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A New Graph over Semi-Direct Products of Groups

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Abstract. In this paper, by establishing a new graph $\Gamma(G)$ over the semi-direct product of groups, we will first state and prove some graph-theoretical properties, namely, diameter, maximum and minimum degrees, girth, degree sequence, domination number, chromatic number, clique number of $\Gamma(G)$. In the final section we will show that $\Gamma(G)$ is actually a perfect graph.

1. Introduction and Preliminaries

In this paper, as we indicated in the title, we will mainly define a new graph over semi-direct products of groups and then will investigate the graph theoretical properties for this new graph. In detailed, for any two subgroups A and K of G satisfying $A \leq G$, $K \lhd G$ and $A \cap K = \{1_G\}$, we will say that G is a semi-direct product of A and K and then will define a new simple undirected graph $\Gamma(G)$ in terms of G. After that, as a quite similar way in [13, 20], we will investigate some graph-theoretical properties (such as the diameter, maximum and minumum degrees, girth, chromatic number, clique number, domination number, degree sequence and irregularity index) over $\Gamma(G)$.

In the literature, there are some important studies that are interested in special graph varieties related to algebraic structures. The most important class of graphs associated to algebraic structures is that of Cayley graphs, because these graphs have valuable applications (cf. [24]) and are related to automata theory (cf. [25, 26]). Cayley graphs of groups have been considered in many articles (see, for example, [7, 22, 23, 31]). Well-known classes of graphs related to algebraic structures include power graphs (cf. [1, 27]) and zero-divisor graphs (cf. [2, 3, 14]). In fact our new graph $\Gamma(G)$ will be constructed in the light of these thoughts. However the graph $\Gamma(G)$ in here is different than the previous special type of graphs and corresponding works on them since it will be built up on the semi-direct product of two groups in the meaning of the vertex and edge sets. This will give us an opportunity for extending the theory used in here to iterated semi-direct products (cf. [12]) and wreath products (cf. [34]) for future studies. We also note that although a similar approximation has been recently applied by the second author in another joint paper [20], the results in here are more general than the previous study since the submonoids considered in [20] were very special while the subgroups in here are not. We finally indicate that one of the reason to define the graph $\Gamma(G)$ is the following. In some studies, for instance, [21, 28, 32] while the diameters in the related graphs are given as inequalities, the same parameter in here will be given as a direct equation (see Theorem 2.1).

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In here we also establish up similar conditions for the graph parameter girth (see Theorem 2.2), and for maximum and minumum degrees (see Theorem 2.3).

Since our new graph construction will be based on the semi-direct products of two groups (see, for instance, [8, 34]), let us remind it very briefly: For any two finite groups A and K with presentations $\mathcal{P}_A = [X\,;\, \mathbf{r}]$ and $\mathcal{P}_K = [Y\,;\, \mathbf{s}]$, and also for all $a \in A$, let us consider the homomorphism $\varphi : A \to Aut(K)$, $a \mapsto \varphi_a$. The semi-direct product $G = K \rtimes_\varphi A$ of K by A is defined as the set of elements that are all ordered pairs (a,k) $(a \in A,\ k \in K)$ satisfying the multiplication $(a,k)(a',k') = (aa',(k\varphi_{a'})k')$, and has a presentation $\mathcal{P} = [X,\ Y\,;\, \mathbf{r},\ \mathbf{s},\ \mathbf{t}]$, where $\mathbf{t} = \{yx\lambda_{yx}^{-1}x^{-1} \mid y \in Y,\ x \in X\}$ and λ_{yx} is a word on Y representing the element $(k_y)\varphi_{a_x}$ of K $(a \in A,\ k \in K,\ x \in X,\ y \in Y)$ (cf. [19, Proposition 10.1]). Throughout this paper, all elements z_i $(i = 1, 2, \ldots, k)$ in the generating set $X \cup Y$ of G will be formed as $z_i \neq z_1^{\varepsilon_1} z_2^{\varepsilon_2} \ldots z_k^{\varepsilon_k}$, where $k \geq 2$ according to the *Normal Form Theorem* (*NFT*) (see [11]). Also the homomorphism φ will always be not identity id_G unless stated otherwise. We should finally note that, in some sources of the literature, the definition of the semi-direct product does not always given as in above. But we actually preferred to use it as this way since group presentations are very economical way to describe the generating set and relations (which contains the canonical elements, see [11] for the details) among these generators. A generating set for a group is essential for defining a new graph in this paper (see below).

1.1. *A new graph* $\Gamma(G)$ *based on semi-direct products*

As we described in the previous section, since the semi-direct product G is constructed on two finite groups A and K, the generating set $X \cup Y$ of it certainly contains a finite number of elements. This situation occurs in the definition of $\Gamma(G) = (V, E)$ associated with G since the *vertex set V* consists of the whole elements of G (not only generaters, and so $V = G$) and the *edge set E* is obtained by the following steps:

(I) Each of the vertices in this graph must be adjoined to the vertex 1_G (except 1_G itself since the graph is assumed to be simple).

(II) **(*i*)** For any two vertices $w_1 = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ \ldots\ x_m^{\varepsilon_m}$ and $w_2 = y_1^{\delta_1} y_2^{\delta_2} \ \ldots\ y_n^{\delta_n}$ (where $n \geqslant 2$, ε_i and δ_j are integers) and for all $x_i,\ y_j \in X \cup Y$ $(1 \leq i \leq m\ , 1 \leq j \leq n)$, if $x_i \neq y_j$, then w_1 is adjoined to the w_2 (shortly, $w_1 \sim w_2$).

(*ii*) As a consequence of (i), for any two vertices $w_1 = x_i^k$ and $w_2 = x_j^t$ $(1 \leq i,\ j \leq n,\ i \neq j$, and k, t are integers), we can directly take $w_1 \sim w_2$. However, to adjoin w_1 and w_2 while $i = j$, it must be $k \neq t$.

In the rest of this paper the notation Γ will always denote a general undirected simple graph while $\Gamma(G)$ will denote the graph defined in the above steps, and also all results in this paper will be based on $\Gamma(G)$. In Figures 1 and 2, we will present some $\Gamma(G)$ graph examples that are obtained by the above definition.

2. Some Special Properties of $\Gamma(G)$

In this section, by considering $\Gamma(G)$, we will mainly deal with the special graph properties, namely diameter, girth, maximum and minimum degrees, domination numbers and finally irregularity index. It is quite well known that, for any graph Γ, most of these properties can be obtained by checking the distance or the total number of the vertices (see [15]). So we will follow the same idea to prove the results in this section.

We first recall that, for any Γ, the *distance* (i.e. *length of the shortest path*) between two vertices w_1, w_2 of Γ is denoted by $d_\Gamma(w_1,\ w_2)$, and so the *diameter* of Γ is defined by the set $diam(\Gamma) = \sup\{d_\Gamma(w_1,\ w_2) :$ w_1 and w_2 are vertices of $\Gamma\}$. By taking into account $\Gamma(G)$ as Γ, we obtain the following result.

Theorem 2.1. *Let G be a semi-direct product of any two finite groups, and let us consider* $\Gamma(G)$. *Then* $diam\Gamma(G) = 2$.

Proof. The proof will be presented in three cases by considering the vertices of $\Gamma(G)$.

Case 1: Let $w_1 = x_i^{\varepsilon_i}$ and $w_2 = y_j^{\delta_j}$ be any two vertices in $\Gamma(G)$. Thus, by (II)-(*ii*) in Section 1.1, $w_1 \sim w_2$ and so $d_G(w_1,\ w_2) = 1$.

Case 2: Assume that $w_1 = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ and $w_2 = y_j^{\delta_j}$ are any two vertices in $\Gamma(G)$. By (II)-(i) in Section 1.1, for each $i = 1, 2, \ldots, n$, if $x_i \neq y_j$ then $w_1 \sim w_2$ which implies $d_G(w_1, w_2) = 1$. On the other hand, for at least one $i = 1, 2, \ldots, n$, if $\exists\, x_i = y_j$ then $w_1 \nsim w_2$ but since $w_1 \sim 1_G$ and $w_2 \sim 1_G$ (by (I) in Section 1.1), we get $w_1 \sim 1_G \sim w_2$ and this clearly gives $d_G(w_1, w_2) = 2$.

Case 3: As the next step of Case 2, let us consider two vertices $w_1 = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ and $w_2 = y_1^{\delta_1} y_2^{\delta_2} \ldots y_n^{\delta_n}$ in $V(\Gamma(G))$. By a similar approximation, for all i, $j = 1, 2, \ldots, n$, if $x_i \neq y_j$ then $w_1 \sim w_2$ which implies $d_G(w_1, w_2) = 1$ by (II)-(i) in Section 1.1. However, for at least one i, $j = 1, 2, \ldots, n$, if $\exists\, x_i = y_j$ then $w_1 \nsim w_2$ but since $w_1 \sim 1_G$ and $w_2 \sim 1_G$ (by (I) in Section 1.1), we get $w_1 \sim 1_G \sim w_2$ and so $d_G(w_1, w_2) = 2$.

At this point we accurately note that since G is a semi-direct product (and so $\varphi \neq id_G$), it cannot be abelian and therefore cannot be a cyclic group. Therefore the generating set of G definitely does not consist of one element x_i. This implies that *Case 1* cannot be the only case that occurs for all pairs of vertices w_1 and w_2. In other words, the generating set of G must contain another element y_j together with x_i, and then the vertices of the related graph must be formed as $w = x_i^k y_j^t$ (where $1 \leq k \leq n$ and $1 \leq t \leq m$ such that n, m stand for the orders of x_i and y_j, respectively). Notice that the length of these elements can be at least two with the form $w = x_i y_j$. Hence, by the adjoining steps defined in Section 1.1, this situation always shows that $diam(\Gamma(G)) = 2$. □

For our next result, let us recall another graph parameter *girth*. It is quite well known that the *girth* of any Γ, denoted by $girth(\Gamma)$, is the length of the shortest cycle contained in it. However, if Γ does not contain any cycle (i.e. *acyclic*), then the girth of it is assumed to be infinity. Thus we have the following theorem.

Theorem 2.2. *For a semi-direct product G of any two finite groups,* $girth(\Gamma(G)) = 3$.

Proof. By considering the vertices of $\Gamma(G)$, the proof is given in two main cases as similar as in the proof of Theorem 2.1.

Case 1: For a vertex $w_1 = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ ($n \geq 2$) in $\Gamma(G)$, if there does not exist an adjacent vertex $w_2 = y_1^{\delta_1} y_2^{\delta_2} \ldots y_m^{\delta_m}$, in other words $\exists\, x_i = y_j$ ($1 \leq i \leq n$, $1 \leq j \leq m$), we cannot discuss about the cycle in $\Gamma(G)$. However, if there exists $\forall\, x_i \neq y_j$ ($1 \leq i \leq n$, $1 \leq j \leq m$), then we only get $w_1 \sim 1_G$ and $w_2 \sim 1_G$ (equivalently, $w_1 \sim 1_G \sim w_2 \sim w_1$) by (I) in Section 1.1. Thus $girth(\Gamma(G)) = 3$.

Case 2: For a vertex $w_1 = x_i^{\varepsilon_i}$ in $\Gamma(G)$, since G is a group and so there must be a cyclic subgroup $< x_i >$ of G, we certainly have another vertex (actually an element of G) $w_2 = x_i^t$ (or if $o(x_i) = 2$ then, by Section 1.1, there is $w_3 = x_j$ in $\Gamma(G)$) in $\Gamma(G)$. Hence, by the edge set steps in Section 1.1, $w_1 \sim w_2$ ($or\ w_3$), and then $w_1 \sim 1_G$ and $w_2 \sim 1_G$ (i.e. $w_1 \sim 1_G \sim w_2$ ($or\ w_3$) $\sim w_1$). This also implies $girth(\Gamma(G)) = 3$. □

For any simple graph Γ, the *degree* of a vertex w of Γ, denoted by $deg_\Gamma(w)$, is the number of vertices adjacent to w. Among all degrees, the *maximum degree* $\Delta(\Gamma)$ and the *minimum degree* $\delta(\Gamma)$ of Γ is the number of the largest and the smallest degrees, respectively, in Γ. Our next result is actually about these two parameters.

Theorem 2.3. *Let G be a semi-direct product of order n, and let* $\Gamma(G)$ *be the graph obtained from G as defined in Section* 1.1. *Then the maximum and minimum degrees of* $\Gamma(G)$ *are determined by* $\Delta(\Gamma(G)) = n - 1$ *and* $\delta(\Gamma(G)) = 1$, *respectively.*

Proof. Since G has total n element included 1_G, by (I) in Section 1.1, for all $w \in \Gamma(G)$ ($w \neq 1_G$), we have $w \sim 1_G$. Therefore 1_G has total $n - 1$ adjacent vertex, and so $deg_G(1_G) = n - 1$. (Recall also that for a simple graph with n vertices, the maximum degree can be $n - 1$). Thus $\Delta(\Gamma(G)) = n - 1$.

Now assume that the group G (having n elements) is generated by $\langle x_1, x_2, \ldots, x_k \rangle$. (Recall that the generating set must contains finite number of elements by our assumption). Therefore, for all $\varepsilon_i \neq 0$, we certainly have a vertex $w_1 = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_k^{\varepsilon_k}$. Thus, by the definition in Section 1.1, since there should not be existed an adjacent vertex $1_G \neq w_2 = y_1^{\delta_1} y_2^{\delta_2} \ldots y_k^{\delta_k}$ (for all $x_i \neq y_j$ and $i, j = 1, 2, \ldots, k$) with w_1, the vertex w_1 can only be adjacent with 1_G. As a result of this, $\delta(\Gamma(G)) = 1$. □

For any simple graph Γ, there also exists the term *degree sequence* $DS(\Gamma)$ which is a sequence of degrees of the vertices in Γ. In [33], a new graph parameter *irregularity index*, denoted by $t(\Gamma)$, is defined for simple graphs. In fact $t(\Gamma)$ is the number of distinct terms in the set $DS(\Gamma)$. In the following, we will prove a result on degree sequence and irregularity index by considering a semi-direct product of two finite cyclic groups. We note that although most of the results over irregularity index are stated as inequalities, our next theorem establishes an exact equality.

Theorem 2.4. *Let us take two finite cyclic groups $C_m = \langle x\rangle$ and $C_n = \langle y\rangle$ of orders m and n, respectively, and also let us consider a semi-direct product $G = C_m \rtimes_\varphi C_n$ such that $\varphi \neq id_G$. Under these conditions, the degree sequence and irregularity index of the graph $\Gamma(G)$ are*

$$DS(\Gamma(G)) = \left\{ \underbrace{1,\ 1, \ldots,\ 1}_{((m-1)(n-1))\ times},\ \underbrace{(m+n-2),\ (m+n-2), \ldots,\ (m+n-2)}_{(m+n-2)\ times},\ (mn-1) \right\}$$

and $t(\Gamma(G)) = 3$, respectively.

Proof. We should keep in our mind that the order of G is mn, and thus $|V(\Gamma(G)| = mn$. Now, the proof will be discussed under some cases by considering the type of these mn vertices.

• By (I) in Section 1.1, for all $w \in V(\Gamma(G))$, we have $1_G \sim w$ and since the graph is simple, $1_G \nsim 1_G$ which implies that $deg_{\Gamma(G)}(1_G) = mn - 1$.

• Since $G = \langle x\rangle \rtimes_\varphi \langle y\rangle$, we certainly have some vertices of the form x^k $(1 \leq k < m)$ and y^t $(1 \leq t < n)$ in the related graph $\Gamma(G)$. Therefore, by (I) in Section 1.1, each $x^k \sim 1_G$ and $y^t \sim 1_G$. Moreover, by (II)-(*ii*) in Section 1.1, $x^k \sim x$, $x^k \sim x^2, \ldots, x^k \sim x^{k-1}$, $x^k \sim x^{k+1}, \ldots, x^k \sim x^{m-1}$, $y^t \sim y$, $y^t \sim y^2, \ldots, y^t \sim y^{t-1}$, $y^t \sim y^{t+1}, \ldots, y^t \sim y^{n-1}$, $x^k \sim y$, $x^k \sim y^2, \ldots, x^k \sim y^{n-1}$ and $y^t \sim x$, $y^t \sim x^2, \ldots, y^t \sim x^{m-1}$. Notice that $x^k \nsim x^k$ and $y^t \nsim y^t$. In addition, by (II)-(*i*) in Section 1.1, $x^k \nsim x^s y^t$ $(1 \leq s < m, 1 \leq t < n)$ and similarly $y^t \nsim y^l x^u$ $(1 \leq t < n, 1 \leq u < m)$.

In fact these above processes show that the degree of the vertices of the form x^k are $(m+n-2)$ (i.e., the number of elements in C_m plus the number of elements in C_n minus the number of non-adjoining elements). The same value is obtained for the vertices of the form y^t. Hence the total number of such vertices is equal to $(m+n-2)\,[m+n-2]$.

• Finally, let us consider the vertices of the form $x^k y^t$ $(1 \leq k < m,\ 1 \leq t < n)$. By (I) in Section 1.1, vertices of this type can only be adjoined with 1_G. In fact the total number of these vertices is obtained by subtract the total number of vertices of the form x^k and y^t, and identity element from the order of G. That is $mn - (m+n-2) - 1 = mn - m - n + 1 = [(m-1)(n-1)]$.

Hence, these all above cases and by the definition of degree sequence, we clearly obtain the set $DS(\Gamma(G))$ as depicted in the theorem. Nevertheless, it is easily seen that the irregularity index $t(\Gamma(G)) = 3$, as required. □

The adjacency spectrum $spec(\Gamma)$ of a graph Γ is the multiset of eigenvalues of its adjacency matrix. Two graphs Γ and Γ′ are called cospectral if $spec(\Gamma) = spec(\Gamma')$ and Γ′ is called to be a cospectral mate for Γ. Moreover, a graph Γ is called determined by the spectrum if Γ is isomorphic to all its cospectral mates. Additionally, for the graphs Γ and Γ, if $spec(\Gamma) = spec(\Gamma')$ then $\Gamma \cong \Gamma'$. In fact the characterization of Γ graphs in terms of the determination of its spectrum is very difficult and goes back to about half of a century and it is originated in chemistry [16, 35].

According to the above paragraph, we can state the following remark and theorem.

Remark 2.5. For any two cyclic groups C_m and C_n having orders m and n, respectively, if $(m, n) \neq 1$ then we have two groups G and G' such that $G \cong C_m \rtimes_\varphi C_n$ and $G' \cong C_m \rtimes_{\varphi_{id}} C_n$. Then, by Section 1.1, when we obtain two graphs $\Gamma(G)$ and $\Gamma(G')$, it is easy to see that $spec(\Gamma(G)) = spec(\Gamma(G'))$. This yields $\Gamma(G) \cong \Gamma(G')$.

Theorem 2.6. *For any two different finite semi-direct product groups G and G', their graphs $\Gamma(G)$ and $\Gamma(G')$ can be isomorphic.*

After introducing adjacency spectrum and Theorem 2.6, we can ontain the following corollary as a consequence of Theorem 2.4.

Corollary 2.7. *Let us take two finite cyclic groups $C_m = \langle x \rangle$ and $C_n = \langle y \rangle$ of orders m and n, respectively, and also let us consider a semi-direct product $G_1 = C_m \rtimes_{id_{G_1}} C_n$ (i.e., $\varphi = id_{G_1}$) and let us assume that $hcf(m,n) = k$. Therefore,*

$$\text{either} \quad DS(\Gamma(G_1)) = \{\underbrace{(mn-1),\ (mn-1),\ldots,\ (mn-1)}_{mn\ times}\} \ \text{and}\ t(\Gamma(G_1)) = 1$$

$$\text{or} \quad DS(\Gamma(G)) = DS(\Gamma(G_1)) \ \text{and}\ t(\Gamma(G)) = t(\Gamma(G_1)),\ \text{where } G \text{ is the group as stated in Theorem 2.4.}$$

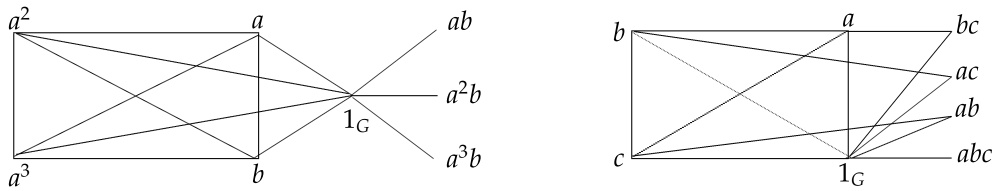
Proof. Depends on k, we have two cases. Of course the first one is the trivial case which is the situation $k = 1$. This clearly implies that the group G_1 is isomorphic to the cyclic group C_{mn} of order mn. Hence, by (II)-(*ii*) in Section 1.1, we see that the graph $\Gamma(C_{mn})$ is complete with mn vertices. Thus the degree sequence of $\Gamma(C_{mn})$ consists of mn-times the $mn - 1$ elements, and the irregularity index is $t(\Gamma(C_{mn})) = 1$.

On the other hand, if $k \neq 1$, then G_1 is a finite group of order mn. Notice that since the condition $\varphi = id_{G_1}$ still holds, the finite group G is not the same with the group $G_1 \cong C_{mn}$ investigated in the above paragraph. However, by considering Remark 2.5 and Theorem 2.6, we certainly have $\Gamma(G_1) \cong \Gamma(G)$ and so, as a consequence of Theorem 2.4, we obtain the equalities $DS(\Gamma(G)) = DS(\Gamma(G_1))$ and $t(\Gamma(G)) = t(\Gamma(G_1))$, as required. □

Again, for any Γ, a subset $\emptyset \neq D$ of the vertex set $V(\Gamma)$ is called a *dominating set* if every vertex $V(\Gamma)\backslash D$ is joined to at least one vertex of D by an edge. Additionally, the *domination number* $\gamma(\Gamma)$ is the number of vertices in a smallest dominating set for Γ.

Theorem 2.8. *For a semi-direct product G of two finite groups, let us consider the graph $\Gamma(G)$. Then $\gamma(\Gamma(G)) = 1$.*

Proof. As we used in most of the proofs in above results, for all $w \in G$, $1_G \sim w$ in $\Gamma(G)$. According to the definition of $\Gamma(G)$, the smallest dominating set is $\{1_G\}$ which implies $\gamma(\Gamma(G)) = 1$, as required. □



(a) $G \cong D_8$ has two generaters a and b **(b)** G has three generaters a, b and c

Figure 1: Examples of graphs defined in Section 1.1

Example 2.9. It is known that the dihedral group D_8 is isomorphic to the semi-direct product $C_4 = \langle a\,;\,a^4\rangle \rtimes_\varphi C_2 = \langle b\,;\,b^2\rangle$ with a presentation $D_8 = \langle a,b\,;\,a^4, b^2, b^{-1}ab = a^{-1}\rangle$. Since D_8 has two generaters a and b, the vertex set $V(\Gamma(D_8)) = \{1_G, a, a^2, a^3, b, ab, a^2b, a^3b\}$ and the graph can be drawn as in Figure 1-(**(a)**). Thus, as an application of the above results, we get $diam(\Gamma(D_8)) = 2$, $girth(\Gamma(D_8)) = 3$, $\Delta(\Gamma(D_8)) = 7$, $\delta(\Gamma(D_8)) = 1$, $DS(\Gamma(D_8)) = \{1,1,1,4,4,4,4,7\}$, $t(\Gamma(D_8)) = 3$ and $\gamma(\Gamma(D_8)) = 1$.

Example 2.10. Consider Klein-4 group $\mathcal{V}_4$ with a presentation $\mathcal{P}_{\mathcal{V}_4} = \langle a,b\,;\,a^2, b^2, (ab)^2\rangle$. Also let us consider the semi-direct product $G \cong \mathcal{V}_4 \rtimes_\varphi C_2$, where $C_2 = \langle c\,;\,c^2\rangle$. Since G has three generators a, b and c, the vertex set is defined by $\{1_G, a, b, c, ab, ac, bc, abc\}$ and thus the graph can be presented as in Figure 1-**(b)**. Clearly, Theorem 2.4 cannot be applied for this example since $\mathcal{V}_4$ is not cyclic. But, by considering the graph in Figure 1-**(b)**, it is easy to see that $DS(\Gamma(G)) = \{1,2,2,2,4,4,4,7\}$ and so $t(\Gamma(G)) = 4$. Moreover, as an application of the above theorems (except Theorems 2.4 and 2.6), we obtain $diam(\Gamma(G)) = 2$, $girth(\Gamma(G)) = 3$, $\Delta(\Gamma(G)) = 7$, $\delta(\Gamma(G)) = 1$ and $\gamma(\Gamma(G)) = 1$.

3. Perfectness Property of $\Gamma(G)$

Throughout this section it is assumed that a semi-direct product having finite order with a generating set $\langle x_1, x_2, \ldots, x_k \rangle$. Our main goal in this section is to show that the graph $\Gamma(G)$ defined in Section 1.1 is actually perfect.

Perfect graphs form an important class of graphs, in particular, related to the boundary value problems and various other applications (see, for instance, [10, 17, 30]). For example, perfect graphs have been used in chemistry. This is why perfect graphs are worth considering here too.

Since *perfect graphs* are directly related to the terms *coloring*, *clique* and *clique numbers*, let us start this section by reminding these graph parameters. The coloring of Γ is to be an assignment of colors (elements of some set) to the vertices of Γ, one color to each vertex, so that adjacent vertices are assigned distinct colors. If n colors are used, then the coloring is referred to as an *n-coloring*. If there exists an n-coloring of Γ, then the graph is called *n-colorable*. The minimum number n for which Γ is n-colorable is called the *chromatic number* of Γ, and is denoted by $\chi(\Gamma)$. In addition, the other graph parameter clique is depending on the vertices. Basically each of the maximal complete subgraphs of Γ is called a *clique*. Moreover, the largest number of vertices in any clique of Γ is called the *clique number* and denoted by $\omega(\Gamma)$. In general, it is well known that $\chi(\Gamma) \geq \omega(\Gamma)$ for any graph Γ ([15]). For every induced subgraph $\Gamma_H \subseteq \Gamma$, if $\chi(\Gamma_H) = \omega(\Gamma_H)$, then Γ is called a *perfect graph* (cf. [30]).

In here, the equality of the chromatic and clique numbers will be stated in a unique theorem but proved separately for the graph $\Gamma(G)$, and so it will be obtained the perfectness of this special graph. We recall that the notation $o(x_i)$ denotes the order of an element x_i (where $1 \leq i \leq k$). Since G has finite order, each of these orders are finite as well.

Theorem 3.1. *The chromatic and clique numbers of* $\Gamma(G)$ *are given by*

$$\chi(\Gamma(G)) = \left\{ \left[\sum_{i=1}^{k} o(x_i) \right] - (k-1) \right\} = \omega(\Gamma(G)).$$

Proof. For the semi-direct product $G = \langle x_1, x_2, \ldots, x_k \rangle$, let us first consider the element x_1. Actually it does not only a vertex in the related graph $\Gamma(G)$ but also it defines a subgroup $H_1 = \langle x_1 \rangle$ of G and, by (II)-(*ii*), we obtain a complete subgraph $\Gamma(H_1)$ with $o(x_1)$ vertices. Similarly, by (II)-(*ii*), each of these generators construct a complete subgraph as $\Gamma(H_i)$ for $i = \{1, 2, \cdots, k\}$. Moreover, again by (II)-(*ii*), since for all these distinct one generator vertices are adjoining each other, they construct a complete subgraph with total $\{(o(x_1) + o(x_2) + \cdots + o(x_k)) - (k-1)\}$ vertices. (In other words, the collection of subgraphs $\Gamma(H_i)$ gives a complete subgraph, say $\Gamma(H^k)$). In fact this final complete subgraph $\Gamma(H^k)$ is $\left\{\left[\sum_{i=1}^{k} o(x_i)\right] - (k-1)\right\}$-colorable. Because, for all $w = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_k^{\varepsilon_k}$ such that $(k \geqslant 2)$, there exists $y_j^{\delta_j}$ such that $x_i^{\varepsilon_i} = y_j^{\delta_j}$. By (II)-(*ii*), this implies the vertex w cannot adjoint with $y_j^{\delta_j}$ and so by the meaning of coloring w and $y_j^{\delta_j}$ could be the same color in $\Gamma(G)$. After all these progress, we see that the chromatic number of $\Gamma(G)$ must be equal to the total number of vertices of the subgraph $\Gamma(H^k)$, i.e. $\chi(\Gamma(G)) = \left\{\left[\sum_{i=1}^{k} o(x_i)\right] - (k-1)\right\}$, as required.

On the other hand, to determine the clique number, let us consider again the complete subgraph $\Gamma(H^k)$ as obtained in the above paragraph. We know that the total number of vertices $V(\Gamma(H^k))$ is equal to the $\left\{\left[\sum_{i=1}^{k} o(x_i)\right] - (k-1)\right\}$. However we still need to show that $\Gamma(H^k)$ is the maximal complete subgraph. Therefore, without loss of generalization, let us assume that $\Gamma(H^k)$ is not maximal. So there exist another complete subgraph, say $\Gamma(T^k)$ such that $\left|V(\Gamma(H^k))\right| < \left|V(\Gamma(T^k))\right|$. But, by (II)-(*i*), this case implies that when we had a vertex of the form $w = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_k^{\varepsilon_k}$ $(k \geqslant 2)$ obtained from the vertex set of $\Gamma(T^k)$, we should have a vertex $y_j^{\delta_j}$ such that $\exists\, x_i^{\varepsilon_i} = y_j^{\delta_j}$. However, by (II)-(*ii*), this situation shows w and $y_j^{\delta_j}$ cannot be adjoint, and then gives a contradiction to $\Gamma(T^k)$ be a complete (sub)graph. Therefore $\Gamma(H^k)$ is the maximal complete subgraph with $\left|V(\Gamma(H^k))\right|$ number of vertices.

This completes the proof. □

As a consequence of Theorem 3.1, we have the following result.

Corollary 3.2. *The graph* $\Gamma(G)$ *is perfect.*

We recall that any graph Γ is called *Berge* if no induced subgraph of Γ is an odd cycle of length at least five or the complement of one (see [9]). The following lemma (which is named as *Strong Perfect Conjecture* in some sources) proved by Chudnovsky et al. ([10]) that actually figures out a relationship between perfect and Berge graphs.

Lemma 3.3. ([10]) *A graph is perfect if and only if it is Berge.*

Therefore, by using this relationship, we can give the next result on the perfectness of $\Gamma(G)$ which is sharper than the result given in Corollary 3.2.

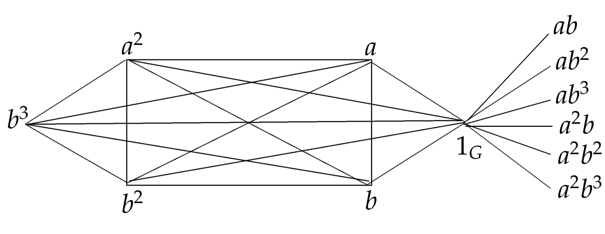
Theorem 3.4. *For the group* $G = \langle x_1, x_2, \ldots, x_k\rangle$, *if* $k \geq 3$, *then* $\Gamma(G)$ *is perfect. However, it is also perfect if* $o(x_1) + o(x_2) \geq 6$ *while* $k = 2$.

Proof. Assume that $k = 3$. Then, by (I) and (II)-(*ii*), we obtain a 4-lenght cycle by only using the generators x_1, x_2, x_3 (without their powers) and the identity element 1_G. Additionally, by (II)-(*i*), we get $x_1 \sim x_2x_3$ for $x_1, x_2x_3 \in G$, and then by (I), since $1_G \sim x_2x_3$, we obtain a 5-lenght cycle $1_G \sim x_2x_3 \sim x_1 \sim x_2 \sim x_3 \sim 1_G$. It is easy to see by considering (II)-(*ii*), one can obtain a cycle of length five only using the generators without their powers. Thus $\Gamma(G)$ is Berge and so by Lemma 3.3 it is perfect.

Moreover, for $k = 2$, since $\Gamma(G)$ is constructed by $G = \langle x_1, x_2\rangle$, for all powers of these generators, the vertices of the type $x_1^{\varepsilon_1}$, $x_2^{\varepsilon_2}$ can only be connected with 1_G by (I) and (II)-(*ii*). Thus the required cycles will be obtained by the generators x_1, x_2 and their powers. But to get an odd cycle of length at least five, we need at least 5 vertices included 1_G. Hence we easily see that the condition $o(x_1) + o(x_2) \geq 6$ must be held to get a (at least) 5-lenght cycle by using (II)-(*ii*). Thus just under this condition $\Gamma(G)$ is Berge (and so perfect Lemma 3.3) for $k = 2$. □

Example 3.5. Let us consider the semi-direct product of $C_3 = \langle a\,;\, a^3\rangle$ by $C_4 = \langle b\,;\, b^4\rangle$ with a presentation $G = \langle a, b\,;\, a^3, b^4, b^{-1}ab = x^{-1}\rangle$. In fact G is the metacyclic group of order 12 (see *[34]*). Considering Section *1.1*, one can draw the graph $\Gamma(G)$ of G as in Figure 2. Therefore we have the following facts for $\Gamma(G)$:

- $V(\Gamma(G)) = \{1_G, a, a^2, b, b^2, b^3, ab, ab^2, ab^3, a^2b, a^2b^2, a^2b^3\}$ and so $|V(\Gamma(G))| = 12$.
- $diam(\Gamma(G)) = 2$, $girth(\Gamma(G)) = 3$, $\gamma(\Gamma(G)) = 1$, $\Delta(\Gamma(G)) = 11$ and $\delta(\Gamma(G)) = 1$.
- $DS(\Gamma(G)) = (1, 1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 11)$ and so $t(\Gamma(G)) = 3$.
- Since $\chi(\Gamma(G)) = 6 = \omega(\Gamma(G))$ (or, by Theorem 3.4, since $o(a) + o(b) = 7$ while the number of generators is 2), it is perfect and so Berge (by Lemma 3.3).



G has two generaters *a* and *b*

Figure 2: Another example of the graph defined in Section 1.1

Conjecture 3.6. *As it is well known in graph theory, topological indices are very popular characterization methods for simple graphs. Therefore one may adapt some special indices (for example, Zagreb indices* ([18, 29, 36])) *or irregularity index* ([4]) *to the our new graph defined in this paper. On the other hand, as another future project, since the main idea in the paper* [13] *was to defined a new graph over a special algebraic structure, namely finite monogenic monoids, and since our main goal in here is quite similar with that paper (just by replacing monoids with groups), one may also study some special graph products (for instance, lexicographic, tensor, cartesian etc.) as in the papers* [5, 6]*, and so investigate the effect of these products to the related groups.*

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