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Suborbital Graphs for a Special Subgroup of the $SL(3,\mathbb{Z})$

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Abstract. In this paper we examine some properties of suborbital graphs for the group $SL^*(3,\mathbb{Z})$. We first introduce an invariant equivalence relation by using the congruence subgroup $SL^*(3,\mathbb{Z})$ instead of $\Gamma_0(n)$ and obtain some results for the newly constructed subgraphs $F_{u,n}$ whose vertices form the block $[\infty]$. We obtain edge and circuit conditions and some relations between lengths of circuits in $F_{u,n}$ and elliptic elements of $\Gamma_0(n)$.

1. Introduction

Let $\hat{\mathbb{Z}}$ denote the set $(\mathbb{Z}\times\mathbb{Z})\cup\{\infty\}$ and $SL(3,\mathbb{Z})$ the special linear group of all matrices with integer coefficients with determinant 1. Also

$$SL^*(3,\mathbb{Z}) := \left\{\begin{pmatrix} a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{pmatrix} : a,b,c,d\in\mathbb{Z},\ ad-bc=1\right\}$$

is subgroup of $SL(3,\mathbb{Z})$.

Let $PSL(3,\mathbb{Z})$ be the group $SL(3,\mathbb{Z})/\{\pm I\}$. Then there is a homomorphism $\mu : SL(3,\mathbb{Z})\mapsto PSL(3,\mathbb{Z})$ with kernel $\{\pm I\}$. It is known that G.A. Jones, D. Singerman and K. Wicks [6] used the notion of the imprimitive action [3, 4] for a Γ-invariant equivalence relation induced on $\mathbb{Q}\cup\{\infty\}$ by the congruence subgroup

$$\Gamma_0(n) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : c\equiv 0 \,(\mathrm{mod}\ n)\right\}$$

to obtain some suborbital graphs and their properties.

In this study, we consider the action of the group $SL^*(3,\mathbb{Z})$ on the set $\hat{\mathbb{Z}}$ in the spirit of the theory of permutation groups, and graph arising from this action in hyperbolic geometric terms.

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2. The Action of $SL^*(3,\mathbb{Z})$ on $\hat{\mathbb{Z}}$

Any element of $\hat{\mathbb{Z}}$ is represented as $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$, with $x, y \in \mathbb{Z}$ and also ∞ is represented as $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$. The action of $SL^*(3,\mathbb{Z})$ on $\hat{\mathbb{Z}}$ now becomes

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \to \begin{pmatrix} ax+by \\ cx+dy \\ 0 \end{pmatrix}.$$

Theorem 2.1. *The action of $SL^*(3,\mathbb{Z})$ on $\hat{\mathbb{Z}}$ is transitive.*

Proof. It is enough to prove that the orbit containing ∞ is $\hat{\mathbb{Z}}$. If $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \in \hat{\mathbb{Z}}$ then there exist $\alpha, \beta \in \mathbb{Z}$ with $a\alpha - b\beta = 1$. Then the element $\begin{pmatrix} a & \beta & 0 \\ b & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is in $SL^*(3,\mathbb{Z})$ and sends ∞ to $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$. $\square$

We now consider the imprimitivity of the action of $SL^*(3,\mathbb{Z})$ on $\hat{\mathbb{Z}}$, beginning with a general discussion of primitivity of permutation groups. Let (G,Ω) be a transitive permutation group, consisting of a group G acting on a set Ω transitively. An equivalence relation $\approx$ on Ω is called *G-invariant* if, whenever $\alpha, \beta \in \Omega$ satisfy $\alpha \approx \beta$, then $g(\alpha) \approx g(\beta)$ for all $g \in G$. The equivalence classes are called blocks, and the block containing α is denoted by $[\alpha]$.

We call (G,Ω) *imprimitive* if Ω admits some G-invariant equivalence relation different from

(i) the identity relation, $\alpha \approx \beta$ if and only if $\alpha = \beta$;

(ii) the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Omega$.

Otherwise (G,Ω) is called *primitive*. These two relations are supposed to be trivial relations. Clearly, a primitive group must be transitive, for if not the orbits would form a system of blocks. The converse is false, but we have the following useful result in [3].

Lemma 2.2. *Let (G,Ω) be a transitive permutation group. (G,Ω) is primitive if and only if G_α,the stabilizer of $\alpha \in \Omega$, is a maximal subgroup of G for each $\alpha \in \Omega$.*

From the above lemma we see that whenever, for some α, $G_\alpha < H < G$, then Ω admits some G-invariant equivalence relation other than the trivial cases. Because of the transitivity, every element of Ω has the form $g(\alpha)$ for some $g \in G$. Thus one of the non-trivial G-invariant equivalence relation on Ω is given as follows:

$$g(\alpha) \approx g'(\alpha) \text{ if and only if } g' \in gH.$$

The number of blocks (equivalence classes) is the index $|G : H|$ and the block containing α is just the orbit $H(\alpha)$.

We can apply these ideas to case where G is the $SL^*(3,\mathbb{Z})$ and Ω is $\hat{\mathbb{Z}}$.

Lemma 2.3. *The stabilizer of ∞ in $SL^*(3,\mathbb{Z})$ is the set $\left\{ \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \lambda \in \mathbb{Z} \right\}$ denoted by $SL^*(3,\mathbb{Z})_\infty$.*

Proof. The stabilizer of a point in $\hat{\mathbb{Z}}$ is a infinite cyclic group. Since the action is transitive, stabilizers of any two points are conjugate. Therefore it is enough to look at the stabilizer of ∞ in $SL^*(3,\mathbb{Z})$.

$$T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}a&b&0\\c&d&0\\0&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

and so $\begin{pmatrix}a\\c\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$. Then $a=1, c=0$ and as $detT=1, d=1$. Therefore $b=\lambda\in\mathbb{Z}$. So $T=\begin{pmatrix}1&\lambda&0\\0&1&0\\0&0&1\end{pmatrix}$. This shows that the stabilizer of ∞ in $SL^*(3,\mathbb{Z})$ is $\left\langle\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\right\rangle$. □

Definition 2.4. $SL^*(3,\mathbb{Z})_0 := \{T\in SL^*(3,\mathbb{Z})|\ c\equiv 0(modn), n\in\mathbb{Z}\}$ is a subgroup of $SL^*(3,\mathbb{Z})$.

We must point out that the above equivalence relation is different from the one in [6]. Here let us take the group $SL^*(3,\mathbb{Z})_0$ instead of $\Gamma_0(n)$.

It is clear that $SL^*(3,\mathbb{Z})_\infty < SL^*(3,\mathbb{Z})_0 < SL^*(3,\mathbb{Z})$. We shall define an equivalence relation $\approx$ induced on $\hat{\mathbb{Z}}$ by $SL^*(3,\mathbb{Z})$. Now let $\begin{pmatrix}r\\s\\0\end{pmatrix},\begin{pmatrix}x\\y\\0\end{pmatrix}\in\hat{\mathbb{Z}}$. Corresponding to these there are two matrices

$$T_1 := \begin{pmatrix}r&*&0\\s&*&0\\0&0&1\end{pmatrix}, T_2 := \begin{pmatrix}x&*&0\\y&*&0\\0&0&1\end{pmatrix}$$

in $SL^*(3,\mathbb{Z})$ for which $T_1(\infty)=\begin{pmatrix}r\\s\\0\end{pmatrix}$ and $T_2(\infty)=\begin{pmatrix}x\\y\\0\end{pmatrix}$. We get the following imprimitive $SL^*(3,\mathbb{Z})$-invariant equivalence relation on $\hat{\mathbb{Z}}$ by $SL^*(3,\mathbb{Z})_0$ as

$$\begin{pmatrix}r\\s\\0\end{pmatrix}\approx\begin{pmatrix}x\\y\\0\end{pmatrix} \quad \text{if and only if} \quad T_1^{-1}T_2\in SL^*(3,\mathbb{Z})_0,$$

and so from the above we can easily verify that $\begin{pmatrix}r\\s\\0\end{pmatrix}\approx\begin{pmatrix}x\\y\\0\end{pmatrix}$ if and only if ; $ry-sx\equiv 0 \pmod n$. Here, the number $\psi(n)$ of blocks is $|SL^*(3,\mathbb{Z}) : SL^*(3,\mathbb{Z})_0|$.

Theorem 2.5. *The index* $|SL^*(3,\mathbb{Z}) : SL^*(3,\mathbb{Z})_0| = n\prod_{p|n}\left(1+\frac{1}{p}\right)$, *where the product is over the distinct primes p dividing* $n\in\mathbb{Z}$.

Proof. By our general discussion of imprimitivity, the number of equivalence classes under $\approx_m$ is given by $\psi(n)=|SL^*(3,\mathbb{Z}) : SL^*(3,\mathbb{Z})_0|$, the following formula for $\psi(n)$ is well-known but for completeness we will sketch a proof here. Firstly, we show that ψ multiplicative function. Let $n = lm$ with $(l, m) = 1$. Then, $v \approx_n w$ if and only if $v \approx_l w$ and $v \approx_m w$, so by counting equivalence classes we have

$$\psi(n) = \psi(l)\psi(m)$$

as required. Now the function $n \to n \prod_{p|n} \left(1 + \dfrac{1}{p}\right)$ on the right-hand side is clearly also multiplicative, so to prove the theorem it is sufficient to consider the case where n is a power of some prime p.

If $v = \begin{pmatrix} r \\ s \\ 0 \end{pmatrix} \in \hat{\mathbb{Z}}$ and is therefore a unit $modn$ we see that $\begin{pmatrix} r \\ s \\ 0 \end{pmatrix} \approx_n \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ or $\begin{pmatrix} r \\ s \\ 0 \end{pmatrix} \approx_n \begin{pmatrix} 1 \\ j \\ 0 \end{pmatrix}$ for some $i \in \mathbb{Z}_n$ or $j \in \mathbb{Z}_n$. Hence $2n$ classes are distinct.The number of such coincident pairs is Euler's function $\phi(n) = n(1 - \frac{1}{p})$, so the number of distinct classes is $2n - \phi(n) = n(1 + \frac{1}{p})$ as required. Consequently we have $\psi(n) = |SL^*(3, \mathbb{Z}) : SL^*(3, \mathbb{Z})_0| = n \prod_{p|n} \left(1 + \dfrac{1}{p}\right)$. $\square$

3. Suborbital Graphs of $SL^*(3, \mathbb{Z})$ on $\hat{\mathbb{Z}}$

In [9], Sims introduced the idea of the suborbital graphs of a permutation group G acting on a set Ω, these are graphs with vertex-set Ω, on which G induces automorphisms. We summarize Sims' theory as follows:

Let (G, Ω) be transitive permutation group. Then G acts on $\Omega \times \Omega$ by

$$g(\alpha, \beta) = (g(\alpha), g(\beta))$$

where $g \in G, \alpha, \beta \in \Omega$. The orbits of this action are called *suborbitals* of G. The orbit containing (α, β) is denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a *suborbital graph* $G(\alpha, \beta)$: its vertices are the elements of Ω, and there is a directed edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$. A directed edge from γ to δ is denoted by $\gamma \to \delta$. If $(\gamma, \delta) \in O(\alpha, \beta)$, then we will say that there exists an edge $\gamma \to \delta$ in $G(\alpha, \beta)$. This theory reveals the relationship between graphs and permutation groups. In this paper our calculation concerns $SL^*(3, \mathbb{Z})$, so we can draw this edge as a hyperbolic geodesic in the upper half-space $\mathbb{H}^3 := \{(x, y, z) | x, y, z \in \mathbb{R}, z \geq 0\}$.

The orbit $O(\beta, \alpha)$ is also a suborbital graph and it is either equal to or disjoint from $O(\alpha, \beta)$. In the latter case $G(\beta, \alpha)$ is just $G(\alpha, \beta)$ with the arrows reserved and we call, in this case, $G(\alpha, \beta)$ and $G(\beta, \alpha)$ *paired suborbital graphs*. In the former case $G(\alpha, \beta) = G(\beta, \alpha)$ and the graph consists of pairs of oppositely directed edges; it is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call *self paired*.

The above ideas are also described in a paper by Neumann[7] and in books by Tsuzuku [10] and by Bigg and White [3], the emphasis being on applications to finite groups.

In this study, G and Ω will be $SL^*(3, \mathbb{Z})$ and $\hat{\mathbb{Z}}$, respectively. Since $SL^*(3, \mathbb{Z})$ acts transitively on $\hat{\mathbb{Z}}$, each suborbital contains a pair (∞, v) for some $v \in \hat{\mathbb{Z}}$; writing $v = \frac{u}{n}$, we denote this suborbital by $O_{u,n}$ and the corresponding suborbital graph by $G_{u,n}$.

Definition 3.1. By a directed circuit in $G_{u,n}$ we mean that a sequence $v_1, v_2, \ldots, v_m$ of different vertices such that $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_m \longrightarrow v_1$, where $m \geq 3$; an anti-directed circuit will denote a configuration like the above with at least an arrow (not all) reversed.

If $m = 2$, then we will call the configuration $v_1 \longrightarrow v_2 \longrightarrow v_1$ a self paired edge: it consists of a loop based at each vertex.

If $m = 3$ or $m = 4$, then the circuit, directed or not, is called a triangle or quadrilateral.
We call a graph a *forest* if it does not contain any circuits.

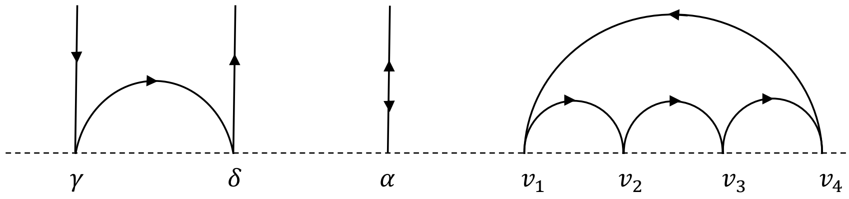

Figure 1: Circuits

3.1. *Graph $G_{u,n}$*

We now investigate the suborbital graphs for the action $SL^*(3,\mathbb{Z})$ on $\hat{\mathbb{Z}}$. We use the following theorem frequently in our calculation.

Theorem 3.2. *Let $r,s,x,y \in \mathbb{Z}^+$ and then only the following occur*

(I) *there exists an edge* $\begin{pmatrix} r\\ s\\ 0\end{pmatrix} \to \begin{pmatrix} x\\ y\\ 0\end{pmatrix}$ *or* $\begin{pmatrix} -r\\ -s\\ 0\end{pmatrix} \to \begin{pmatrix} -x\\ -y\\ 0\end{pmatrix}$ *in $G_{u,n}$ if and only if* $x \equiv -ur \,(\mathrm{mod}\ n)$, $\quad y \equiv -us\,(\mathrm{mod}\ n)$ *and* $ry - sx = -n$,

(II) *there exists an edge* $\begin{pmatrix} r\\ s\\ 0\end{pmatrix} \to \begin{pmatrix} -x\\ -y\\ 0\end{pmatrix}$ *or* $\begin{pmatrix} -r\\ -s\\ 0\end{pmatrix} \to \begin{pmatrix} x\\ y\\ 0\end{pmatrix}$ *in $G_{u,n}$ if and only if* $x \equiv ur\,(\mathrm{mod}\ n)$, $\quad y \equiv us\,(\mathrm{mod}\ n)$ *and* $ry - sx = n$,

(III) *there exists an edge* $\begin{pmatrix} -r\\ s\\ 0\end{pmatrix} \to \begin{pmatrix} -x\\ y\\ 0\end{pmatrix}$ *or* $\begin{pmatrix} r\\ -s\\ 0\end{pmatrix} \to \begin{pmatrix} x\\ -y\\ 0\end{pmatrix}$ *in $G_{u,n}$ if and only if* $x \equiv -ur\,(\mathrm{mod}\ n)$, $\quad y \equiv -us\,(\mathrm{mod}\ n)$ *and* $ry - sx = n$,

(IV) *there exists an edge* $\begin{pmatrix} -r\\ s\\ 0\end{pmatrix} \to \begin{pmatrix} x\\ -y\\ 0\end{pmatrix}$ *or* $\begin{pmatrix} r\\ -s\\ 0\end{pmatrix} \to \begin{pmatrix} -x\\ y\\ 0\end{pmatrix}$ *in $G_{u,n}$ if and only if* $x \equiv ur\,(\mathrm{mod}\ n)$, $\quad y \equiv us\,(\mathrm{mod}\ n)$ *and* $ry - sx = -n$.

Proof. Let r,s,x,y in positive integer.We suppose that there exists an edge $\begin{pmatrix} r\\ s\\ 0\end{pmatrix} \to \begin{pmatrix} x\\ y\\ 0\end{pmatrix}$ in $G_{u,n}$ and $\left(\begin{pmatrix} r\\ s\\ 0\end{pmatrix}, \begin{pmatrix} x\\ y\\ 0\end{pmatrix}\right) \in O_{u,n}$. Therefore there exist some T in $SL^*(3,\mathbb{Z})$ such that T sends the pair $\left(\begin{pmatrix} -1\\ 0\\ 0\end{pmatrix}, \begin{pmatrix} u\\ n\\ 0\end{pmatrix}\right)$ to the pair $\left(\begin{pmatrix} r\\ s\\ 0\end{pmatrix}, \begin{pmatrix} x\\ y\\ 0\end{pmatrix}\right)$, that is $T\begin{pmatrix} -1\\ 0\\ 0\end{pmatrix} = \begin{pmatrix} r\\ s\\ 0\end{pmatrix}$ and $T\begin{pmatrix} u\\ n\\ 0\end{pmatrix} = \begin{pmatrix} x\\ y\\ 0\end{pmatrix}$. Now let $T := \begin{pmatrix} a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{pmatrix}$, $a,b,c,d \in \mathbb{Z}$. Then we have that $\begin{pmatrix} -a\\ -c\\ 0\end{pmatrix} = \begin{pmatrix} r\\ s\\ 0\end{pmatrix}$

and $\begin{pmatrix} au+bn \\ cu+dn \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. Therefore $-a = r, -c = s$, $au + bn = x$ and $cu + dn = y$. Hence, we write that

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & u & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} r & x & 0 \\ s & y & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From the determinant, we get $-n = ry - sx$. Thus, we obtain that $x \equiv -ur(modn)$, $y \equiv -us(modn)$ and $ry - sx = -n$.

Conversely, we assume that $x \equiv -ur(modn)$, $y \equiv -us(modn)$ and $ry - sx = -n$. Then there exist $b, d \in \mathbb{Z}$ such that $x = -ur + bn$, $y = -us + dn$. Taking $a = -r$ and $c = -s$, then $x = au + bn$, $y = cu + dn$ and so $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & u & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} r & x & 0 \\ s & y & 0 \\ 0 & 0 & 1 \end{pmatrix}$. As $ry-sx = -n$, we have $ad-bc = 1$, so $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL^*(3,\mathbb{Z})$ and hence $\begin{pmatrix} r \\ s \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ in $G_{u,n}$. The proof for $\begin{pmatrix} -r \\ -s \\ 0 \end{pmatrix} \to \begin{pmatrix} -x \\ -y \\ 0 \end{pmatrix}$ is similar. We can prove cases (II), (III) and (IV) similarly. □

Theorem 3.3. *$G_{u,n}$ is self-paired if and only if $u^2 + 1 \equiv 0(modn)$.*

Proof. We suppose that $G_{u,n}$ is self-paired. If $\infty \to \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$, then it must also be $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \to \infty$. From the edge $\begin{pmatrix} u \\ n \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, we have that $1 \equiv -u^2(modn)$. Therefore, $u^2 + 1 \equiv 0(modn)$.

Conversely, assume that $u^2 + 1 \equiv 0(modn)$. There exists some integer b such that $u^2 + 1 \equiv bn$. Hence $-u^2 + bn = 1$. Let $T := \begin{pmatrix} u & -b & 0 \\ n & -u & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then $T(\infty) = \begin{pmatrix} u \\ n \\ 0 \end{pmatrix}$, $T\begin{pmatrix} u \\ n \\ 0 \end{pmatrix} = \infty$ and $detT = 1$. □

If $\begin{pmatrix} r \\ s \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ in $G_{u,n}$, then Theorem 3.2 implies that $ry - sx = \pm n$, so $\begin{pmatrix} r \\ s \\ 0 \end{pmatrix} \approx \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. Thus each connected component of $G_{u,n}$ lies in a single block for $\approx$, of which there are $\psi(n)$, so we have:

Corollary 3.4. *$G_{u,n}$ has at least $\psi(n)$ connected components; in particular, $G_{u,n}$ is not connected if n is not a unit.*

3.2. Subgraph $F_{u,n}$

As we saw, each $G_{u,n}$ is a disjoint union of $\psi(n)$ subgraphs, the vertices of each subgraph forming a single block with respect to the relation $\approx$. Since $SL^*(3,\mathbb{Z})$ acts transitively on $\hat{\mathbb{Z}}$, it permutes these blocks transitively, so the subgraphs are all isomorphic. We let $F_{u,n}$ be the subgraph of $G_{u,n}$ whose vertices form the block

$$[\infty] := \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix} = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \mid x, y \in \mathbb{Z} \quad \text{and} \quad y \equiv 0 \,(\text{mod } n) \right\},$$

so that $G_{u,n}$ consists of $\psi(n)$ disjoint copies of $F_{u,n}$.

Theorem 3.5. *Let* $r, s, x, y \in \mathbb{Z}^+$ *and* $\begin{pmatrix} r \\ s \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in [\infty]$. *Then*

(I) *there exists an edge* $\begin{pmatrix} (-1)^i r \\ (-1)^i s \\ 0 \end{pmatrix} \to \begin{pmatrix} (-1)^i x \\ (-1)^i y \\ 0 \end{pmatrix}$ *in* $F_{u,n}$ *where* $i = 0$ *or* $i = 1$ *if and only if* $x \equiv -ur \pmod{n}$ *and* $ry - sx = -n$,

(II) *there exists an edge* $\begin{pmatrix} (-1)^i r \\ (-1)^i s \\ 0 \end{pmatrix} \to \begin{pmatrix} (-1)^j x \\ (-1)^j y \\ 0 \end{pmatrix}$ *in* $F_{u,n}$ *where* $i = 0, j = 1$ *or* $i = 1, j = 0$ *if and only if* $x \equiv ur \pmod{n}$ *and* $ry - sx = n$,

(III) *there exists an edge* $\begin{pmatrix} (-1)^i r \\ (-1)^j s \\ 0 \end{pmatrix} \to \begin{pmatrix} (-1)^i x \\ (-1)^j y \\ 0 \end{pmatrix}$ *in* $F_{u,n}$ *where* $i = 1, j = 0$ *or* $i = 0, j = 1$ *if and only if* $x \equiv -ur \pmod{n}$ *and* $ry - sx = n$,

(IV) *there exists an edge* $\begin{pmatrix} (-1)^i r \\ (-1)^j s \\ 0 \end{pmatrix} \to \begin{pmatrix} (-1)^j x \\ (-1)^i y \\ 0 \end{pmatrix}$ *in* $F_{u,n}$ *where* $i = 1, j = 0$ *or* $i = 0, j = 1$ *if and only if* $x \equiv ur \pmod{n}$ *and* $ry - sx = -n$.

An automorphism of the graph $F_{u,n}$ is a permutation of $[\infty]$ which takes edges to edges. In view of this it can easily seen that $SL^*(3,\mathbb{Z})_0 < AutF_{u,n}$.

Theorem 3.6. $SL^*(3,\mathbb{Z})_0$ *permutes the vertices and the edges of* $F_{u,n}$ *transitively.*

Proof. Suppose that $u, v \in [\infty]$. As $SL^*(3,\mathbb{Z})$ acts on $\hat{\mathbb{Z}}$ transitively, $g(u) = v$ for some $g \in SL^*(3,\mathbb{Z})$. Since $u \approx \infty$ and $\approx$ is $SL^*(3,\mathbb{Z})$-invariant equivalence relation, $g(u) \approx g(\infty)$; that is $v \approx g(\infty)$. Thus, as $v \approx g(\infty)$, $g \in SL^*(3,\mathbb{Z})_0$.

Assume that $v, w \in [\infty]$; $k_1, k_2 \in [\infty]$ and $v \to w$, $k_1 \to k_2 \in F_{u,n}$. Then (v, w), $(k_1, k_2) \in O\left(\infty, \begin{pmatrix} u \\ n \\ 0 \end{pmatrix}\right)$.

Therefore, for some $S, T \in SL^*(3,\mathbb{Z})$;

$$S(\infty) = v,\ S\begin{pmatrix} u \\ n \\ 0 \end{pmatrix} = w;\ T(\infty) = k_1,\ T\begin{pmatrix} u \\ n \\ 0 \end{pmatrix} = k_2.$$

Hence $S, T \in SL^*(3,\mathbb{Z})_0$ as $S(\infty), T(\infty) \in [\infty]$. Furthermore $TS^{-1}(v) = k_1$ and $TS^{-1}(w) = k_2$; that is $TS^{-1} \in SL^*(3,\mathbb{Z})_0$. $\square$

Theorem 3.7. $F_{u,n}$ *contains directed triangles if and only if* $u^2 + u + 1 \equiv 0 (mod n)$.

Proof. Suppose that $F_{u,n}$ contains a directed triangle. Because of the transitive action, the form of directed triangle can be taken as $\infty \to \begin{pmatrix} u \\ n \\ 0 \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 n \\ 0 \end{pmatrix} \to \infty$. Since $\begin{pmatrix} u \\ n \\ 0 \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 n \\ 0 \end{pmatrix}$, then $uy_0 - x_0 = -1$ and $x_0 \equiv -u^2 (modn)$. From $\begin{pmatrix} x_0 \\ y_0 n \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $y_0 = 1$ is obtained. Hence $x_0 = u + 1$. Consequently, we have that $u^2 + u + 1 \equiv 0(modn)$.

Conversely, assume that $u^2 + u + 1 \equiv 0(modn)$. Then by Theorem 3.2 the circuit $\infty \to v_1 \to v_2 \to \infty$ is a directed triangle in $F_{u,n}$. $\square$

3.3. Some results

Corollary 3.8. *Transformations* $\phi_1 = \begin{pmatrix} u & -\frac{u^2+u+1}{n} \\ n & -u-1 \end{pmatrix}$, $\phi_2 = \begin{pmatrix} u & \frac{u^2+u+1}{n} \\ -n & -u-1 \end{pmatrix}$, $\phi_3 = \begin{pmatrix} -u & \frac{u^2+u+1}{n} \\ -n & u+1 \end{pmatrix}$, $\phi_4 = \begin{pmatrix} -u & -\frac{u^2+u+1}{n} \\ n & u+1 \end{pmatrix}$ *in* $\Gamma_0(n)$*, which are defined by means of the congruence* $u^2 + u + 1 \equiv 0(modn)$*, are elliptic element of order* 3.

And also $\varphi_1 := \begin{pmatrix} u & -\frac{u^2+u+1}{n} & 0 \\ n & -u-1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $det\varphi_1 = 1$. *Furthermore, it is easily seen that*

$$\varphi_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -u \\ n \\ 0 \end{pmatrix}, \varphi_1 \begin{pmatrix} -u \\ n \\ 0 \end{pmatrix} = \begin{pmatrix} -u-1 \\ n \\ 0 \end{pmatrix}, \varphi_1 \begin{pmatrix} -u-1 \\ n \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Similarly, the others are also illustrated. The transformations ϕ_i*, where* $1 \le i \le 4$*, establish a connection between circuits in the graph and elliptic elements in the group* $\Gamma_0(n)$*.*

Example 3.9. Let $n = 3, u = 1$. Then we have eight triangles in $F_{1,3}$:

$$\begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} \to \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \to \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \to \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \to \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix}.$$

These are pictured as,

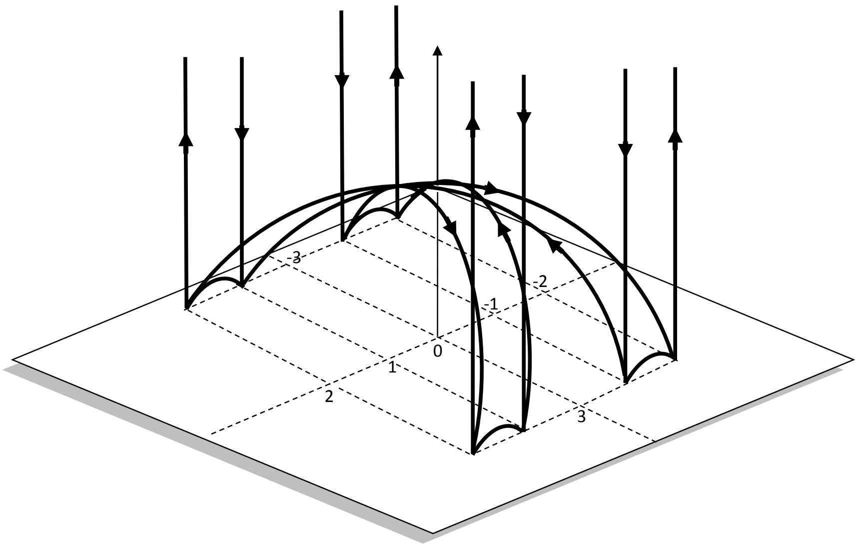

Figure 2: Triangles in $F_{1,3}$

Example 3.10. Let $n = 2, u = 1$. Then, since $u^2 + u + 1 \equiv 0 \pmod{n}$ does not hold , there are not any triangles in $F_{1,2}$. But there are 2-gons in $F_{1,2}$:

$$\begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \to \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}.$$

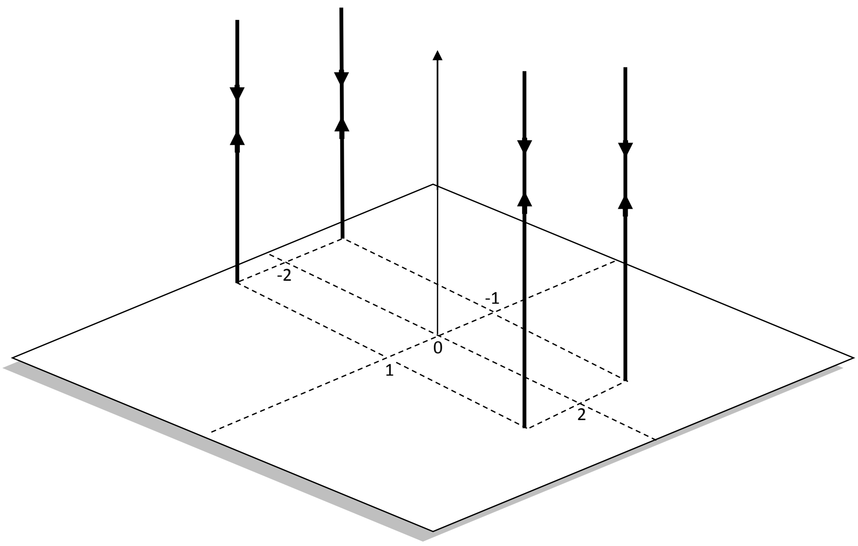

Figure 3: Self paired edges in $F_{1,2}$

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References

[1] M. Akbaş, D. Singerman, The signature of the normalizer of $\Gamma_0(N)$, London Math. Soc. Lecture Note Series 165 (1992) 77–86.
[2] M. Akbaş, On suborbital graphs for the modular group, Bull. London Math. Soc. 33 (2001) 647–652.
[3] N.L. Bigg, A.T. White, Permutation groups and combinatorial structures, Cambridge University Press, Cambridge, 1979.
[4] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, Oxford, 1979.
[5] G.A. Jones, D. Singerman, Complex functions: an algebraic and geometric viewpoint, Cambridge University Press, Cambridge, 1987.
[6] G.A. Jones, D. Singerman, K. Wicks, The modular group and generalized Farey graphs, London Math. Soc. Lecture Note Series 160 (1991) 316–338.
[7] P.M. Neumann, Finite Permutation Groups, Edge-Coloured Graphs and Matrices, Topics in Group Theory and Computation, Ed. M.P.J. Curran, Academic Press, London-New York-San Fransisco, 1977.
[8] B. Schoeneberg, Elliptic Modular Functions, Springer, Berlin, 1974.
[9] C.C. Sims, Graphs and finite permutation groups, Math. Z. 95 (1967) 76–86.
[10] T. Tsuzuku, Finite Groups and Finite Geometries, Cambridge University Press, Cambridge, 1982.
[11] B.Ö. Güler, M. Beşenk, A.H. Değer, S. Kader, Elliptic elements and circuits in suborbital graphs, Hacettepe J. Math. Stat. 40 (2011) 203–210.
[12] Y. Kesicioğlu, M. Akbaş, M. Beşenk, Connectedness of a suborbital graph for congruence subgroups, J. Ineq. Appl. 1 (2013) 117–124.