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On Statistical Convergence of Double Sequences of Closed Sets

Yurdal Sever[a], Özer Talo[b]

[a] *Afyon Kocatepe University, Faculty of Arts and Sciences, Department of Mathematics 03200 Afyonkarahisar, Turkey*
[b] *Celal Bayar University, Faculty of Art and Sciences, Department of Mathematics 45040 Manisa, Turkey*

Abstract. In this paper, we introduce the concepts of statistical inner and statistical outer limits for double sequences of closed sets and give some formulas for finding these limits. Also, we give the Kuratowski statistical convergence of double sequences of sets by means of the statistical inner and statistical outer limits of a double sequence of closed sets.

1. Introduction

Recent years have witnessed a rapid development on applications of set-valued and variational analysis. In set-valued and variational analysis, limits of sequences of sets have the leading role. The concept of set convergence provides ways for approximation of set-valued mappings and extension of real valued functions by using convergence of graphs and epigraphs (see [1, 5, 12]). Kuratowski [7] defined the convergence of sequences of sets via the inner and outer limits of sequences of sets. Talo et al. [17] introduced the Kuratowski statistical convergence for sequences of closed sets. In this paper we extend this notion from single to double sequences of sets.

2. Definitions and Notation

A double sequence $x = (x_{jk})$ is said to be convergent in Pringsheim's sense [11] if for given $\varepsilon > 0$ there exists an integer n_0 such that $|x_{jk} - l| < \varepsilon$ whenever $j, k > n_0$. In this case we write $\lim_{j,k\to\infty} x_{jk} = l$, where j and k tending to infinity independently of each other.

Statistical convergence of sequences was introduced by Fast [6] and was extended to the double sequences by Mursaleen and Edely [9]. Let $E \subseteq \mathbb{N}^2$ and $E(m, n) = \{(j, k) : j \le m, k \le n\}$. Then, the double natural density of E is defined by

$$\delta_2(E) = \lim_{m,n\to\infty} \frac{|E(m, n)|}{mn}$$

if the limit on the right hand-side exists, where the vertical bars denote the cardinality of the enclosed set.

A real double sequence $x = (x_{jk})$ is said to be statistically (or briefly st_2-) convergent to the number L if for every $\varepsilon > 0$, the set $\{(j, k) : |x_{jk} - L| > \varepsilon\}$ has double natural density zero. In this case, we write $st_2\text{-}\lim x = L$.

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Email addresses: yurdalsever@hotmail.com (Yurdal Sever), ozertalo@hotmail.com (Özer Talo)

Clearly, a convergent double sequence is also st_2-convergent but the converse is not true, in general. Also, note that a st_2-convergent double sequence does not need to be bounded.

The limit as $k, l \to \infty$ with $(k, l) \in K \subseteq \mathbb{N}^2$ will be indicated by $\lim_{(k,l)\in K}$. Now, we quote two lemmas which are needed in the text.

Lemma 2.1. ([9, Theorem 2.1]) *A real double sequence $x = (x_{jk})$ is statistically convergent to a number L if and only if there exists a subset $K = \{(j, k)\} \subseteq \mathbb{N}^2$ such that $\delta_2(K) = 1$ and $\lim_{(j,k)\in K} x_{jk} = L$.*

The concepts of statistical limit superior and statistical limit inferior of double sequences of real numbers were introduced by Çakan and Altay [4], as follows:

Definition 2.2. ([4]) Define the sets A_x and B_x by

$$A_x = \{a \in \mathbb{R} : \delta_2(\{(j, k) : x_{jk} > a\}) \neq 0\} \text{ and } B_x = \{b \in \mathbb{R} : \delta_2(\{(j, k) : x_{jk} < b\}) \neq 0\},$$

where $\delta_2(E) \neq 0$ means that either $\delta_2(E) > 0$ or E does not have double natural density. Then, the statistical limit superior $st_2\text{-}\limsup x$ and the statistical limit inferior $st_2\text{-}\liminf x$ of a real double sequence $x = (x_{jk})$ are defined by

$$st_2\text{-}\limsup x = \begin{cases} \sup A_x & , \text{ if } A_x \neq \emptyset, \\ -\infty & , \text{ if } A_x = \emptyset \end{cases} \quad \text{and} \quad st_2\text{-}\liminf x = \begin{cases} \inf B_x & , \text{ if } B_x \neq \emptyset, \\ \infty & , \text{ if } B_x = \emptyset. \end{cases}$$

Lemma 2.3. ([4]) *Let $x = (x_{jk})$ be a double sequence of real numbers. Then, the following statements hold:*

(a) $st_2\text{-}\limsup x = \beta \Leftrightarrow$ *for any* $\varepsilon > 0$, $\delta_2(\{(j, k) : x_{jk} > \beta - \varepsilon\}) \neq 0$ *and* $\delta_2(\{(j, k) : x_{jk} > \beta + \varepsilon\}) = 0$.

(b) $st_2\text{-}\liminf x = \alpha \Leftrightarrow$ *for any* $\varepsilon > 0$, $\delta_2(\{(j, k) : x_{jk} < \alpha + \varepsilon\}) \neq 0$ *and* $\delta_2(\{(j, k) : x_{jk} < \alpha - \varepsilon\}) = 0$.

The idea of statistical convergence can be extended to double sequences of points of a metric space. We say that a double sequence $x = (x_{jk})$ of points of a metric space (X, d) statistically converges to a point $\xi \in X$ if the following equality holds for each $\varepsilon > 0$:

$$\delta_2\Big(\{(j, k) \in \mathbb{N}^2 : d(x_{jk}, \xi) \geq \varepsilon\}\Big) = 0.$$

A point $\xi \in X$ is called a statistical cluster point of $x = (x_{jk})$ if the following condition holds for each $\varepsilon > 0$:

$$\delta_2\Big(\Big\{(j, k) \in \mathbb{N}^2 : d(x_{jk}, \xi) < \varepsilon\Big\}\Big) \neq 0.$$

The set of all statistical cluster points of x will be denoted by $\Gamma_2(x)$.

Let (X, d) be a metric space and $A \subset X$, $x \in X$. Then the distance from x to A with respect to d is given by $d(x, A) := \inf_{a\in A} d(x, a)$, where we set $d(x, \emptyset) := \infty$. The open ball with center x and radius $\varepsilon > 0$ in X is denoted by $B(x, \varepsilon)$, i.e.,

$$B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Let us recall some definitions and basic properties of Kuratowski convergence. We use the following notation:

$$\begin{aligned} \mathcal{N} &:= \{N \subseteq \mathbb{N} : \mathbb{N} \backslash N \text{ finite}\} \\ &:= \{\text{subsequences of } \mathbb{N} \text{ containing all n beyond some } n_0\} \\ \mathcal{N}^\# &:= \{N \subseteq \mathbb{N} : N \text{ infinite}\} = \{\text{all subsequences of } \mathbb{N}\}. \end{aligned}$$

Definition 2.4. (Inner and outer limits) Let (X, d) be a metric space. For a sequence (A_n) of closed subsets of X; the outer limit is the set

$$\begin{aligned} \limsup_{n\to\infty} A_n &:= \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{N}^\#,\ \forall n \in N : A_n \cap B(x, \varepsilon) \neq \emptyset\Big\}, \\ &:= \Big\{x \mid \exists N \in \mathcal{N}^\#,\ \forall n \in N,\ \exists x_n \in A_n : \lim_{n\in N} x_n = x\Big\}, \end{aligned}$$

while the inner limit is the set

$$\begin{aligned}\liminf_{n\to\infty} A_n &:= \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{N},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\},\\ &:= \Big\{x \mid \exists N \in \mathcal{N},\ \forall n \in N,\ \exists x_n \in A_n : \lim_{n\in N} x_n = x\Big\}.\end{aligned}$$

The limit of a sequence (A_n) of closed subsets of X exists if the outer and inner limit sets are equal, that is,

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n.$$

For more information about inner and outer limits of sequences of sets, we refer to [2, 3, 8, 10, 12, 13, 15, 16].

Sever et al. [14] extended the definitions of various kinds of convergence from ordinary (single) sequences to double sequences of closed sets. Also, Talo et al. [17] introduced the Kuratowski statistical convergence for sequences of closed sets. They gave the definition of statistical outer and statistical inner limits of a sequence of sets by using the following collections of subsets of $\mathbb{N}$

$$\mathcal{S} := \{N \subseteq \mathbb{N} : \delta(N) = 1\} \quad \text{and} \quad \mathcal{S}^{\#} := \{N \subseteq \mathbb{N} : \delta(N) \neq 0\}.$$

Definition 2.5. Let (X, d) be a metric space. The statistical outer limit and the statistical inner limit of a sequence (A_n) of closed subsets of X are defined as

$$st\text{-}\limsup_{n\to\infty} A_n := \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{S}^{\#},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\}$$

and

$$st\text{-}\liminf_{n\to\infty} A_n := \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{S},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\}.$$

respectively. The statistical limit of a sequence (A_n) exists if its statistical outer and statistical inner limits are identical. In this situation, we say that the sequence (A_n) is Kuratowski statistically convergent and we write

$$st\text{-}\liminf_{n\to\infty} A_n = st\text{-}\limsup_{n\to\infty} A_n = st\text{-}\lim_{n\to\infty} A_n.$$

Since $\mathcal{N} \subseteq \mathcal{S}$ and $\mathcal{S}^{\#} \subseteq \mathcal{N}^{\#}$, it is clear that

$$\liminf_{n\to\infty} A_n \subseteq st\text{-}\liminf_{n\to\infty} A_n \subseteq st\text{-}\limsup_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n. \tag{1}$$

We extend this definition from single to double sequences of sets and give some formulas for the statistical inner and the statistical outer limits of a double sequence of sets.

3. Main Results

In this section, we introduce the Kuratowski statistical convergence of double sequences of closed sets. For operational reasons, it will be more convenient to work with the following collections of subsets of $\mathbb{N}^2$

$$\mathcal{S}_2 := \{N \subseteq \mathbb{N}^2 : \delta_2(N) = 1\} \quad \text{and} \quad \mathcal{S}_2^{\#} := \{N \subseteq \mathbb{N}^2 : \delta_2(N) \neq 0\}.$$

Firstly, we define the statistical inner and the statistical outer limits of a double sequence of closed sets, as follows.

Definition 3.1. Let (X, d) be a metric space. The statistical outer limit and the statistical inner limit of a double sequence (A_{kl}) of the closed subsets of X are defined as

$$st_2\text{-}\limsup_{k,l\to\infty} A_{kl} \quad := \quad \left\{ x \mid \forall \varepsilon > 0,\ \exists N \in \mathcal{S}_2^{\#},\ \forall\ (k,l) \in N : A_{kl} \cap B(x,\varepsilon) \neq \emptyset \right\},$$

and

$$st_2\text{-}\liminf_{k,l\to\infty} A_{kl} \quad := \quad \left\{ x \mid \forall \varepsilon > 0,\ \exists N \in \mathcal{S}_2,\ \forall\ (k,l) \in N : A_{kl} \cap B(x,\varepsilon) \neq \emptyset \right\}.$$

respectively. When the statistical inner and outer limits are equal to the same set A, this set is defined to be the statistical limit of double sequence (A_{kl}). In this situation, we say that the double sequence (A_{kl}) is Kuratowski statistically convergent to A and we write

$$st_2\text{-}\liminf_{k,l\to\infty} A_{kl} = st_2\text{-}\limsup_{k,l\to\infty} A_{kl} = st_2\text{-}\lim_{k,l\to\infty} A_{kl} = A.$$

Moreover, the inclusion

$$st_2\text{-}\liminf_{k,l\to\infty} A_{kl} \subseteq st_2\text{-}\limsup_{k,l\to\infty} A_{kl}$$

always holds and implies that $st_2\text{-}\lim_{k,l\to\infty} A_{kl} = A$ if and only if the inclusion

$$st_2\text{-}\limsup_{k,l\to\infty} A_{kl} \subseteq A \subseteq st_2\text{-}\liminf_{k,l\to\infty} A_{kl}$$

holds.

Example 3.2. Let us define the double sequence (A_{kl}) of closed sets by

$$A_{kl} = \begin{cases} [-4,1] & , \text{ if } k \text{ is even and } l \text{ is square,} \\ [-1,4] & , \text{ if } k \text{ is odd and } l \text{ is square,} \\ [-3,2] & , \text{ if } k \text{ is even and } l \text{ is nonsquare,} \\ [-2,3] & , \text{ if } k \text{ is odd and } l \text{ is nonsquare.} \end{cases}$$

Then, $st_2\text{-}\liminf_{k,l\to\infty} A_{kl} = [-2,2]$, $st_2\text{-}\limsup_{k,l\to\infty} A_{kl} = [-3,3]$. So, the double sequence (A_{kl}) is not Kuratowski statistically convergent.

Example 3.3. Let A and B be two different nonempty closed sets in X. Define

$$A_{kl} := \begin{cases} A & , \ k \text{ and } l \text{ are both squares,} \\ B & , \ \text{otherwise.} \end{cases}$$

Then, $st_2\text{-}\lim_{k,l\to\infty} A_{kl} = B$.

Theorem 3.4. *Let (X, d) be a metric space and (A_{kl}) be a double sequence of closed subsets of X. Then*

$$st_2\text{-}\liminf_{k,l\to\infty} A_{kl} = \bigcap_{N \in \mathcal{S}_2^{\#}} cl \bigcup_{(k,l)\in N} A_{kl} \quad \text{and} \quad st_2\text{-}\limsup_{k,l\to\infty} A_{kl} = \bigcap_{N \in \mathcal{S}_2} cl \bigcup_{(k,l)\in N} A_{kl}$$

Proof. Since the proof of the second equality is obtained by the similar way, we prove only the first equality. Let $x \in st_2\text{-}\liminf_{k,l\to\infty} A_{kl}$ and $N \in \mathcal{S}_2^{\#}$ be arbitrary. For every $\varepsilon > 0$, there exists $N_1 \in \mathcal{S}_2$ such that for every $(k,l) \in N_1$

$$A_{kl} \cap B(x,\varepsilon) \neq \emptyset.$$

Since $N \cap N_1 \neq \emptyset$, there exists $(k_0, l_0) \in N \cap N_1$ such that $A_{k_0 l_0} \cap B(x, \varepsilon) \neq \emptyset$. Therefore,

$$\left(\bigcup_{(k,l)\in N} A_{kl} \right) \cap B(x, \varepsilon) \neq \emptyset.$$

This means that $x \in \mathrm{cl} \bigcup_{(k,l)\in N} A_{kl}$. This holds for any $N \in \mathcal{S}_2^{\#}$. Consequently,

$$x \in \bigcap_{N\in\mathcal{S}_2^{\#}} \mathrm{cl} \bigcup_{(k,l)\in N} A_{kl}.$$

For the reverse inclusion, suppose that $x \notin st_2 - \liminf_{k,l\to\infty} A_{kl}$. Then, there exists $\varepsilon > 0$ such that

$$\delta_2\Big(\Big\{(k,l) \in \mathbb{N}^2 : A_{kl} \cap B(x,\varepsilon) \neq \emptyset\Big\}\Big) \neq 1$$

and so, the set

$$N = \Big\{(k,l) \in \mathbb{N}^2 : A_{kl} \cap B(x,\varepsilon) = \emptyset\Big\}$$

does not have double natural density zero, i.e. $N \in \mathcal{S}_2^{\#}$. Thus

$$\left(\bigcup_{(k,l)\in N} A_{kl} \right) \cap B(x, \varepsilon) = \emptyset.$$

This means that $x \notin \mathrm{cl} \bigcup_{(k,l)\in N} A_{kl}$ which completes the proof. □

As a result of Theorem 3.4, both the statistical outer and the statistical inner limits of a double sequence (A_{kl}) are closed sets.

Theorem 3.5. *Let (X, d) be a metric space and (A_{kl}) be a double sequence of closed subsets of X. Then, we have*

$$\begin{aligned} st_2\text{-}\limsup_{k,l\to\infty} A_{kl} &= \Big\{x \mid st_2\text{-}\liminf_{k,l\to\infty} d(x, A_{kl}) = 0\Big\}, \\ st_2\text{-}\liminf_{k,l\to\infty} A_{kl} &= \Big\{x \mid st_2\text{-}\lim_{k,l\to\infty} d(x, A_{kl}) = 0\Big\}. \end{aligned}$$

Proof. For any closed set A in X, we have

$$d(x, A) < \varepsilon \quad \Leftrightarrow \quad A \cap B(x,\varepsilon) \neq \emptyset. \tag{2}$$

Suppose that $st_2\text{-}\liminf_{k,l\to\infty} d(x, A_{kl}) = 0$. Then, for every $\varepsilon > 0$ we have

$$\delta_2\Big(\Big\{(k,l) \in \mathbb{N}^2 : d(x, A_{kl}) < \varepsilon\Big\}\Big) \neq 0.$$

By (2), we get $x \in st_2\text{-}\limsup_{k,l\to\infty} A_{kl}$.

Conversely, if $x \in st_2\text{-}\limsup_{k,l\to\infty} A_{kl}$, then for every $\varepsilon > 0$ there exists $N \in \mathcal{S}_2^{\#}$ such that $A_{kl} \cap B(x,\varepsilon) \neq \emptyset$ for every $(k,l) \in N$. Since

$$N \subseteq \Big\{(k,l) \in \mathbb{N}^2 : A_{kl} \cap B(x,\varepsilon) \neq \emptyset\Big\},$$

we get

$$\delta_2\Big(\Big\{(k,l) \in \mathbb{N}^2 : A_{kl} \cap B(x,\varepsilon) \neq \emptyset\Big\}\Big) \neq 0.$$

By (2) and Lemma 2.3, we have $st_2\text{-}\liminf_{k,l\to\infty} d(x, A_{kl}) = 0$.

Similarly, for any closed set A

$$d(x, A) \geq \varepsilon \Leftrightarrow A \cap B(x,\varepsilon) = \emptyset. \tag{3}$$

Now, the second equality can be obtained by using (3). □

Theorem 3.6. *Let (X, d) be a metric space and (A_{kl}) be a double sequence of closed subsets of X. Then*

$$st_2\text{-}\liminf_{k,l\to\infty} A_{kl} = \Big\{x \mid \exists N \in \mathcal{S}_2,\ \forall\,(k,l) \in N,\ \exists y_{kl} \in A_{kl} : \lim_{(k,l)\in N} y_{kl} = x\Big\}. \tag{4}$$

Proof. Let $x \in st_2\text{-}\liminf_{k,l\to\infty} A_{kl}$. Then, $st_2\text{-}\lim_{k,l\to\infty} d(x, A_{kl}) = 0$. By Lemma 2.1 there exists a subset $N \in \mathcal{S}_2$ such that

$$\lim_{(k,l)\in N} d(x, A_{kl}) = 0.$$

Since A_{kl} is closed, for $(k,l) \in N$, there exists $y_{kl} \in A_{kl}$ such that $d(x, y_{kl}) \le 2d(x, A_{kl})$. Now, we define the sequence $\{y_{kl} \mid y_{kl} \in A_{kl},\ (k,l) \in N\}$. Then, $\lim_{(k,l)\in N} y_{kl} = x$.

Conversely, if x is an element of the set given by the right side of the equality (4), then there exist $N \in \mathcal{S}_2$ and a sequence $\{y_{kl} \mid y_{kl} \in A_{kl},\ (k,l) \in N\}$ such that $\lim_{(k,l)\in N} y_{kl} = x$. Given $\varepsilon > 0$, then we can choose a number $n_0 \in \mathbb{N}$ such that for each $k, l > n_0$, $(k,l) \in N$, we have $y_{kl} \in B(x, \varepsilon)$. Define the set M by $M = N \setminus \{(m,n) \in N : m \le m_0 \text{ or } n \le n_0\}$. Then, $M \in \mathcal{S}_2$ and $y_{kl} \in A_{kl} \cap B(x, \varepsilon)$. This means that the sets A_{kl} and $B(x, \varepsilon)$ are not disjoint. Hence, $x \in st_2\text{-}\liminf_{k,l\to\infty} A_{kl}$. □

Corollary 3.7. *Let X be a normed linear space and (A_{kl}) be a double sequence of closed subsets of X. If there is a set $K \in \mathcal{S}_2$ such that A_{kl} is convex for each $(k,l) \in K$, then $st_2\text{-}\liminf_{k,l\to\infty} A_{kl}$ is convex.*

Proof. Let $st_2\text{-}\liminf_{k,l\to\infty} A_{kl} = A$. If x_1 and x_2 belong to A, by Theorem 3.6, for all $(k,l) \in N$ in some set $N \in \mathcal{S}_2$ we can find the points y^1_{kl} and y^2_{kl} in A_{kl} such that $\lim_{k,l\to\infty} y^1_{kl} = x_1$ and $\lim_{k,l\to\infty} y^2_{kl} = x_2$. Since $K \in \mathcal{S}_2$, we have $M \in \mathcal{S}_2$ with $M = N \cap K$. Then, for arbitrary $\lambda \in [0,1]$ and $(k,l) \in M$, let us define

$$y^\lambda_{kl} := (1-\lambda)y^1_{kl} + \lambda y^2_{kl} \quad \text{and} \quad x_\lambda := (1-\lambda)x_1 + \lambda x_2.$$

Then, $\lim_{(k,l)\in M} y^\lambda_n = x_\lambda$. By Theorem 3.6, we obtain $x_\lambda \in A$. This means that A is convex. □

Theorem 3.8. *Let (X, d) be a metric space and (A_{kl}) be a double sequence of closed subsets of X. Then, we have*

$$st_2\text{-}\limsup_{k,l\to\infty} A_{kl} = \Big\{x \mid \exists N \in \mathcal{S}^\#_2,\ \forall\,(k,l) \in N,\ \exists y_{kl} \in A_{kl} : x \in \Gamma_2(y)\Big\}. \tag{5}$$

Proof. Let x be an arbitrary point in $st_2\text{-}\limsup_{k,l\to\infty} A_{kl}$. By Theorem 3.5, we have

$$st_2\text{-}\liminf_{k,l\to\infty} d(x, A_{kl}) = 0.$$

By Lemma 2.3, for every $\varepsilon > 0$ the set

$$\Big\{(k,l) \in \mathbb{N}^2 : d(x, A_{kl}) < \frac{\varepsilon}{2}\Big\}$$

does not have double natural density zero. Since A_{kl} is closed, for $(k,l) \in \mathbb{N}^2$ there exists $y_{kl} \in A_{kl}$ such that $d(x, y_{kl}) \le 2d(x, A_{kl})$. Now, we define the sequence

$$\{y_{kl} \mid y_{kl} \in A_{kl},\ (k,l) \in \mathbb{N}^2\}.$$

Then, clearly, x is a statistical cluster point of (y_{kl}). That is, $x \in \Gamma_2(y)$.

On the other hand, if x is an element of the set given by the right side of the equality (5), then there exist $N \in \mathcal{S}^\#_2$ and a sequence $\{y_{kl} \mid y_{kl} \in A_{kl}, (k,l) \in N\}$ such that $x \in \Gamma_2(y)$. That is, the set $\Big\{(k,l) \in \mathbb{N}^2 : d(x, y_{kl}) < \varepsilon\Big\}$ does not have double natural density zero for $\varepsilon > 0$. The inequality $d(x, y_{kl}) \ge d(x, A_{kl})$ yields the inclusion

$$\{(k,l) \in \mathbb{N}^2 : d(x, y_{kl}) < \varepsilon\Big\} \subseteq \{(k,l) \in \mathbb{N}^2 : d(x, A_{kl}) < \varepsilon\Big\}.$$

So, the set $N' = \{(k,l) \in \mathbb{N}^2 : d(x, A_{kl}) < \varepsilon\Big\}$ does not have double natural density zero. That is, $N' \in \mathcal{S}^\#_2$. By (2), for every $(k,l) \in N'$ we obtain $A_{kl} \cap B(x, \varepsilon) \neq \emptyset$. This means that $x \in st_2\text{-}\limsup_{k,l\to\infty} A_{kl}$. □

By Theorem 3.6 and Theorem 3.8, note that when $A_{kl} \neq \emptyset$ for all $k, l \in \mathbb{N}$ the statistical outer and inner limit sets can be described equivalently in terms of the sequences $(y_{kl})_{k,l \in \mathbb{N}}$ that can be formed by selecting a $y_{kl} \in A_{kl}$ for each $(k, l) \in \mathbb{N}^2$: the set of all cluster points of such sequences is $st_2\text{-}\limsup_{k,l\to\infty} A_{kl}$, while the set of all limits of such sequences is $st_2\text{-}\liminf_{k,l\to\infty} A_{kl}$.

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