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Generalized Weighted Statistical Convergence of Double Sequences and Applications

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Abstract. In this paper we introduce the concept generalized weighted statistical convergence of double sequences. Some relations between weighted $(\lambda,\mu)-$statistical convergence and strong $(\overline{N}_{\lambda\mu},p,q,\alpha,\beta)-$summablity of double sequences are examined. Furthermore, we apply our new summability method to prove a Korovkin type theorem.

1. Introduction

The idea of statistical convergence was formerly defined under the name "almost convergence" by Zygmund [37] in the first edition of his celebrated monograph published in Warsaw in 1935. The concept was formally introduced by Fast [14] and was reintroduced by Schonberg [35] and also, independently, by Buck [3]. Later the idea was associated with summability theory by Connor [6], Cakallı [7], Et et al. ([10–12, 18]), Duman and Orhan [8], Fridy [15], Işık [19], Mohiuddine et al. ([1, 22–24]), Mursaleen et al. ([26–28]), Šalat [32], Savaş ([33, 34]) and many others.

Let $\mathbb{N}$ be the set of all natural numbers, $K \subseteq \mathbb{N}$ and $K(n) = \{k \leq n : k \in K\}$. The natural density of K is defined by $\delta(K) = \lim_n \frac{1}{n}|K(n)|$, if the limit exists. The vertical bars indicate the number of the elements in enclosed set. A sequence $x = (x_k)$ is said to be statistically convergent to L if the set $K(\varepsilon) = \{k \leq n : |x_k - L| \geq \varepsilon\}$ has natural density zero. A sequence $x = (x_k)$ is said to be statistically Cauchy sequence if for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that

$$\lim_n \frac{1}{n}|k \leq n : |x_k - x_N| \geq \varepsilon| = 0.$$

The notion of weighted statistical convergence was introduced by Karakaya and Chishti [20] as follows:

Let (p_n) be a sequence of positive real numbers such that $P_n = p_0 + p_1 + ... + p_n \to \infty$ as $n \to \infty$ and $p_n \neq 0$, $p_0 > 0$. A sequence $x = (x_k)$ is said to be weighted statistical convergent if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n}\left|\{k \leq n : p_k|x_k - L| \geq \varepsilon\}\right| = 0.$$

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In this case we write $S_{\overline{N}} - \lim x = L$. We shall denote the set of all weighted statistical convergent sequences by $S_{\overline{N}}$. Mursaleen *et al.* [27] was modified the definition of weighted statistical convergence such as:

A sequence $x = (x_k)$ is said to be weighted statistical convergent if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n} \left|\{k \leq P_n : p_k |x_k - L| \geq \varepsilon\}\right| = 0.$$

Recently Ghosal [17] was added to the definition of weighted statistical convergence the condition $\liminf p_n > 0$.

Let $t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k x_k$, $n = 0, 1, 2, 3...$. The sequence $x = (x_k)$ is said to be $\left(\overline{N}, p_n\right)-$summable to L if $\lim_{n\to\infty} t_n = L$. A sequence $x = (x_k)$ is said to be $\left(\overline{N}, p_n\right)-$statistically summable to L if $st-\lim_{n\to\infty} t_n = L$ [25]. In this case we write $\overline{N}(st) - \lim x = L$.

A double sequence $x = \left(x_{jk}\right)_{j,k=0}^{\infty}$ is said to be convergent in the Pringsheim sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left|x_{jk} - L\right| < \varepsilon$, whenever $j, k > N$. In this case we write $P - \lim x = L$ [31].

A double sequence $x = \left(x_{jk}\right)_{j,k=0}^{\infty}$ is bounded if there exists a positive number M such that $\left|x_{jk}\right| < M$ for all $i, j \in \mathbb{N}$. We denote the set of all bounded double sequence by ℓ_∞^2.

Let $K \subseteq \mathbb{N}\times\mathbb{N}$ and $K(m,n) = \{(j,k) : j \leq m, k \leq n\}$. The double natural density of K is defined by

$$\delta_2(K) = P - \lim_{m,n} \frac{1}{mn} |K(m,n)|, \text{ if the limit exists.}$$

A double sequence $x = \left(x_{jk}\right)$ is said to be statistically convergent to L if for every $\varepsilon > 0$ the set $\left\{(j,k), j \leq m \text{ and } k \leq n : \left|x_{jk} - L\right| \geq \varepsilon\right\}$ has double natural density zero [28]. In this case we write $st_2 - \lim x = L$ and we denote the set of all statistically convergent double sequence by st_2. A convergent double sequence is also $st-$convergent, but the converse is not true general. Also a $st-$convergent double sequence need not be bounded. For this consider a sequence $x = \left(x_{jk}\right)$ defined by

$$x_{jk} = \begin{cases} jk & \text{if } j \text{ and } k \text{ are square} \\ 1 & \text{otherwise} \end{cases},$$

then $st_2 - \lim x = 1$, but $x = \left(x_{jk}\right)$ neither convergent nor bounded.

Let $p = \left\{p_j\right\}_{j=0}^{\infty}$ and $q = \{q_k\}_{k=0}^{\infty}$ be sequences of non-negative numbers that are not all zero and let $Q_n = q_1 + q_2 + q_3 + ... + q_n$, $q_1 > 0$ and $P_m = p_1 + p_2 + ... + p_m$, $p_1 > 0$. The weighted mean $t_{mn}^{\alpha\beta}$ was defined by

$$t_{mn}^{11} = \frac{1}{P_m Q_n}\sum_{j=0}^{m}\sum_{k=0}^{n} p_j q_k x_{jk}$$

$$t_{mn}^{10} = \frac{1}{P_m}\sum_{j=0}^{m} p_j x_{jn},\ t_{mn}^{01} = \frac{1}{Q_n}\sum_{k=0}^{n} q_k x_{mk}$$

where $m, n \geq 0$ and $(\alpha,\beta) = (1,1), (1,0)$ or $(0,1)$. If $t_{mn}^{\alpha\beta}$ convergent to L as $\min(m,n) \to \infty$ then; we say that a double sequence $x = \left(x_{jk}\right)$ is $\left(\overline{N}, p, q, \alpha, \beta\right)-$summable to L and we show that $\lim_{m,n\to\infty} t_{mn}^{\alpha\beta} = L$. In this case we write $x_{ij} \to L\left(\overline{N}, p, q, \alpha, \beta\right)$ ([4],[5]).

2. Main Results

In this section we generalize the concept of weighted statistical convergence for double sequences.

Definition 2.1. Let K be a subset of $\mathbb{N}\times\mathbb{N}$. We define the double weighted density of K by

$$\delta_{\overline{N}_2}(K)=\lim_{m,n}\frac{1}{P_mQ_n}\left|K_{P_mQ_n}(m,n)\right|,\text{ provided the limit exists,}$$

where $K_{P_mQ_n}(m,n)=\left\{(j,k),j\leq P_m\text{ and }k\leq Q_n:p_jq_k\left|x_{jk}-L\right|\geq\varepsilon\right\}$, $\liminf p_n>0$, $\liminf q_m>0$. We say that a double sequence $x=\left(x_{jk}\right)$ is said to be weighted statistically convergent $\left(\text{or } S_{\overline{N}_2}-\text{convergent}\right)$ to L if for every $\varepsilon>0$

$$\lim_{m,n\to\infty}\frac{1}{P_mQ_n}\left|\left\{(j,k),j\leq P_m\text{ and }k\leq Q_n:p_jq_k\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|=0.$$

In this case we write $S_{\overline{N}_2}-\lim x=L$.

Definition 2.2. Let $\lambda=(\lambda_m)$ and $\mu=(\mu_n)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and

$$\lambda_{m+1}\leq\lambda_m+1,\ \lambda_1=0$$
$$\mu_{n+1}\leq\mu_n+1,\ \mu_1=0.$$

Let $p=\left(p_j\right)$ and $q=(q_k)$ be two sequence of non-negative numbers such that $p_0>0,q_0>0$ and

$$P_{\lambda_m}=\sum_{j\in J_m}p_j\to\infty\ m\to\infty$$
$$Q_{\mu_n}=\sum_{k\in I_n}q_k\to\infty\ n\to\infty$$

where, $J_m=[m-\lambda_m+1,m]$, $I_n=[n-\mu_n+1,n]$ and we define generalized weighted mean

$$\sigma_{mn}^{11}=\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_kx_{jk}$$
$$\sigma_{mn}^{10}=\frac{1}{P_{\lambda_m}}\sum_{j\in J_m}p_jx_{jn},\sigma_{mn}^{01}=\frac{1}{Q_{\mu_n}}\sum_{k\in I_n}q_kx_{mk}$$

A double sequence $x=\left(x_{jk}\right)$ is said to be $\left(\overline{N}_{\lambda\mu},p,q,\alpha,\beta\right)$−summable to L, if $\lim\limits_{m,n\to\infty}\sigma_{mn}^{\alpha\beta}=L$, where $(\alpha,\beta)=(1,1),(1,0)$ or $(0,1)$.

Definition 2.3. A double sequence $x=\left(x_{jk}\right)$ is said to be strongly $\left(\overline{N}_{\lambda\mu},p,q,\alpha,\beta\right)$−summable (or $\left[\overline{N}_{\lambda\mu},p,q,\alpha,\beta\right]$ -summable) to L, if

$$\lim_{m,n\to\infty}\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_k\left|x_{jk}-L\right|.$$

In this case we write $x_{jk}\to L\left[\overline{N}_{\lambda\mu},p,q,\alpha,\beta\right]$.

If we take $p_j=1$ and $q_k=1$ for all $j,k\in\mathbb{N}$ in the above definition $\left(\overline{N}_{\lambda\mu},p,q,\alpha,\beta\right)$−summability reduces to (V,λ,μ)−summability which were studied Mursaleen *et al.* [29]. Also if we take $p_j=1,q_k=1$ for all $j,k\in\mathbb{N}$ and $\lambda_m=m,\mu_n=n$ for all $n,m\in\mathbb{N}$, then; $\left(\overline{N}_{\lambda\mu},p,q,\alpha,\beta\right)$−summability reduces to $(C,1,1)$−summability.

Definition 2.4. A double sequence $x = \left(x_{jk}\right)$ is said to be weighted (λ, μ) −statistically convergent (or $S_{\overline{N}_{(\lambda,\mu)}}$ −convergent) to L if for every $\varepsilon > 0$

$$\lim_{m,n\to\infty} \frac{1}{P_{\lambda_m} Q_{\mu_n}} \left|\left\{(j,k)\,;\; j \le P_{\lambda_m} \text{ and } k \le Q_{\mu_n} : p_j q_k \left|x_{jk} - L\right| \ge \varepsilon\right\}\right| = 0.$$

In this case we write $S_{\overline{N}_{(\lambda,\mu)}} - \lim x = L$. We denote the set of all weighted (λ, μ) −statistically convergent double sequences by $S_{\overline{N}_{(\lambda,\mu)}}$.

Definition 2.5. A double sequence $x = \left(x_{jk}\right)$ is said to be $\left(\overline{N}^2_{\lambda\mu}, p, q, \alpha, \beta\right)$−statistically summable to L if $st_2 - \lim\limits_{m,n\to\infty} \sigma^{\alpha\beta}_{mn} = L$. In this case we write $\overline{N}^2_{\lambda\mu}(st) - \lim x = L$.

Theorem 2.6. *Let $p_j q_k \left|x_{jk} - L\right| \le M$ for all $j, k \in \mathbb{N}$. If a double sequence $x = \left(x_{jk}\right)$ is $S_{\overline{N}_{(\lambda,\mu)}}$−convergent to L then; it is $\left(\overline{N}^2_{\lambda\mu}, p, q, \alpha, \beta\right)$−statistically summable, but the converse is not true.*

Proof. Let $p_j q_k \left|x_{jk} - L\right| \le M$ for all $j, k \in \mathbb{N}$ and $x = \left(x_{jk}\right)$ is $S_{\overline{N}_{(\lambda,\mu)}}$−convergent to L and set

$$K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon) = \left\{(j,k)\,;\; j \le P_{\lambda_m} \text{ and } k \le Q_{\mu_n} : p_j q_k \left|x_{jk} - L\right| \ge \varepsilon\right\}.$$

Then, we can write

$$\begin{aligned}
\left|\sigma^{\alpha\beta}_{mn} - L\right| &= \left|\frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{j\in J_m} \sum_{k\in I_n} p_j q_k x_{jk} - L\right| \\
&\le \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{\substack{j\in J_m \\ (j,k)\in K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)}} \sum_{k\in I_n} p_j q_k \left|x_{jk} - L\right| + \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{\substack{j\in J_m \\ (j,k)\notin K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)}} \sum_{k\in I_n} p_j q_k \left|x_{jk} - L\right| \\
&\le \frac{M}{P_{\lambda_m} Q_{\mu_n}} \left|K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)\right| + \varepsilon \to \varepsilon + 0
\end{aligned}$$

as $m, n \to \infty$ which implies that $\sigma^{\alpha\beta}_{mn} \to L$.

For the converse, consider a sequence defined by $x = \left(x_{jk}\right) = \left((-1)^j\right)$ for all $k \in \mathbb{N}$. Let $p_j = 1, q_k = 1, \lambda_m = m, \mu_n = n$ for all $j, k, n, m \in \mathbb{N}$. Then, the sequence $x = \left(x_{jk}\right)$ is $\left(\overline{N}^2_{\lambda\mu}, p, q, \alpha, \beta\right)$−statistically summable to zero, but $x = \left(x_{jk}\right)$ is not $S_{\overline{N}_{(\lambda,\mu)}}$−convergent. □

Definition 2.7. A double sequence $x = \left(x_{jk}\right)$ is said to be $\left[\overline{N}, p, q, \alpha, \beta\right]_r$ −summable to L if

$$\lim_{m,n\to\infty} \frac{1}{P_m Q_n} \sum_{j=1}^{m} \sum_{k=1}^{n} p_j q_k \left|x_{jk} - L\right|^r = 0, \qquad (0 < r < \infty)$$

and we write $x_{ij} \to L\left[\overline{N}, p, q, \alpha, \beta\right]_r$.

Theorem 2.8. *If $\liminf\limits_{m,n}\left(\frac{P_{\lambda_m} Q_{\mu_n}}{P_m Q_n}\right) > 0$, then; $S_{\overline{N}_2} \subset S_{\overline{N}_{(\lambda,\mu)}}$.*

Proof. Let $x = \left(x_{jk}\right)$ be $S_{\overline{N}_2}$−convergent to L. We may write

$$\begin{aligned}
&\frac{1}{P_m Q_n}\left|\left\{(j,k), j \leq P_m \text{ and } k \leq Q_n : p_j q_k \left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \\
&\geq \frac{1}{P_m Q_n}\left|\left\{(j,k);\ j \leq P_{\lambda_m} \text{ and } k \leq Q_{\mu_n} : p_j q_k \left|x_{jk} - L\right| \geq \varepsilon\right\}\right| \\
&= \frac{P_{\lambda_m} Q_{\mu_n}}{P_m Q_n} \frac{1}{P_{\lambda_m} Q_{\mu_n}}\left|\left\{(j,k);\ j \leq P_{\lambda_m} \text{ and } k \leq Q_{\mu_n} : p_j q_k \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|.
\end{aligned}$$

Since $\liminf_{m,n}\left(\frac{P_{\lambda_m} Q_{\mu_n}}{P_m Q_n}\right) > 0$, taking limit as $m, n \to \infty$, we get $S_{\overline{N}_{(\lambda,\mu)}} - \lim x = L$. □

Theorem 2.9. *If a double sequence $x = \left(x_{jk}\right)$ is $\left[\overline{N}_{\lambda\mu}, p, q, \alpha, \beta\right]$−summable to L, then; it is $S_{\overline{N}_{(\lambda,\mu)}}$−statistically convergent to L and the inclusion is strict.*

Proof. Let $x = \left(x_{jk}\right)$ be $\left[\overline{N}_{\lambda\mu}, p, q, \alpha, \beta\right]$−summable to L. Then, for $\varepsilon > 0$ we have

$$\begin{aligned}
&\left|\frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{j \in J_m} \sum_{k \in I_n} p_j q_k \left|x_{jk} - L\right|\right| \\
&= \left|\frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{\substack{j \in J_m\ k \in I_n \\ (j,k) \in K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)}} p_j q_k \left|x_{jk} - L\right| + \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{\substack{j \in J_m\ k \in I_n \\ (j,k) \notin K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)}} p_j q_k \left|x_{jk} - L\right|\right| \\
&\geq \frac{1}{P_{\lambda_m} Q_{\mu_n}}\left|\left\{(j,k), j \leq P_{\lambda_m} \text{ and } k \leq Q_{\mu_n} : p_j q_k \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

and this implies that $S_{\overline{N}_{(\lambda,\mu)}} - \lim x = L$.

To show the inclusion is strict consider the following example:

Let $\lambda_m = m, \mu_n = n, p_j = j, q_k = k$ for all $j, k, n, m \in \mathbb{N}$ and define a sequence by

$$x_{jk} = \begin{cases} \sqrt{jk} & j \text{ and } k \text{ square} \\ 0 & \text{otherwise} \end{cases}.$$

Then,

$$P_m = \sum_{j=1}^{m} p_j = \frac{m(m+1)}{2}, Q_n = \sum_{k=1}^{n} q_k = \frac{n(n+1)}{2}$$

and so we have

$$\begin{aligned}
&\frac{1}{P_m Q_n}\left|\left\{(j,k), j \leq P_m \text{ and } k \leq Q_n : p_j q_k \left|x_{jk} - 0\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{\sqrt{\frac{m(m+1)}{2}}\sqrt{\frac{n(n+1)}{2}}} \to 0 \text{ as } m, n \to \infty,
\end{aligned}$$

but

$$\frac{1}{P_m Q_n} \sum_{j=1}^{m} \sum_{k=1}^{n} p_j q_k \left|x_{jk}\right| \to \infty.$$

□

Theorem 2.10. *Let $p_jq_k\left|x_{jk}-L\right|\leq M$ for all $j,k\in\mathbb{N}$. If a double sequence $x=\left(x_{jk}\right)$ is weighted (λ,μ) −statistically convergent to L ,then; it is $\left[\overline{N}_{\lambda\mu},p,q,\alpha,\beta\right]$ summable to L, hence $x=\left(x_{jk}\right)$ is $\left(\overline{N},p,q,\alpha,\beta\right)$ −summable to L.*

Proof. The first implication is obvious. On the other hand, we have

$$\begin{aligned}
&\frac{1}{P_mQ_n}\sum_{j=1}^{m}\sum_{k=1}^{n}p_jq_k\left|x_{jk}-L\right| \\
&=\frac{1}{P_mQ_n}\sum_{j=1}^{m-\lambda_m}\sum_{k=1}^{n-\mu_n}p_jq_k\left|x_{jk}-L\right|+\frac{1}{P_mQ_n}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_k\left|x_{jk}-L\right| \\
&\leq\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j=1}^{m-\lambda_m}\sum_{k=1}^{n-\mu_n}p_jq_k\left|x_{jk}-L\right|+\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_k\left|x_{jk}-L\right| \\
&\leq\frac{2}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_k\left|x_{jk}-L\right|
\end{aligned}$$

Hence x is $\left(\overline{N},p,q,\alpha,\beta\right)$-summable to L. □

Definition 2.11. A double sequence $x=\left(x_{jk}\right)$ is said to be strongly weighted $(\lambda,\mu)_r$ −convergent if

$$\lim_{m,n}\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_k\left|x_{jk}-L\right|^r=0,\quad(0<r<\infty).$$

In this case we write $x_{jk}\to L\left[\overline{N}_{\lambda,\mu},p,q,\alpha,\beta\right]_r$.

Theorem 2.12. *Let a double sequence $x=\left(x_{jk}\right)$ is strongly weighted $(\lambda,\mu)_r$ −convergent to L. If the following conditions are provided, then; $x=\left(x_{jk}\right)$ is weighted (λ,μ) −statistically convergent to L.*

*Case*1 : $0<r<1$ *and* $\left|x_{jk}-L\right|<1$
*Case*2 : $1\leq r<\infty$ *and* $1\leq\left|x_{jk}-L\right|<\infty$

Proof. Since $p_jq_k\left|x_{jk}-L\right|^r\geq p_jq_k\left|x_{jk}-L\right|$ for *Case*1 and *Case*2 then we can write

$$\begin{aligned}
\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_k\left|x_{jk}-L\right|^r&\geq\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_k\left|x_{jk}-L\right| \\
&\geq\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{\substack{j\in J_m\\ (j,k)\in K_{P_{\lambda_m}Q_{\mu_n}}(\varepsilon)}}\sum_{k\in I_n}p_jq_k\left|x_{jk}-L\right| \\
&\geq\frac{\varepsilon}{P_{\lambda_m}Q_{\mu_n}}\left|K_{P_{\lambda_m}Q_{\mu_n}}(\varepsilon)\right|.
\end{aligned}$$

Taking limit as $m,n\to\infty$, we get $x=\left(x_{jk}\right)$ is weighted (λ,μ) −statistically convergent to L, where $K_{P_{\lambda_m}Q_{\mu_n}}(\varepsilon)=\left\{(j,k),\,j\in J_m\text{ and }k\in I_n:p_jq_k\left|x_{jk}-L\right|\geq\varepsilon\right\}$. □

Theorem 2.13. *Let a double sequence $x=\left(x_{jk}\right)$ is weighted (λ,μ) −statistically convergent to L and $p_jq_k\left|x_{jk}-L\right|\leq M$. If the following cases are provided then; $x=\left(x_{jk}\right)$ is $\left[\overline{N}_{\lambda,\mu},p,q,\alpha,\beta\right]_r$ −summable to L.*

*Case*1 : $0 < r < 1$ *and* $1 < M < \infty$
*Case*2 : $1 \leq r < \infty$ *and* $0 \leq M < 1$

Proof. Suppose that $x = \left(x_{jk}\right)$ is weighted (λ, μ) −statistically convergent to L . Since $p_j q_k \left|x_{jk} - L\right| \leq M$ we can write

$$\begin{aligned}
&\frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{j \in J_m} \sum_{k \in I_n} p_j q_k \left|x_{jk} - L\right|^r \\
&= \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{\substack{j \in J_m \\ (j,k) \in K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)}} \sum_{k \in I_n} p_j q_k \left|x_{jk} - L\right|^r + \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{\substack{j \in J_m \\ (j,k) \in K^c_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)}} \sum_{k \in I_n} p_j q_k \left|x_{jk} - L\right|^r \\
&\leq \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{\substack{j \in J_m \\ (j,k) \in K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)}} \sum_{k \in I_n} p_j q_k \left|x_{jk} - L\right| + \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{\substack{j \in J_m \\ (j,k) \in K^c_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)}} \sum_{k \in I_n} p_j q_k \left|x_{jk} - L\right| \\
&\leq \frac{M \left|K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)\right|}{P_{\lambda_m} Q_{\mu_n}} + \frac{\varepsilon}{P_{\lambda_m} Q_{\mu_n}} \left|K^c_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon)\right| \to 0 \quad \text{as } m, n \to \infty,
\end{aligned}$$

where $K^c_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon) = \left\{(j,k), j \in J_m \text{ and } k \in I_n : p_j q_k \left|x_{jk} - L\right| < \varepsilon\right\}$. Hence $x = \left(x_{jk}\right)$ is $\left[\overline{N}_{\lambda,\mu}, p, q, \alpha, \beta\right]_r$ −summable to limit L. □

3. Applications

Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$. We know that $C[a,b]$ is a Banach space with norm $\left\|f\right\|_\infty = \sup_{x \in [a,b]} \left|f(x)\right|$, $f \in [a,b]$. The classical Korovkin type approximation theorem states as follows [21]:

Suppose that (T_n) be a sequence positive linear operators from $C[a,b]$ into $C[a,b]$.
Then, $\lim_n \left\|T_n(f;x) - f(x)\right\|_\infty = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \left\|T_n(f_i;x) - f_i(x)\right\|_\infty = 0$ for $i = 0,1,2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

By $C(K)$ we denote the space of all continuous real valued functions on any compact subset of the real two dimensional space. This space is equipped with the supremum norm $\left\|f\right\|_{C(K)} = \sup_{(x,y) \in K} \left|f(x,y)\right|$, $f \in C(K)$. Let L be a linear operator from $C(K)$ into $C(K)$. Then, as usual we say that L is positive linear operator provided that $f \geq 0$ implies $Lf \geq 0$. Also we denote the value of Lf at a point$(x,y) \in K$ by $L(f;x,y)$.

Recently Korovkin type approximation theorems have been studied in ([2],[9],[13],[16],[22],[27],[30]).

Theorem 3.1. *Let $\{T_{mn}\}$ is a double sequence of positive linear operators from $C(K)$ into $C(K)$. Then, for all $f \in C(K)$*

$$\overline{N}^2_{\lambda\mu}(st) - \lim_{m,n} \left\|T_{mn} f - f\right\|_{C(K)} = 0 \tag{3.1}$$

if and only if

$$\overline{N}^2_{\lambda\mu}(st) - \lim_{m,n} \left\|T_{mn} f_i - f_i\right\|_{C(K)} = 0 \tag{3.2}$$

$(i = 0,1,2,3)$, where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$.

Proof. Since each $f_i \in C(K)$ $(i = 0, 1, 2, 3)$ condition (3.1) follows immediately (3.2). Assume that (3.2) holds. Let $f \in C(K)$ we can write $\left|f(x, y)\right| \leq M$ where $M = \left\|f\right\|_{C(K)}$. Since f is continuous on K for every $\varepsilon > 0$, there is $\delta > 0$ such that $\left|f(u, v) - f(x, y)\right| < \varepsilon$ for all $(u, v) \in K$ satisfying $|u - x| < \delta, \left|v - y\right| < \delta$. Hence we get

$$\left|f(u, v) - f(x, y)\right| < \varepsilon + \frac{2M}{\delta^2}\left\{(u - x)^2 + (v - y)^2\right\} \tag{3.3}$$

Since T_{mn} is linear and positive and by (3.3) we obtain

$$\begin{aligned}
&\left|T_{mn}(f; x, y) - f(x, y)\right| \\
&= \left|T_{mn}(f(u, v) - f(x, y); x, y) - f(x, y)\, T_{mn}(f_0; x, y) - f_0(x, y)\right| \\
&\leq T_{mn}\left(\left|f(u, v) - f(x, y)\right|; x, y\right) + M\left|T_{mn}(f_0; x, y) - f_0(x, y)\right| \\
&\leq \left|T_{mn}\left(\varepsilon + \frac{2M}{\delta^2}\left\{(u - x)^2 + (v - y)^2\right\}; x, y\right)\right| + M\left|T_{mn}(f_0; x, y) - f_0(x, y)\right| \\
&\leq \varepsilon + M + \frac{2M}{\delta^2}\left(A^2 + B^2\right)\left|T_{mn}(f_0; x, y) - f_0(x, y)\right| \\
&+ \frac{4M}{\delta^2}A\left|T_{mn}(f_1; x, y) - f_1(x, y)\right| + \frac{4M}{\delta^2}B\left|T_{mn}(f_2; x, y) - f_2(x, y)\right| \\
&+ \frac{2M}{\delta^2}\left|T_{mn}(f_3; x, y) - f_3(x, y)\right| + \varepsilon
\end{aligned}$$

where $A = \max|x|, B = \max\left|y\right|$. Taking supremum over $(x, y) \in K$ we have

$$\left\|T_{mn}f - f\right\|_{C(K)} \leq R\left\{\left\|T_{mn}f_0 - f_0\right\|_{C(K)} + \left\|T_{mn}f_1 - f_1\right\|_{C(K)} + \left\|T_{mn}f_2 - f_2\right\|_{C(K)} + \left\|T_{mn}f_3 - f_3\right\|_{C(K)}\right\} + \varepsilon,$$

where

$$R = \max\left\{\varepsilon + M + \frac{2M}{\delta^2}\left(A^2 + B^2\right), \frac{4M}{\delta^2}A, \frac{4M}{\delta^2}B, \frac{2M}{\delta^2}\right\}.$$

Hence

$$\left\|T_{mn}(f; x, y)\, p_m q_n - f(x, y)\right\|_{C(K)} \leq \varepsilon + R\sum_{i=0}^{3}\left\|T_{mn}(f_i; x, y)\, p_m q_n - f_i(x, y)\right\|_{C(K)} \tag{3.4}$$

Now replace $T_{mn}(.; x, y)\, p_m q_n$ by

$$L_{mn}(.; x, y) = \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{(m,n) \in J_m \times I_n} T_{mn}(.; x, y)\, p_m q_n$$

in (3.4). For a given $r > 0$ choose $\varepsilon' > 0$ such that $\varepsilon' < r$. Define the following sets

$$\begin{aligned}
D &= \left\{(m, n) \in N^2 : \left\|L_{mn}f - f\right\|_{C(K)} \geq r\right\}, \\
D_i &= \left\{(m, n) \in N^2 : \left\|L_{mn}f_i - f_i\right\|_{C(K)} \geq \frac{r - \varepsilon'}{4R}\right\}, i = 0, 1, 2, 3.
\end{aligned}$$

Then, $D \subset \bigcup_{i=0}^{3} D_i$ and so $\delta(D) \leq \delta(D_0) + \delta(D_1) + \delta(D_2) + \delta(D_3)$. Therefore, using conditions (3.4) we get

$$\overline{N}^2_{\lambda\mu}(st) - \lim_{m,n}\left\|T_{mn}f - f\right\|_{C(K)} = 0$$

This completes the proof. □

Corollary 3.2. *Let $\{L_{m,n}\}$ be a sequence of positive linear operators acting from $C(K)$ into itself. Then, for all $f \in C(K)$,*

$$P-\lim_{m,n}\left\|L_{mn}f-f\right\|_{C(K)}=0$$

if and only if

$$P-\lim_{m,n}\left\|L_{mn}f_i-f_i\right\|_{C(K)}=0,(i=0,1,2,3),$$

where $f_0(x,y)=1, f_1(x,y)=x$, $f_2(x,y)=y$ and $f_3(x,y)=x^2+y^2$ [36].

Remark 3.3. We now construct an example of sequence of positive linear operators of two variables satisfying the conditions of Theorem 3.1, but that does not satisfy the conditions of the Korovkin Theorem. For this claim, we consider the following Brenstein operators defined as follows

$$B_{m,n}(f;x,y)=\sum_{k=0}^{m}\sum_{j=0}^{n}f\left(\frac{k}{m},\frac{j}{n}\right)C_m^kx^k(1-x)^{m-k}C_n^jy^j(1-y)^{n-j},$$

where $(x,y)\in[0,1]\times[0,1]$.

Let

$$\begin{aligned}
&B_{m,n}(f_0;x,y)=1\\
&B_{m,n}(f_1;x,y)=x\\
&B_{m,n}(f_2;x,y)=y\\
&B_{m,n}(f_3;x,y)=x^2+y^2+\frac{x-x^2}{m}+\frac{y-y^2}{n}
\end{aligned}$$

Then by Corllary 3.2 we can write for all $f\in C(K)$

$$P-\lim_{m,n}\left\|B_{m,n}f-f\right\|_{C(K)}=0$$

Let the sequence $T_{m,n}:C(K)\to C(K)$ with $T_{m,n}(f;x,y)=(1+u_{mn})B_{m,n}(f;x,y)$ where $u_{mn}=(-1)^n$ for all m. Let $p_m=1$, $q_n=1$, $\lambda_m=m$, $\mu_n=n$. The double sequence (u_{mn}) is neither $P-$convergent nor statistically convertgent, but (u_{mn}) is statistically summable $\left(\overline{N}^2_{\lambda\mu},p,q,\alpha,\beta\right)$ to zero.

$B_{m,n}(1;x,y)=1$, $B_{m,n}(x;x,y)=x$, $B_{m,n}(y;x,y)=y$, $B_{m,n}\left(x^2+y^2;x,y\right)=x^2+y^2+\frac{x-x^2}{m}+\frac{y-y^2}{n}$ and the double sequence $T_{m,n}$ satisfies condition (3.2) for $i=0,1,2,3$. Hence we have

$$\overline{N}^2_{\lambda\mu}(st)-\lim_{m,n}\left\|T_{mn}f-f\right\|_{C(K)}=0.$$

On the other hand, we get $T_{m,n}(f,0,0)=(1+u_{mn})B_{m,n}(f;0,0)$ since $B_{m,n}(f;0,0)=f(0,0)$ and hence

$$\left\|T_{mn}(f;x,y)-f(x,y)\right\|_{C(K)}\geq\left|T_{mn}(f;x,y)-f(x,y)\right|\geq u_{m,n}\left|f(0,0)\right|.$$

We see that $(T_{m,n})$ does not satisfy classical Korovkin type theorem since $\lim u_{mn}$ and $st^2-\lim u_{mn}$ do not exists this proves the claim.

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