of  $\Gamma$  such that, for every  $i \in \{1, \dots, k\}$ ,  $v_{i-1}$  is adjacent to  $v_i$ , and for every  $i \in \{1, \dots, k-1\}$ ,  $v_{i-1} \neq v_{i+1}$ . The graph  $\Gamma$  is called *(G, k)-arc-transitive* (*(G, k)-arc-regular*) if  $G$  is transitive (regular) on the set of  $k$ -arcs of  $\Gamma$ . If  $G = \text{Aut}(\Gamma)$ , then a *(G, k)-arc-transitive* (*(G, k)-arc-regular*) graph is simply called *k-transitive* (*k-regular*). The following result is due to Tutte:

**Theorem 2.1.** [33] *Every cubic symmetric graph is k-regular for some  $k \leq 5$*

In this paper we will occasionally need information about cubic symmetric graphs of small order, and for this purpose use the catalogue [6, Table]. We denote by  $F_nA, F_nB, \dots$  etc. the cubic symmetric graphs on  $n$  points, and simply write  $F_n$  if the graph is uniquely determined by  $n$ . Given an abelian group  $G$ , the *generalized dihedral group*  $\text{Dih}(G)$  is the group  $\langle G, \eta \rangle \cong G \rtimes \langle \eta \rangle$ , where  $\eta$  is an involution and it acts on  $G$  as  $g^\eta = g^{-1}, g \in G$ . We have the following description of cubic symmetric graphs of girth 6:

**Theorem 2.2.** Let  $\Gamma$  be a cubic symmetric graph of girth 6. Then one of the following holds:

- (i)  $\Gamma$  is 1-regular, and  $\text{Aut}(\Gamma)$  contains a regular normal subgroup isomorphic to  $\text{Dih}(L)$ , where  $L \cong \mathbb{Z}_{rm} \times \mathbb{Z}_m$ ,  $r = 3^s p_1^{e_1} \cdots p_t^{e_t}$ ,  $r > 3$  and  $r \geq 11$  if  $m = 1$ ,  $s \in \{0, 1\}$ , and every  $p_i \equiv 1 \pmod{3}$ .
- (ii)  $\Gamma$  is 2-regular, and  $\Gamma \cong \text{GP}(8, 3)$ , or  $\text{Aut}(\Gamma)$  contains a regular normal subgroup isomorphic to  $\text{Dih}(L)$ , where  $L \cong \mathbb{Z}_{rm} \times \mathbb{Z}_m$ ,  $r \in \{1, 3\}$ ,  $m > 1$ , and if  $r = 1$ , then  $m \neq 3$ .
- (iii)  $\Gamma$  is 3-regular, and  $\Gamma \cong \text{F18}$  (the Pappus graph) or  $\text{GP}(10, 3)$  (the Desargues graph).
- (iv)  $\Gamma$  is 4-regular, and  $\Gamma \cong \text{F14}$  (the Heawood graph).

In fact, part (i) is deduced from [23, Theorem 1.2], part (ii) from [23, Theorem 1.1], and parts (iii)-(iv) from [12, Corollary 6.3] (see also [7, Theorem 2.3]).

**Remark 2.3.** Let  $\Gamma$  be a cubic symmetric 0-type abelian bi-Cayley graph. We are going to show below that  $\Gamma$  is isomorphic to one of the graphs given in Table 1. As noted in Remark 1.1,  $\Gamma$  is of girth 4 or 6, and it follows that if the girth is equal to 4, then  $\Gamma \cong K_{3,3}$  or  $Q_3$ . The graph  $K_{3,3}$  is isomorphic to the bi-Cayley graph given in Row no. 5 of Table 1, and  $Q_3$  is isomorphic to the bi-Cayley graph given in Row no. 3 of Table 1 with  $m = 2$ . Assume that  $\Gamma$  is of girth 6. Then by Theorem 2.2,  $\Gamma$  is  $k$ -regular for some  $k \leq 4$ . We consider each case of the theorem separately.

**CASE 1.**  $k = 1$ . In this case  $\text{Aut}(\Gamma)$  contains a regular normal subgroup  $K$  isomorphic to  $\text{Dih}(L)$ , where  $L \cong \mathbb{Z}_{rm} \times \mathbb{Z}_m$ ,  $r = 3^s p_1^{e_1} \cdots p_t^{e_t}$ ,  $r > 3$  and  $r \geq 11$  if  $m = 1$ ,  $s \in \{0, 1\}$ , and every  $p_i \equiv 1 \pmod{3}$ . Consequently, the subgroup  $H \leq K$  that  $H \cong L$  is semiregular and has two orbits on  $V(\Gamma)$ . Notice that, the group  $L$  is characteristic in  $\text{Dih}(L)$ . Thus  $H$  is characteristic in  $K$ , and since  $K \trianglelefteq \text{Aut}(\Gamma)$  we conclude that  $H \trianglelefteq \text{Aut}(\Gamma)$ . Using this and that  $\Gamma$  is symmetric, we find that  $\Gamma$  is bipartite, and the bipartition classes are equal to the orbits of  $H$ . Therefore,  $\Gamma \cong \text{BiCay}(H, S)$  for a subset  $S$  of  $H$ . We may assume without loss of generality that  $1 \in S$ , here 1 denotes the identity element of  $H$ . Since  $\Gamma$  is arc-transitive and  $H$  is normal in  $\text{Aut}(\Gamma)$ , there exist  $\sigma \in \text{Aut}(H)$  and  $h \in H$  with the property that  $S$  is equal to the orbit of 1 under the mapping  $\varphi : x \mapsto x^\sigma h$ ,  $x \in H$ . Thus we may write  $S = \{1, a, b\}$  such that  $1^\varphi = a$ ,  $a^\varphi = b$  and  $b^\varphi = 1$ . It follows from this that  $h = a$ ,  $a^\sigma = a^{-1}b$  and  $b^\sigma = a^{-1}$ . This shows that both elements  $a$  and  $b$  are of the same order. On the other hand  $\Gamma$  is connected, hence  $\langle a, b \rangle = H \cong \mathbb{Z}_{rm} \times \mathbb{Z}_m$ , and thus  $a$  and  $b$  are of order  $rm$ , and  $\langle a^m \rangle = \langle b^m \rangle$ . Then we can write  $\langle a^m \rangle^\sigma = \langle b^m \rangle^\sigma = \langle (b^\sigma)^m \rangle = \langle a^m \rangle$ , and thus  $(a^m)^\sigma = (a^m)^u$  for some integer  $u$ ,  $\text{gcd}(u, r) = 1$ . From this  $(a^m)^u = (a^m)^\sigma = a^{-m}b^m$ , hence  $b^m = a^{m(u+1)}$ . Also,  $(a^m)^{u^2} = (a^m)^{\sigma^2} = (a^{-m}b^m)^\sigma = (a^m)^{-u-1}$ , and this gives that  $u^2 + u + 1 \equiv 0 \pmod{r}$ . To sum up,  $\text{BiCay}(H, \{1, a, b\})$  is one of the graphs described in Row no. 1 of Table 1. In fact, any graph in that row is symmetric, the proof of this claim we leave for the reader.

**CASE 2.**  $k = 2$ . In this case  $\Gamma \cong \text{GP}(8, 3)$ , or  $\text{Aut}(\Gamma)$  contains a regular normal subgroup isomorphic to  $\text{Dih}(L)$ , where  $L \cong \mathbb{Z}_{rm} \times \mathbb{Z}_m$ ,  $r \in \{1, 3\}$ ,  $m > 1$ , and if  $r = 1$ , then  $m \neq 3$ . We have checked by Magma that  $\text{GP}(8, 3)$  admits a bi-Cayley representation given in Row no. 2 of Table 1. Otherwise, copying the same argument as in CASE 1, we derive that  $\Gamma \cong \text{BiCay}(H, S)$ , where  $S = \{1, a, b\}$ , and either  $H = \langle a, b \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_m$ ,  $m > 1$  and  $m \neq 3$ , or  $H = \langle a, b \mid a^{3m} = b^{3m} = 1, a^m = b^m \rangle \cong \mathbb{Z}_{3m} \times \mathbb{Z}_m$ ,  $m > 1$ . Therefore,  $\text{BiCay}(H, \{1, a, b\})$  is one of the graphs described in Row no. 3 of Table 1 in the former case, while it is one of the graphs described in Row no. 4 of Table 1 in the latter case. In fact, any graph in these rows is symmetric, the proof is again left for the reader.

**CASE 3.**  $k = 3$ . In this case  $\Gamma \cong \text{F18}$  (the Pappus graph) or  $\text{GP}(10, 3)$  (the Desargues graph). The Pappus graph admits a bi-Cayley representation given in Row no. 6 of Table 1, and we have checked by Magma that the Desargues graph cannot be represented as a 0-type abelian bi-Cayley graph.

**CASE 4.**  $k = 4$ . In this case  $\Gamma \cong \text{F14}$  (the Heawood graph), which admits a bi-Cayley representation given in Row no. 7 of Table 1.

2.2 *Quotient graphs.* Let  $\Gamma$  be an arbitrary finite graph and  $G \leq \text{Aut}(\Gamma)$  which is transitive on  $V(\Gamma)$ . For a normal subgroup  $N \triangleleft G$  which is not transitive on  $V(\Gamma)$ , the *quotient graph*  $\Gamma_N$  is the graph whose vertices are the  $N$ -orbits on  $V(\Gamma)$ , and two  $N$ -orbits  $\Delta_i, i = 1, 2$ , are adjacent if and only if there exist  $v_i \in \Delta_i, i = 1, 2$ , which are adjacent in  $\Gamma$ . Now, let  $\Gamma = \text{BiCay}(H, R, L, S)$  be a bi-Cayley graph and let  $h \in H$ . Following [37], let  $R(h)$  denote the permutation of  $V(\Gamma) = H_0 \cup H_1$  defined by

$$(x_i)^{R(h)} = (xh)_i, \quad x \in H \text{ and } i \in \{0, 1\}.$$

We set  $R(H) = \{R(h) : h \in H\}$ . Obviously,  $R(H) \leq \text{Aut}(\Gamma)$ , and  $R(H)$  is semiregular with orbits  $H_0$  and  $H_1$ . Notice that, if  $H$  is abelian, then the permutation  $\iota$  of  $V(\Gamma)$  defined by  $(x_0)^\iota = (x^{-1})_1$  and  $(x_1)^\iota = (x^{-1})_0, x \in H$ , is an automorphism of  $\Gamma$ . Furthermore, the group  $\langle R(H), \iota \rangle \leq \text{Aut}(\Gamma)$  is regular on  $V(\Gamma)$ .

The proof of parts (i)-(ii) of the following lemma can be deduced from [26, Theorem 9], and the proof of part (iii) is straightforward, hence it is omitted.

**Lemma 2.4.** *Let  $\Gamma$  be a cubic symmetric graph and let  $N \leq \text{Aut}(\Gamma)$  be a normal subgroup which has more than 2 orbits on  $V(\Gamma)$ . Then the following hold:*

- (i)  $\Gamma_N$  is a cubic symmetric graph.
- (ii)  $N$  is equal to the kernel of  $\text{Aut}(\Gamma)$  acting on the system of blocks consisting of the orbits of  $N$ . Moreover,  $N$  is regular on each of its orbits.
- (iii) Suppose, in addition, that  $\Gamma = \text{BiCay}(H, R, L, S)$  where  $H$  is an abelian group, and that  $N = R(K)$  for some  $K < H$ . Then  $\Gamma_N$  is isomorphic to the bi-Cayley graph  $\text{BiCay}(H/K, R/K, L/K, S/K)$ .

In part (iii) of the above lemma,  $H/K$  denotes the factor group of  $H$  by  $K$ . The elements of  $H/K$  are the cosets  $Kh$ , and for a subset  $X \subseteq H$ , let  $X/K$  denote the subset of  $H/K$  defined by  $X/K = \{Kx : x \in X\}$ .

2.3 *Voltage graphs.* Let  $\Gamma$  be a finite simple graph and  $K$  be a finite group whose identity element is denoted by  $1_K$ . For an arc  $x = (w, w') \in A(\Gamma)$  we set  $x^{-1} = (w', w)$ . A  $K$ -*voltage assignment* of  $\Gamma$  is a mapping  $\zeta : A(\Gamma) \rightarrow K$  with the property  $\zeta(x^{-1}) = \zeta(x)^{-1}$  for every  $x \in A(\Gamma)$ . The values of  $\zeta$  are called *voltages* and  $K$  is called the *voltage group*. Voltages are naturally extended to a directed walk  $\vec{W} = (w_1, \dots, w_n)$  by letting  $\zeta(\vec{W}) = \prod_{i=1}^{n-1} \zeta((w_i, w_{i+1}))$ . Fix a spanning tree  $T$  of  $\Gamma$ . Then every edge not in  $E(T)$  together with the edges in  $E(T)$  span a unique circuit of  $\Gamma$ , and we shall refer to the circuits obtained in this manner as the *base circuits* of  $\Gamma$  relative to  $T$ . The  $K$ -voltage assignment  $\zeta$  is called  $T$ -*reduced* if  $\zeta(x) = 1_K$  whenever  $x$  is an arc belonging to  $A(T)$ . The *voltage graph*  $\Gamma \times_\zeta K$  is defined to have vertex set  $V(\Gamma) \times K$ , and edge set

$$E(\Gamma \times_\zeta K) = \{(w, k), (w', \zeta(x)k) : x = (w, w') \in A(\Gamma) \text{ and } k \in K\}. \tag{1}$$

The voltage group  $K$  induces an automorphism group of  $\Gamma \times_\zeta K$  through the action

$$k_{\text{right}} : (w, l) \mapsto (w, lk), \quad w \in V(\Gamma) \text{ and } k, l \in K.$$

We set  $K_{\text{right}} = \{k_{\text{right}} : k \in K\}$ . Let  $g \in \text{Aut}(\Gamma \times_\zeta K)$  such that it normalizes  $K_{\text{right}}$ . This implies that, if  $(w, k) \in V(\Gamma \times_\zeta K)$  and  $(w, k)^g = (w', k')$ , then  $w'$  does not depend on the choice of  $k \in K$ , and the mapping  $w \mapsto w'$  is a well-defined permutation of  $V(\Gamma)$ . The latter permutation will be called the *projection* of  $g$ , obviously, it belongs to  $\text{Aut}(\Gamma)$ .

On the other hand, an automorphism of  $\text{Aut}(\Gamma)$  is said to *lift* to an automorphism of  $\Gamma \times_\zeta K$  if it is the projection of some automorphism of  $\Gamma \times_\zeta K$ . The following “lifting lemma” is a special case of [29, Theorem 4.2]:

**Theorem 2.5.** *Let  $\Gamma \times_\zeta K$  be a connected voltage graph, where  $K$  is an abelian group, and  $\zeta$  is a  $T$ -reduced  $K$ -voltage assignment. Then  $\sigma \in \text{Aut}(\Gamma)$  lifts to an automorphism of  $\Gamma \times_\zeta K$  if and only if there exists some  $\sigma_* \in \text{Aut}(K)$  such that for every directed base circuit  $\vec{C}$  relative to  $T$ ,  $\sigma_*(\zeta(\vec{C})) = \zeta(\vec{C}^\sigma)$ .*

For more information on voltage graphs the reader is referred to [16, 29].

2.4 BCI-graphs. For a bi-Cayley graph  $\Gamma = \text{BiCay}(H, S)$ , we let  $\mathcal{S}(\text{Aut}(\Gamma))$  denote the set of all semiregular subgroups of  $\text{Aut}(\Gamma)$  whose orbits are  $H_0$  and  $H_1$ . Clearly,  $R(H) \in \mathcal{S}(\text{Aut}(\Gamma))$  always holds. Our main tool in the proof of Theorem 1.5 will be the following lemma:

**Lemma 2.6.** [21, Lemma 2.4] *The following are equivalent for every 0-type bi-Cayley graph  $\Gamma = \text{BiCay}(H, S)$ :*

- (i)  $\text{BiCay}(H, S)$  is a BCI-graph.
- (ii) *The normalizer  $N_{\text{Aut}(\Gamma)}(R(H))$  is transitive on  $V(\Gamma)$ , and every two subgroups in  $\mathcal{S}(\text{Aut}(\Gamma))$ , isomorphic to  $H$ , are conjugate in  $\text{Aut}(\Gamma)$ .*

### 3. Proof of Theorem 1.3

Throughout this section we keep the following notation:

$$\Gamma = \text{BiCay}(H, \{r\}, \{s\}, \{1, t\})$$

is a cubic symmetric graph,  $H = \langle r, s, t \rangle$  is an abelian group, and  $r, s$  are involutions.

The *core* of a subgroup  $A$  in a group  $B$  is the largest normal subgroup of  $B$  contained in  $A$ . In order to derive Theorem 1.3 we analyze the core of  $R(H)$  in  $\text{Aut}(\Gamma)$ .

**Lemma 3.1.** *If  $R(H)$  has trivial core in  $\text{Aut}(\Gamma)$ , then one of the following holds:*

- (i)  $H \cong \mathbb{Z}_2, s = r = t$ , and  $\Gamma \cong K_4$ .
- (ii)  $H \cong \mathbb{Z}_2^2, s \neq r, t = sr$  and  $\Gamma \cong Q_3$ .

*Proof.* If  $\Gamma$  is of girth at most 4, then it is isomorphic to  $K_4$ , or  $K_{3,3}$ , or  $Q_3$  (see Remark 1.1). In the first case we get at once (i), and it is not hard to see that  $K_{3,3}$  is impossible. Furthermore, we compute by Magma [3] that  $Q_3$  is possible,  $H \cong \mathbb{Z}_2^2$ , and  $r, s, t$  must be as given in (ii).

For the rest of the proof we assume that the girth of  $\Gamma$  is larger than 4. Then  $r \neq s$ , for otherwise, we find the 4-circuit  $(1_0, 1_1, r_1, r_0)$ . Then either  $\langle r, s \rangle \cap \langle t \rangle$  is trivial, and

$$H = \langle r, s \rangle \times \langle t \rangle \cong \mathbb{Z}_2^2 \times \mathbb{Z}_n; \tag{2}$$

or  $t$  is of even order, say  $2n, t^n \in \langle r, s \rangle$ , and

$$H = \langle r, s, t \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{2n}. \tag{3}$$

Note that, we have  $|H| = 4n$ .

By Tutte’s Theorem (Theorem 2.1),  $\Gamma$  is  $k$ -regular for some  $k \leq 5$ . The order  $|\text{Aut}(\Gamma)| = |V(\Gamma)| \cdot 3 \cdot 2^{k-1} = |H| \cdot 3 \cdot 2^k$ , and thus  $|\text{Aut}(\Gamma) : R(H)| = 3 \cdot 2^k$ . Consider the action of  $\text{Aut}(\Gamma)$  on the set of its right  $R(H)$ -cosets. Since  $R(H)$  has trivial core in  $\text{Aut}(\Gamma)$ , this action is faithful. Using this and that  $R(H)$  acts as a point stabilizer, we have an embedding of  $R(H)$  into  $S_{3 \cdot 2^{k-1}}$ . We shall write below  $H \leq S_{3 \cdot 2^{k-1}}$ . It was proved in [4, Theorem 1] that, if  $n = 3m + 2$  and  $A \leq S_n$  is an abelian subgroup, then

$$|A| \leq 2 \cdot 3^m, \tag{4}$$

and equality holds if and only if  $A \cong \mathbb{Z}_2 \times \mathbb{Z}_3^m$ .

CASE 1.  $k = 1$ . In this case  $\mathbb{Z}_2^2 \leq H \leq S_5$ . This implies that  $|H| = 4, \Gamma \cong Q_3$  (see [6, Table]), which is in contradiction with the assumption that the girth is larger than 4.

CASE 2.  $k = 2$ . In this case  $H \leq S_{11}$ . Since  $|H| = 4n$ , we obtain by (4) that  $n \leq 13$ . We compute by Magma that, if  $H$  is given as in (2) and  $n \leq 13$ , then  $\Gamma$  is not edge-transitive. Furthermore, if  $H$  is given as in (3)

and  $n \leq 13$ , then  $\Gamma$  is edge-transitive only if  $n = 2$  or  $n = 3$ . Consequently,  $\Gamma \cong GP(8, 3)$  or  $GP(12, 5)$  (see [6, Table]). However, we have checked by Magma that in both cases the possible semiregular subgroups have a non-trivial core in the full automorphism group, and thus this case is excluded.

CASE 3.  $k \geq 3$ . We may assume that  $n > 13$  (see the previous paragraph). We find in  $\Gamma$  the 8-cycle  $(1_0, r_0, r_1, (rs)_1, (rs)_0, s_0, s_1, 1_1)$ . Thus there must be an 8-cycle, say  $C$ , starting with the 3-arc  $(1_0, t_1, t_0, (t^2)_1)$ , let this be written in the form:

$$C = (1_0, t_1, t_0, (t^2)_1, (\delta t^2)_{x'}, (\gamma \delta t^2)_{x''}, (\beta \gamma \delta t^2)_{x'''}, (\alpha \beta \gamma \delta t^2)_{x''''}),$$

where  $x, x', x'', x'''' \in \{0, 1\}$  and  $\alpha, \beta, \gamma, \delta \in \{1, r, s, t, t^{-1}\}$ . Put  $\eta = \alpha\beta\gamma\delta t^2$ . Observe that,  $\eta = t^i r^j s^k$  for some integers  $i, j, k \geq 0$ . Moreover,  $i \leq 4$  and  $i = 0$  if and only if  $C = (1_0, t_1, t_0, (t^2)_1, (t^2 s)_1, (ts)_0, (ts)_1, s_0)$ , and so  $\eta = s$ . On the other hand, since  $1_0 \sim \eta_{x''''}$  and  $\eta_{x''''} \neq t_1$ ,  $\eta \in \{1, r\}$ , and we conclude that  $i > 0$  (recall that  $r \neq s$ ). Now,  $1 = \eta^2 = t^{2i} r^{2j} s^{2k} = t^{2i}$ , which implies that the order of  $t$  is at most 8, and hence  $n \leq 8$  (see (2) and (3)), which contradicts that  $n > 13$ . This completes the proof of the lemma.  $\square$

**Lemma 3.2.** *Let  $R(N)$  be the core of  $R(H)$  in  $\text{Aut}(\Gamma)$ . Then one of the following holds:*

- (i)  $H = N \times \langle r \rangle$ , and  $Nr = Ns = Nt$ .
- (ii)  $H = N \times \langle r, s \rangle$ ,  $r \neq s$ , and  $Nt = Nrs$ .

*Proof.* By Lemma 2.4.(iii), the quotient graph  $\Gamma_{R(N)}$  can be written in the form

$$\Gamma_{R(N)} = \text{BiCay}(H/N, \{Nr\}, \{Ns\}, \{N, Nt\}).$$

We claim that  $R(H/N)$  has trivial core in  $\text{Aut}(\Gamma_{R(N)})$ . This and Lemma 3.1 combined together give (i) and (ii).

Let  $\rho$  be the permutation representation of  $\text{Aut}(\Gamma)$  derived from its action on the set of  $R(N)$ -orbits. By Lemma 2.4.(ii), the kernel  $\ker \rho = R(N)$ ,  $\rho(R(H)) = R(H/N)$ , and any subgroup of  $R(H/N)$  is of the form  $\rho(R(K))$  for some  $N \leq K \leq H$ . Assume that  $\rho(R(K)) \trianglelefteq \text{Aut}(\Gamma_{R(N)})$ . Then  $\rho(R(K)) \trianglelefteq \rho(\text{Aut}(\Gamma))$ , and hence  $R(K) \trianglelefteq \text{Aut}(\Gamma)$ . Thus  $R(K) = R(N)$ , because  $R(N)$  is the core. We find that  $\rho(R(K))$  is trivial, and the claim is proved.  $\square$

In the next lemma we deal with case (i) of Lemma 3.2.

**Lemma 3.3.** *Let  $R(N)$  be the core of  $R(H)$  in  $\text{Aut}(\Gamma)$ , and suppose that  $N \neq 1$  and case (i) of Lemma 3.2 holds. Then one of the following holds:*

- (i)  $H \cong \mathbb{Z}_2^2$ ,  $r = s \neq t$ , and  $\Gamma \cong Q_3$ .
- (ii)  $H = \langle r \rangle \times \langle t \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ , and  $\Gamma \cong \text{BiCay}(H, \{r\}, \{rt^2\}, \{1, t\}) \cong GP(8, 3)$ .

*Proof.* In this case  $H = N \times \langle r \rangle$ , and  $Nr = Ns = Nt$ . Thus  $s = n_1 r$ , and  $t = n_2 r$  for some  $n_1, n_2 \in N$ . Furthermore,  $n_1$  is an involution, and since  $H = \langle r, s, t \rangle$ ,  $N = \langle n_1, n_2 \rangle$ .

Assume for the moment that  $N$  is not a 2-group, and let  $p$  be an odd prime divisor of  $|N|$ . Then  $M = \langle n_1, n_2^p \rangle$  is the unique subgroup in  $N$  of index  $p$ , hence it is characteristic in  $N$ . Using also that  $R(N) \trianglelefteq \text{Aut}(\Gamma)$ , this gives that  $R(M) \trianglelefteq \text{Aut}(\Gamma)$ . The quotient graph  $\Gamma_{R(M)}$  is a cubic symmetric graph on  $4p$  points admitting a 1-type bi-Cayley representation over  $H/M$ . It was proved in [11, Theorem 6.2] that  $\Gamma_{R(M)}$  is isomorphic to one of the graphs:  $GP(10, 2)$ ,  $GP(10, 3)$  and the Coxeter graph  $F28$ . We compute by Magma that none of these graphs has a 1-type bi-Cayley representation. We conclude that  $N$  is a 2-group.

Notice that,  $N \cong \mathbb{Z}_{2^m}$  or  $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-1}}$ . If  $|N| \geq 8$ , then  $N$  has a characteristic subgroup  $M$  such that  $|N : M| = 8$ . Using also that  $R(N) \trianglelefteq \text{Aut}(\Gamma)$ , we find in turn that,  $R(M) \trianglelefteq \text{Aut}(\Gamma)$ , and  $\Gamma_{R(M)}$  is a cubic symmetric graph on 32 points which admits a 1-type bi-Cayley representation over  $H/M$ . Thus  $\Gamma$  is isomorphic to the Dyck graph  $F32$  (see [6, Table]), which can be excluded by the help of Magma. Therefore,  $|N| \in \{2, 4\}$ , and these yield easily cases (i) and (ii) respectively.  $\square$

In the next lemma we deal with case (ii) of Lemma 3.2.

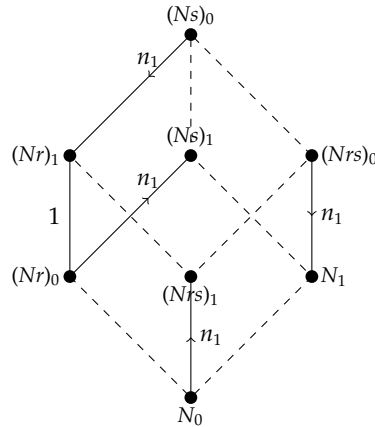


Figure 1: Voltage assignment  $\zeta$  of  $\Gamma_{R(N)}$ .

**Lemma 3.4.** *Let  $R(N)$  be the core of  $R(H)$  in  $\text{Aut}(\Gamma)$ , and suppose that  $N \neq 1$  and case (ii) of Lemma 3.3 holds. Then  $H = \langle r \rangle \times \langle t \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ , and  $\Gamma \cong \text{BiCay}(H, \{r\}, \{rt^3\}, \{1, t\}) \cong GP(12, 5)$ .*

*Proof.* In this case  $H = N \times \langle r, s \rangle$ ,  $r \neq s$ , and  $Nt = Nrs$ . Thus  $t = n_1rs$  for some  $n_1 \in N$ . Since  $H = \langle r, s, t \rangle$ ,  $N = \langle n_1 \rangle$ . Now, by Lemma 2.4.(iii) we may write

$$\Gamma_{R(N)} = \text{BiCay}(H/N, \{Nr\}, \{Ns\}, \{N, Nrs\}) \cong Q_3.$$

We proceed by defining an  $N$ -voltage assignment of the quotient graph  $\Gamma_{R(N)}$ . For this purpose we have depicted  $\Gamma_{R(N)}$  in Fig. 1, where we have also fixed the spanning tree  $T$  specified by the dashed edges. Now, let  $\zeta : A(\Gamma_{R(N)}) \rightarrow N$  be the  $T$ -reduced  $N$ -voltage assignment with its voltages being given in Fig. 1. To simplify notation we set  $\widehat{\Gamma} = \Gamma_{R(N)} \times_{\zeta} N$ . Recall that  $N_{\text{right}}$  is a subgroup of  $\text{Aut}(\widehat{\Gamma})$  (see Subsection 2.3). Next, we prove the following properties:

$$\Gamma \cong \widehat{\Gamma}, \text{ and } N_{\text{right}} \trianglelefteq \text{Aut}(\widehat{\Gamma}). \tag{5}$$

Define the mapping  $f : V(\widehat{\Gamma}) \rightarrow V(\Gamma)$  by

$$f : ((Nx)_0, n) \mapsto (nx)_0 \text{ and } ((Nx)_1, n) \mapsto (nx)_1, \quad x \in \{1, r, s, rs\}, \quad n \in N.$$

Notice that,  $f$  is well-defined because  $\{1, r, s, rs\}$  is a complete set of coset representatives of  $N$  in  $H$ . We prove below that  $f$  is an isomorphism from  $\widehat{\Gamma}$  to  $\Gamma$ . Let  $\widehat{v}_1$  and  $\widehat{v}_2$  be two adjacent vertices of  $\widehat{\Gamma}$ . This means that  $\widehat{v}_1 = ((Nx)_i, n)$  and  $\widehat{v}_2 = ((Ny)_j, \zeta(a)n)$ , where  $a = ((Nx)_i, (Ny)_j)$  is an arc of  $\Gamma_{R(N)}$ . Then  $f(\widehat{v}_1) = (xn)_i$  and  $f(\widehat{v}_2) = (y\zeta(a)n)_j$ .

Let  $i = j = 0$ . Then it can be seen in Fig. 1 that  $y = rx$  and  $\zeta(a) = 1$ . Thus in  $\Gamma$  we find  $f(\widehat{v}_1) = (nx)_0 \sim (rnx)_0 = (y\zeta(a)n)_0 = f(\widehat{v}_2)$ . Let  $i = j = 1$ . Then  $y = sx$ ,  $\zeta(a) = 1$ , and so  $f(\widehat{v}_1) = (nx)_1 \sim (snx)_1 = (y\zeta(a)n)_1 = f(\widehat{v}_2)$ . Finally, let  $i = 0$  and  $j = 1$ . Then  $y = x$  or  $y = rsx$ . In the former case  $\zeta(a) = 1$ , and  $f(\widehat{v}_1) = (nx)_0 \sim (nx)_1 = (y\zeta(a)n)_1 = f(\widehat{v}_2)$ . In the latter case  $\zeta(a) = n_1$ , and

$$f(\widehat{v}_1) = (nx)_0 \sim (tnx)_1 = (n_1rsnx)_1 = (y\zeta(a)n) = f(\widehat{v}_2).$$

By these we have proved that  $f$  is indeed an isomorphism.

For the second part of (5), compute that  $fR(m)f^{-1}$  maps  $((Nx)_i, n)$  to  $((Nx)_i, nm)$  for every  $m \in N$ . Thus  $fR(m)f^{-1} = m_{\text{right}}$ , and so  $fR(N)f^{-1} = N_{\text{right}}$ . Since  $R(N) \trianglelefteq \text{Aut}(\Gamma)$ ,  $N_{\text{right}} = fR(N)f^{-1} \trianglelefteq f\text{Aut}(\Gamma)f^{-1} = \text{Aut}(\widehat{\Gamma})$ , as claimed.



Now, (5) holds, implying that  $\text{Aut}(\widehat{\Gamma})$  projects to an edge-transitive subgroup of  $\text{Aut}(\Gamma_{R(N)})$ . We obtain from this that the automorphism  $\alpha \in \text{Aut}(\Gamma_{R(N)})$  lifts, where

$$\alpha = ((Nr)_0, (Nrs)_1, N_1)(Nr)_1, (Nrs)_0, (Ns)_1).$$

Apply Theorem 2.5 to  $\widehat{\Gamma}$  with  $\sigma = \alpha$  and the following directed base circuits relative to  $T$ :

$$\vec{C} = ((Ns)_0, (Nrs)_0, N_1, (Ns)_1) \text{ and } \vec{C}' = (N_0, (Nr)_0, (Ns)_1, N_1).$$

Let  $\sigma_*$  be the automorphism of  $N$  given in Theorem 2.5. Since  $\zeta(\vec{C}) = \zeta(\vec{C}') = n_1$ ,  $\zeta(\vec{C}^\alpha) = \sigma_*(n_1) = \zeta(\vec{C}'^\alpha)$ , which gives  $n_1^{-2} = n_1$ . Thus  $|N| = 3$ , and this yields easily the statement of the lemma.  $\square$

*Proof of Theorem 1.3.* The theorem follows directly from Lemmas 3.1–3.4.  $\square$

#### 4. Proof of Theorem 1.5

Throughout this section we keep the following notation:

$$\Gamma = \text{BiCay}(H, \{1, a, b\})$$

is a cubic symmetric graph, where  $H = \langle a, b \rangle$  is an abelian group.

Recall that,  $\mathcal{S}(\text{Aut}(\Gamma))$  denotes the set of all semiregular subgroups of  $\text{Aut}(\Gamma)$  whose orbits are  $H_0$  and  $H_1$ .

**Lemma 4.1.** *For every abelian group  $X \in \mathcal{S}(\text{Aut}(\Gamma))$ , there exists an involution  $\tau_X \in \text{Aut}(\Gamma)$  which satisfies the following properties:*

- (i) Every subgroup  $Y \leq X$  is normalized by  $\tau_X$ .
- (ii) The group  $\langle X, \tau_X \rangle$  is regular on  $V(\Gamma)$ .

*Proof.* Let  $\text{BiCay}(X, U)$  be a bi-Cayley representation of  $\Gamma$  arising from  $X$ . Then as a permutation group of  $V(\Gamma)$ ,  $X$  is permutation isomorphic to  $R(X)$  acting on  $V(\text{BiCay}(X, U))$ . Therefore, it is sufficient to show the existence of an involution  $\tau \in \text{Aut}(\text{BiCay}(X, U))$  such that

- $\tau$  normalizes every  $R(Y) \leq R(X)$ ; and
- the group  $\langle R(X), \tau \rangle$  is regular on  $V(\text{BiCay}(X, U))$ .

We claim that the permutation  $\tau$ , defined by  $\tau : x_0 \mapsto (x^{-1})_1$  and  $x_1 \mapsto (x^{-1})_0$ , satisfies both properties. The edge  $\{x_0, (ux)_1\}$ ,  $u \in U$ , is mapped by  $\tau$  to the pair  $\{(x^{-1})_1, u^{-1}x^{-1}\}_0$ . This is an edge of  $\text{BiCay}(X, U)$ , and hence  $\tau \in \text{Aut}(\text{BiCay}(X, U))$ . Furthermore, a direct computation gives that for  $y \in Y$ ,  $\tau^{-1}R(y)\tau = R(y^{-1})$ , hence  $\tau$  normalizes  $R(Y)$ , and the lemma follows.  $\square$

**Lemma 4.2.** *Let  $N \leq \text{Aut}(\Gamma)$  be a normal subgroup such that there exists an  $N$ -orbit properly contained in  $H_0$ , and let  $X$  be an abelian group from  $\mathcal{S}(\text{Aut}(\Gamma))$ . Then  $N < X$ .*

*Proof.* Let  $\Delta$  be an  $N$ -orbit such that  $\Delta \subset H_0$ , and let us consider  $Y = X \cap \text{Aut}(\Gamma)_{\{\Delta\}}$ . Since  $\Delta$  is a block contained in an  $X$ -orbit, we obtain that  $\Delta$  is an  $Y$ -orbit. We write  $\Delta = \text{Orb}_Y(v)$ . Moreover, as  $X$  is semiregular,  $Y$  is regular on  $\Delta$ , and by this and Lemma 2.4.(ii) we have

$$|Y| = |\Delta| = |N|. \tag{6}$$

Let  $\tau_X \in \text{Aut}(\Gamma)$  be the automorphism defined in Lemma 4.1, and set  $L = \langle X, \tau_X \rangle$ . According to Lemma 4.1 the group  $L$  is transitive on  $V(\Gamma)$ , and also  $Y \trianglelefteq L$ . Denote by  $\delta$  the system of blocks induced by  $\Delta$ . Then we may write

$$\delta = \{\Delta^l : l \in L\} = \{\text{Orb}_Y(v)^l : l \in L\} = \{\text{Orb}_Y(v^l) : l \in L\}.$$

From this  $Y \leq \text{Aut}(\Gamma)_\delta$ , where  $\text{Aut}(\Gamma)_\delta$  is the kernel of  $\text{Aut}(\Gamma)$  acting on  $\delta$ . Since  $\text{Aut}(\Gamma)_\delta = N$ , see Lemma 2.4.(ii),  $Y \leq N$ . This and (6) combined together imply that  $N = Y < X$ .  $\square$

For a group  $G$  and a prime  $p$  dividing  $|G|$ , we let  $G_p$  denote a Sylow  $p$ -subgroup of  $G$ .

*Proof of Theorem 1.5.* We have to show that  $\Gamma$  is a BCI-graph. Let  $X \in \mathcal{S}(\text{Aut}(\Gamma))$  such that  $X \cong H$ . By Lemma 2.6 and Lemma 4.1, it is sufficient to show the following

$$X \text{ and } R(H) \text{ are conjugate in } \text{Aut}(\Gamma). \tag{7}$$

Recall that the girth of  $\Gamma$  is 4 or 6, and if it is 4, then  $\Gamma$  is isomorphic to  $K_{3,3}$  or  $Q_3$  (see Remark 1.1). It is easy to see that (7) holds when  $\Gamma \cong K_{3,3}$ , and we have checked by the help of Magma that it also holds when  $\Gamma \cong Q_3$ . Thus assume that  $\Gamma$  is of girth 6. By Theorem 2.2,  $\Gamma$  is  $k$ -regular for some  $k \leq 4$ .

CASE 1.  $k = 1$ . In this case  $\text{Aut}(\Gamma)$  contains a regular normal subgroup  $K$  isomorphic to  $\text{Dih}(L)$ , where  $L \cong \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \dots \times \mathbb{Z}_{r_t}$ ,  $r = 3^s p_1^{e_1} \cdots p_t^{e_t}$ ,  $r > 3$  and  $r \geq 11$  if  $m = 1$ ,  $s \in \{0, 1\}$ , and every  $p_i \equiv 1 \pmod{3}$ . We have proved in CASE 1 of Remark 2.3 that  $\text{Aut}(\Gamma)$  contains a semiregular normal subgroup  $N$  such that  $N \cong L$ , and the orbits of  $N$  are  $H_0$  and  $H_1$ . Notice that,  $X$  contains every proper characteristic subgroup  $K$  of  $N$ . Indeed, since  $N \trianglelefteq \text{Aut}(\Gamma)$ ,  $K \trianglelefteq \text{Aut}(\Gamma)$ , and Lemma 4.2 can be applied for  $N$ , implying that  $K < X$ . In particular, if  $N$  is not a  $p$ -group, then  $N_p < X$  for every prime  $p$  dividing  $|N|$ , and thus  $N = X$ . Since this holds for every  $X \in \mathcal{S}(\text{Aut}(\Gamma))$  with  $X \cong H$ , it holds also for  $X = R(H)$ , and we get  $R(H) = N = X$ . In this case (7) holds trivially. Let  $N$  be a  $p$ -group for a prime  $p$ . Then it follows from the fact that  $N \cong L$  that  $p > 3$ , and thus both  $R(H)$  and  $X$  are Sylow  $p$ -subgroups of  $\text{Aut}(\Gamma)$ . In this case (7) follows from Sylow’s Theorem.

CASE 2.  $k = 2$ . In this case  $\Gamma \cong GP(8, 3)$ , or  $\text{Aut}(\Gamma)$  contains a regular normal subgroup isomorphic to  $\text{Dih}(L)$ , where  $L \cong \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \dots \times \mathbb{Z}_{r_t}$ ,  $r \in \{1, 3\}$ ,  $m > 1$ , and if  $r = 1$ , then  $m \neq 3$ . If  $\Gamma \cong GP(8, 3)$ , then we have checked by Magma that  $H \cong \mathbb{Z}_8$  and (7) holds. Assume that  $\Gamma \not\cong GP(8, 3)$ . We have proved in CASE 2 of Remark 2.3 that  $\text{Aut}(\Gamma)$  contains a semiregular normal subgroup  $N$  such that  $N \cong L$ , and the orbits of  $N$  are  $H_0$  and  $H_1$ . Now, repeating the argument in CASE 1 above, we obtain that  $N = X = R(H)$  if  $N$  is not a  $p$ -group. Let  $N$  be a  $p$ -group for a prime  $p$ . If  $p > 3$ , then both  $R(H)$  and  $X$  are Sylow  $p$ -subgroups of  $\text{Aut}(\Gamma)$ , and (7) follows from Sylow’s Theorem. We are left with the case that  $p \in \{2, 3\}$ .

Let  $p = 2$ . Since  $N \cong L$ , we find that  $N \cong \mathbb{Z}_{2^e} \times \mathbb{Z}_{2^e}$ ,  $e \geq 1$ . Define  $K = \{x \in N : o(x) \leq 2^{e-1}\}$ . Then  $K$  is characteristic in  $N$  and thus  $K \trianglelefteq \text{Aut}(\Gamma)$ . By Lemma 4.2,  $K \leq X \cap R(H)$ . By Lemma 2.4.(iii), the quotient graph  $\Gamma_K$  is a 0-type Bi-Cayley graph over the group  $N/K \cong \mathbb{Z}_2^2$ . Then  $\Gamma_K \cong Q_3$  and both  $N/K$  and  $R(H)/K$  are semiregular on  $V(\Gamma_K)$  having orbits the two bipartition classes of  $\Gamma_K$ . Since  $X \cong R(H)$ ,  $X/K \cong R(H)/K$ . A direct computation, using Magma, gives that there are two possibilities:  $X/K \cong R(H)/K \cong \mathbb{Z}_2^2$  or  $\mathbb{Z}_4$ . Furthermore, In the former case  $X/K = R(H)/K$ , which together with  $K < X \cap R(H)$  yield that  $X = R(H)$ , and (7) holds trivially. Suppose that the latter case holds and consider  $\text{Aut}(\Gamma)$  acting on the set of  $K$ -orbits. The kernel of this action is equal to  $K$ , see Lemma 2.4.(ii), and thus the image  $\text{Aut}(\Gamma)/K$  is a subgroup of  $\text{Aut}(\Gamma_K)$  which is transitive on the set of 2-arcs of  $\Gamma_K$ . However,  $\Gamma_K$  is 2-regular (it is, in fact, isomorphic to  $Q_3$ ), and we obtain that  $\text{Aut}(\Gamma)/K = \text{Aut}(\Gamma_K)$ . We compute by Magma that  $X/K$  and  $R(H)/K$  are conjugate in  $\text{Aut}(\Gamma_K) = \text{Aut}(\Gamma)/K$ , and so (7) follows from this and the fact that  $K < X \cap R(H)$ .

Let  $p = 3$ . Observe first that  $|N| > 3$ . For otherwise,  $\Gamma \cong K_{3,3}$ , contradicting that the girth is 6. Since  $N \cong L$ , we find that  $N \cong \mathbb{Z}_{3^{e+\varepsilon}} \times \mathbb{Z}_{3^e}$ ,  $e \geq 1$ ,  $\varepsilon \in \{0, 1\}$ , and if  $\varepsilon = 0$ , then  $e \geq 2$ . Let  $\varepsilon = 0$ . Define  $K = \{x \in N : o(x) \leq 3^{e-2}\}$ . Then  $K$  is characteristic in  $N$  and thus  $K \trianglelefteq \text{Aut}(\Gamma)$ . By Lemma 4.2,  $K \leq X \cap R(H)$ . By Lemma 2.4.(iii), the quotient graph  $\Gamma_K$  is a 0-type Bi-Cayley graph of the group  $N/K \cong \mathbb{Z}_9^2$ . It follows that  $\Gamma_K$  is the unique cubic symmetric graph on 162 points of girth 6 (see [6, Table]). A direct computation, using Magma, gives that  $X/K = R(H)/K = N/K$ , which together with  $K < X \cap R(H)$  yield that  $X = R(H)$ , and (7) holds trivially. Let  $\varepsilon = 1$ . Define  $K = \{x \in N : o(x) \leq 3^{e-1}\}$ . Then  $K$  is characteristic in  $N$  and thus  $K \trianglelefteq \text{Aut}(\Gamma)$ . By Lemma 4.2,  $K \leq X \cap R(H)$ . By Lemma 2.4.(iii), the quotient graph  $\Gamma_K$  is a 0-type Bi-Cayley graph of the group  $N/K \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ . It follows that  $\Gamma_K$  is the unique cubic symmetric graph on 54 points (see [6, Table]). A direct computation, using Magma, gives that  $X/K = R(H)/K = N/K$ , which together with  $K < X \cap R(H)$  yield that  $X = R(H)$ , and (7) holds also in this case.

CASE 3.  $k = 3$ . In this case  $\Gamma \cong F18$  (the Pappus graph) or  $GP(10, 3)$  (the Desargues graph). We have checked by Magma that in the former case  $H \cong \mathbb{Z}_3^2$  and (7) holds, and the latter case cannot occur.

CASE 4.  $k = 4$ . In this case  $\Gamma \cong F14$  (the Heawood graph), and (7) follows at once because  $X$  and  $R(H)$  are Sylow 7-subgroups of  $\text{Aut}(\Gamma)$ .  $\square$

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