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On the Unified Family of Generalized Apostol-type Polynomials of Higher order and Multiple Power Sums

Veli Kurt

Akdeniz University, Faculty of Sciences Department of Mathematics, Antalya, TR-07058, Turkey

Abstract. In last last decade, many mathematicians studied the unification of the Bernoulli and Euler polynomials. Firstly Karande B. K. and Thakare N. K. in [6] introduced and generalized the multiplication formula. Ozden *et. al.* in [14] defined the unified Apostol-Bernoulli, Euler and Genocchi polynomials and proved some relations. M. A. Ozarslan in [13] proved the explicit relations, symmetry identities and multiplication formula. El-Desouky *et. al.* in ([3], [4]) defined a new unified family of the generalized Apostol-Euler, Apostol-Bernoulli and Apostol-Genocchi polynomials and gave some relations for the unification of multiparameter Apostol-type polynomials and numbers. In this study, we give some symmetry identities and recurrence relations for the unified Apostol-type polynomials related to multiple alternating sums.

1. Introduction, Definitions and Notations

Apostol-Bernoulli polynomials of higher order $\mathcal{B}_n^{(\alpha)}(x,\lambda)$, Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x,\lambda)$ and Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x,\lambda)$ are defined following equations, in Luo [11] respectively:

$$\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!}=\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{xt},\ \left(\left|t+\log\lambda\right|<2\pi,\ 1^{\alpha}:=1\right),$$

$$\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!}=\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}e^{xt},\ \left(\left|t+\log\lambda\right|<\pi,\ 1^{\alpha}:=1\right)$$

and

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!}=\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}e^{xt},\ \left(\left|t+\log\lambda\right|<\pi,\ 1^{\alpha}:=1\right),$$

where α and λ are arbitrary real or complex parameters and $x\in\mathbb{R}$. When $\lambda=1$ in the above relations gives the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$.

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Email address: vkurt@akdeniz.edu.tr (Veli Kurt)

The following unified Apostol-Bernoulli, Euler and Genocchi polynomials are defined by Ozarslan and Ozden in ([13], [14]) as

$$f_{a,b}^{(\alpha)}(x;t,a,b) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta}^{(\alpha)}(x,k,a,b)\frac{t^n}{n!}, \tag{1}$$
$$k \in \mathbb{N}_0, a, b \in \mathbb{R}\backslash\{0\}, \alpha, \beta \in \mathbb{C},$$

(for details on this subject, see Ozarslan [13]).

Remark 1.1. *Setting $k = a = b = 1$ and $\beta = \lambda$ in (1), we get*

$$\mathcal{P}_{n,\lambda}^{(\alpha)}(x,1,1,\lambda) = \mathcal{B}_n^{(\alpha)}(x,\lambda)$$

where $\mathcal{B}_n^{(\alpha)}(x,\lambda)$ are Apostol-Bernoulli polynomials of higher order.

Remark 1.2. *Choosing $k + 1 = -a = b = 1$ and $\beta = \lambda$ in (1), we get*

$$\mathcal{P}_{n,\lambda}^{(\alpha)}(x,0,-1,1) = \mathcal{E}_n^{(\alpha)}(x,\lambda)$$

where $\mathcal{E}_n^{(\alpha)}(x,\lambda)$ are Apostol-Euler polynomials of higher order.

Remark 1.3. *Letting $k = -2a = b = 1$ and $2\beta = \lambda$ in (1), we get*

$$\mathcal{P}_{n,\frac{\lambda}{2}}^{(\alpha)}\left(x,1,-\frac{1}{2},1\right) = \mathcal{G}_n^{(\alpha)}(x,\lambda)$$

where $\mathcal{G}_n^{(\alpha)}(x,\lambda)$ are Apostol-Genocchi polynomials of higher order.

Recently, Garg *et. al.* in ([5] and [20]) introduced the following generalization of the Hurwitz-Lerch zeta functions $\Phi(z,s,a)$;

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) = \sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^n}{(n+a)^s},$$

$$\left(\begin{array}{c}\mu\in\mathbb{C}, a, \upsilon\in\mathbb{C}\backslash\mathbb{Z}_0^-, \rho,\sigma\in\mathbb{R}, \rho<\sigma \text{ when } s,z\in\mathbb{C},\ (|z|<1)\\ \rho=\sigma \text{ and } Res(s-\mu+\nu)>0 \text{ when } |z|=1\end{array}\right).$$

It is obvious that

$$\Phi_{\mu,1}^{(1,1)}(z,s,a) = \Phi^*(z,s,a) = \sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(n+a)^s} \tag{2}$$

(for details on this subject, see ([5], [20]).

The multiple power sums and λ-multiple power sum are defined by Luo in [12] as follows:

$$S_k^{(l)}(m,\lambda) = \sum_{\substack{0\le\nu_1<\cdots<\nu_m=l\\ \nu_1+\cdots+\nu_m=m}}\binom{l}{\nu_1,\nu_2,\cdots,\nu_m}\lambda^{\nu_1+2\nu_2+\cdots+m\nu_m}(\nu_1+2\nu_2+\cdots+m\nu_m)^k. \tag{3}$$

From (3), we have

$$\left(\frac{1-\lambda^m e^{mt}}{1-\lambda e^t}\right)^l = \lambda^{(-l)}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}(-l)^{n-k}S_k^{(l)}(m,\lambda)\right\}\frac{t^n}{n!}, \tag{4}$$

where the radius of convergence $\left|\lambda e^{t}\right| < 1$.

From (4); for $l = 1$, we have

$$\frac{1-\lambda^{m}e^{mt}}{1-\lambda e^{t}} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}S_{k}(m,\lambda)\right\}\frac{t^{n}}{n!}, \tag{5}$$

where the radius of convergence $\left|\lambda e^{t}\right| < 1$.

The generalized Stirling numbers $S(n,\nu,a,b,\beta)$ of the second kinds of order ν are defined in [21] by follows:

$$\sum_{n=0}^{\infty} S(n,\nu,a,b,\beta)\frac{t^{n}}{n!} = \frac{\left(\beta^{b}e^{t}-a^{b}\right)^{\nu}}{\nu!}. \tag{6}$$

2. Explicit Relations for the Unified Family of Generalized Apostol-type Polynomials

In this section, we aim to obtain the explicit relations of the polynomials $\mathcal{P}_{n,\beta}^{(\alpha)}(x,k,a,b)$ and give the relation between the unified family of generalized Apostol-type polynomials and the Stirling numbers of second kind $S(n,\nu,a,b,\beta)$ of order ν.

Theorem 2.1. *The following relation is true for the unified Apostol-type polynomials:*

$$\mathcal{P}_{n-k\alpha,\beta}^{(\alpha-\gamma)}(x,k,a,b) = 2^{(k-1)\gamma}\frac{(n-k\gamma)!}{n!}\sum_{l=0}^{n}\binom{n}{l}\mathcal{P}_{l,\beta}^{(\alpha)}(x,k,a,b)\sum_{p=0}^{\gamma}\binom{\gamma}{p}\beta^{bp}\left(-a^{b}\right)^{\gamma-p}p^{n-l}, \tag{7}$$

where $\gamma > 0$.

Proof. From (1), we write as

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}^{(\alpha-\gamma)}(x,k,a,b)\frac{t^{n}}{n!} &= \left(\frac{2^{1-k}t^{k}}{\beta^{b}e^{t}-a^{b}}\right)^{(\alpha-\gamma)}e^{xt} \\ &= \left(\frac{2^{1-k}t^{k}}{\beta^{b}e^{t}-a^{b}}\right)^{\alpha}\left(\beta^{b}e^{t}-a^{b}\right)^{\gamma}t^{-k\gamma}2^{(k-1)\gamma}e^{xt}. \end{aligned}\tag{8}$$

On the other hand,

$$\left(\beta^{b}e^{t}-a^{b}\right)^{\gamma} = \sum_{p=0}^{\gamma}\binom{\gamma}{p}\beta^{bp}e^{pt}\left(-a^{b}\right)^{\gamma-p} = \sum_{n=0}^{\infty}\sum_{p=0}^{\gamma}\binom{\gamma}{p}\beta^{bp}\left(-a^{b}\right)^{\gamma-p}p^{n}\frac{t^{n}}{n!}.$$

Substituting this equation in the right-hand side of (8), we write as

$$\begin{aligned}&\sum_{n=0}^{\infty}n(n-1)...(n-k\gamma+1)\mathcal{P}_{n-k\gamma,\beta}^{(\alpha-\gamma)}(x,k,a,b)\frac{t^{n}}{n!} \\ =\ & 2^{(k-1)\alpha}\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}^{(\alpha)}(x,k,a,b)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}\sum_{p=0}^{\gamma}\binom{\gamma}{p}\beta^{bp}\left(-a^{b}\right)^{\gamma-p}p^{n}\frac{t^{n}}{n!}.\end{aligned}$$

By using the Cauchy product and comparing the coefficients of $\frac{t^{n}}{n!}$ on the above equation. We have (7). □

Theorem 2.2. *There is the following recurrence relation for the unified Apostol-type polynomials $\mathcal{P}_{n,\beta}^{(\alpha)}(x,k,a,b)$;*

$$\begin{aligned} &\mathcal{P}_{n,\beta}(x,k,a,b) \\ = &\frac{-\beta^b}{1-k-x}\left\{\sum_{s=0}^{n-1}\binom{n-1}{s}\mathcal{P}_{n-s,\beta}(1,1,a,b)\mathcal{P}_{s,\beta}(x,k,a,b)\right\}. \end{aligned} \tag{9}$$

Proof. By using (1), we take the derivative according to t for $\alpha = 1$. We write as

$$\frac{d}{dt}\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}(x,k,a,b)\frac{t^n}{n!} = \frac{d}{dt}\left(\frac{2^{1-k}t^k e^{xt}}{\beta^b e^t - a^b}\right)$$

$$= 2^{1-k}\left\{\frac{\left(kt^{k-1}+xt^k\right)e^{xt}\left(\beta^b e^t - a^b\right)}{\left(\beta^b e^t - a^b\right)^2} - \frac{\beta^b e^t t^k e^{xt}}{\left(\beta^b e^t - a^b\right)^2}\right\}.$$

In the above equality, making the necessary operations, we have (9). □

3. Some Symmetry Identities for the Unified Generalized Apostol-type Polynomials

W. Wang *et. al.* in [23] and Z. Zhang *et. al.* in [24] proved some symmetry identities and recurrence relations for the Apostol-type polynomials. Kurt in ([7], [8]) gave some symmetry identities for the Apostol-type polynomials related to multiple alternating sums.

In this section, we give some symmetry identities for the unified Apostol-type polynomials.

Theorem 3.1. *There is the following relation between the unified Apostol-type polynomials and the Hurwitz-Lerch zeta functions $\Phi^*(z,s,a)$;*

$$\begin{aligned} &c^k\sum_{s=0}^{n-k\alpha}\binom{n-k\alpha}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}(-1)^{s-r-q}S_q\left(d,\left(\frac{\beta}{a}\right)^b\right) \\ &\times\mathcal{P}_{r,\beta}^{(\alpha-1)}(dy,k,a,b)\,c^r d^{n-s}\Phi_\alpha^*\left(\left(\frac{\beta}{a}\right)^b, s+kn-n, cx\right) \\ = \;&d^k\sum_{s=0}^{n-k\alpha}\binom{n-k\alpha}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}(-1)^{s-r-q}S_q\left(c,\left(\frac{\beta}{a}\right)^b\right) \\ &\times\mathcal{P}_{r,\beta}^{(\alpha-1)}(cx,k,a,b)\,d^r c^{n-s}\Phi_\alpha^*\left(\left(\frac{\beta}{a}\right)^b, s+kn-n, dy\right). \end{aligned} \tag{10}$$

Proof. Using the generalized binomials theorem, we get

$$(1+w)^{(-\alpha)} = \sum_{r=0}^{\infty}\binom{\alpha+r-1}{r}(-w)^r,\ |w|<1.$$

Using (1), (2) and (4) in above equation:

$$f(t) = \frac{t^{\alpha(2k-1)}2^{(1-k)(2\alpha-1)}e^{cdxt}\left(\beta^{bd}e^{cdt}-a^{bd}\right)^\alpha e^{cdyt}}{\left(\beta^b e^{dt}-a^b\right)^\alpha\left(\beta^b e^{ct}-a^b\right)^\alpha}$$

$$= c^{(1-\alpha)k}2^{(1-\alpha)k}a^{b(d-\alpha+1)}(-1)^{\alpha}t^{k\alpha}\sum_{m=0}^{\infty}\binom{m+\alpha-1}{m}\left(\frac{\beta}{a}\right)^{mb}e^{mdt}e^{cdxt}\frac{a^b}{\beta^b}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}(-1)^{p-q}$$

$$\times S_q\left(d,\left(\frac{\beta}{a}\right)^b\right)\frac{t^p}{p!}\sum_{r=0}^{\infty}\mathcal{P}_{r,\beta}^{(\alpha-1)}(dy,k,a,b)\,c^r\frac{t^r}{r!}.$$

After taking the Cauchy product, we have

$$f(t) = \sum_{n=k\alpha}^{\infty}\left\{\frac{n!}{(n-k\alpha)!}c^{(1-\alpha)k}2^{(1-\alpha)k}\beta^{-b}a^{b(d-\alpha)}(-1)^{\alpha}\sum_{s=0}^{n-k\alpha}\binom{n-k\alpha}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}\right.$$

$$\left.\times(-1)^{s-r-q}S_q\left(d,\left(\frac{\beta}{a}\right)^b\right)\mathcal{P}_{r,\beta}^{(\alpha-1)}(dy,k,a,b)\,c^r d^{n-k\alpha-s}\Phi_{\alpha}^{*}\left(\left(\frac{\beta}{a}\right)^b,s+kn-n,cx\right)\right\}\frac{t^n}{n!}.$$

We also set

$$f(t) = \frac{t^{\alpha(2k-1)}2^{(1-k)(2\alpha-1)}e^{cdyt}\left(\beta^{bd}e^{cdt}-a^{bd}\right)^{\alpha}e^{cdxt}}{\left(\beta^b e^{ct}-a^b\right)^{\alpha}\left(\beta^b e^{dt}-a^b\right)^{\alpha}}.$$

Using (1), (2) and (4) in above equation, we get;

$$= d^{(1-\alpha)k}2^{(1-\alpha)k}a^{b(d-\alpha+1)}(-1)^{\alpha}t^{k\alpha}\sum_{m=0}^{\infty}\binom{m+\alpha-1}{m}\left(\frac{\beta}{a}\right)^{mb}e^{mct}e^{cdyt}\left(\frac{a}{\beta}\right)^{b}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}(-1)^{p-q}$$

$$\times S_q\left(c,\left(\frac{\beta}{a}\right)^b\right)\frac{t^p}{p!}\sum_{r=0}^{\infty}\mathcal{P}_{r,\beta}^{(\alpha-1)}(cx,k,a,b)\,d^r\frac{t^r}{r!}.$$

Therefore we have

$$f(t) = \sum_{n=k\alpha}^{\infty}\left\{\frac{n!}{(n-k\alpha)!}d^{(1-\alpha)k}2^{(1-\alpha)k}\beta^{-b}a^{b(d-\alpha)}(-1)^{\alpha}\sum_{s=0}^{n-k\alpha}\binom{n-k\alpha}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}\right.$$

$$\left.\times(-1)^{s-r-q}S_q\left(c,\left(\frac{\beta}{a}\right)^b\right)\mathcal{P}_{r,\beta}^{(\alpha-1)}(cx,k,a,b)\,d^r c^{n-k\alpha-s}\Phi_{\alpha}^{*}\left(\left(\frac{\beta}{a}\right)^b,s+kn-n,c,dy\right)\right\}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the above equation, we have (10). □

Remark 3.2. *Let c, $d \in \mathbb{N}$, m, r, s, $q \in \mathbb{N}_0$. For $k = a = b = 1$, $\beta = \lambda$ in (10), we have the following symmetry identities for Apostol-Bernoulli polynomials of higher order:*

$$c\sum_{s=0}^{n-\alpha}\binom{n-\alpha}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}(-1)^{s-r-q}S_q(d,\lambda)\,\mathcal{B}_n^{(\alpha-1)}(dy;\lambda)\,c^r d^{n-s}\Phi_{\alpha}^{*}(\lambda,s,cx)$$

$$= d\sum_{s=0}^{n-\alpha}\binom{n-\alpha}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}(-1)^{s-r-q}S_q(c,\lambda)\,\mathcal{B}_n^{(\alpha-1)}(cx;\lambda)\,d^r c^{n-s}\Phi_{\alpha}^{*}(\lambda,s,dy).$$

Remark 3.3. *Let c, $d \in \mathbb{N}$, m, r, s, $q \in \mathbb{N}_0$. For $k = 0$, $a = -1$, $b = 1$, $\beta = \lambda$ in (10), we have the following symmetry identities for Apostol-Euler polynomials of higher order:*

$$\sum_{s=0}^{n}\binom{n}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}(-1)^{s-r-q}S_q(d,-\lambda)\,\mathcal{E}_n^{(\alpha-1)}(dy;\lambda)\,c^r d^{n-s}\Phi_{\alpha}^{*}(\lambda,s-n,cx)$$

$$= \sum_{s=0}^{n}\binom{n}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}(-1)^{s-r-q}S_q(c,-\lambda)\,\mathcal{E}_n^{(\alpha-1)}(cx;\lambda)\,d^r c^{n-s}\Phi_{\alpha}^{*}(\lambda,s-n,dy).$$

Remark 3.4. *Let $c, d \in \mathbb{N}$, $m, r, s, q \in \mathbb{N}_0$. For $k = 1$, $a = -\frac{1}{2}$, $b = 1$, $\beta = \frac{\lambda}{2}$ in (10), we have the following symmetry identities for the generalized Apostol-Genocchi polynomials of higher order:*

$$\begin{aligned} & c\sum_{s=0}^{n-\alpha}\binom{n-\alpha}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}(-1)^{s-r-q}S_q(d,-\lambda)\mathcal{G}_n^{(\alpha-1)}(dy;\lambda)c^r d^{n-s}\Phi_\alpha^*(\lambda,s,cx) \\ = \; & d\sum_{s=0}^{n-\alpha}\binom{n-\alpha}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{s-r}\binom{s-r}{q}(-1)^{s-r-q}S_q(c,-\lambda)\mathcal{G}_n^{(\alpha-1)}(cx;\lambda)d^r c^{n-s}\Phi_\alpha^*(\lambda,s,dy). \end{aligned}$$

Theorem 3.5. *The unified Apostol-type polynomials satisfy the following symmetry identities:*

$$\begin{aligned} & \sum_{p=0}^{n}\binom{n}{p}\mathcal{P}_{n-p,\beta}^{(\alpha)}(cx,k,a,b)\,d^{n-p-k\alpha}c^p\sum_{r=0}^{p}\binom{p}{r}(-\alpha)^{p-r}S_r^{(\alpha)}\left(d,\left(\frac{\beta}{a}\right)^b\right) \\ = \; & \sum_{p=0}^{n}\binom{n}{p}\mathcal{P}_{n-p,\beta}^{(\alpha)}(dx,k,a,b)\,c^{n-p-k\alpha}d^p\sum_{r=0}^{p}\binom{p}{r}(-\alpha)^{p-r}S_r^{(\alpha)}\left(c,\left(\frac{\beta}{a}\right)^b\right). \end{aligned} \tag{11}$$

Proof. Let

$$g(t)=\frac{\left(2^{1-k}t^k\right)^\alpha e^{cdxt}\left(\beta^{bd}e^{cdt}-a^{bd}\right)^\alpha}{\left(\beta^b e^{dt}-a^b\right)^\alpha\left(\beta^b e^{ct}-a^b\right)^\alpha}=\frac{1}{d^{k\alpha}}\left(\frac{2^{1-k}(dt)^k}{\beta^b e^{dt}-a^b}\right)^\alpha e^{cdxt}a^{(d-1)b\alpha}\left(\frac{\left(\frac{\beta}{a}\right)^{bd}e^{dct}-1}{\left(\frac{\beta}{a}\right)^b e^{ct}-1}\right)^\alpha.$$

By using same method in Theorem 3.4, we get the proof of Theorem 3.5. We omit the proof. □

Remark 3.6. *Let c, $d \in \mathbb{N}$, m, r, s, $q \in \mathbb{N}_0$. For $k = a = b = 1$, $\beta = \lambda$ in (11), we have the following symmetry identities for Apostol-Bernoulli polynomials of higher order and the multiple alternating sums:*

$$\begin{aligned} & \sum_{p=0}^{n}\binom{n}{p}\mathcal{B}_{n-p}^{(\alpha)}(cx,\lambda)\,d^{n-2p-\alpha}c^p\sum_{r=0}^{p}\binom{p}{r}(-\alpha)^{p-r}S_r^{(\alpha)}(d,\lambda) \\ = \; & \sum_{p=0}^{n}\binom{n}{p}\mathcal{B}_{n-p}^{(\alpha)}(dx,\lambda)\,c^{n-2p-\alpha}d^p\sum_{r=0}^{p}\binom{p}{r}(-\alpha)^{p-r}S_r^{(\alpha)}(c,\lambda). \end{aligned}$$

Theorem 3.7. *For all c, d, m, $\gamma \in \mathbb{N}$, n, p, $r \in \mathbb{N}_0$, there is the following symmetry identity:*

$$\begin{aligned} & d^k c^{k(m+1)}\sum_{\gamma=0}^{n}\binom{n}{\gamma}\Bigg\{\mathcal{P}_{n-\gamma,\beta}^{(m+1)}(cx,k,a,b)\,d^{n-\gamma}\sum_{p=0}^{\gamma}\binom{\gamma}{p} \\ & \times\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}S_r^{(m)}\left(d,\left(\frac{\beta}{a}\right)^b\right)\mathcal{P}_{\gamma-p,\beta}(dy,k,a,b)\,c^{\gamma-p}\Bigg\} \\ = \; & c^k d^{k(m+1)}\sum_{\gamma=0}^{n}\binom{n}{\gamma}\Bigg\{\mathcal{P}_{n-\gamma,\beta}^{(m+1)}(dy,k,a,b)\,c^{n-\gamma}\sum_{p=0}^{\gamma}\binom{\gamma}{p} \\ & \times\sum_{r=0}^{p}\binom{p}{r}(-m)^{p-r}S_r^{(m)}\left(c,\left(\frac{\beta}{a}\right)^b\right)\mathcal{P}_{\gamma-p,\beta}(cx,k,a,b)\,d^{\gamma-p}\Bigg\}. \end{aligned} \tag{12}$$

Proof. Let

$$h(t) = \frac{t^{k(m+2)} 2^{(1-k)(m+2)} e^{cdxt} \left(\beta^{bd} e^{cdt} - a^{bd}\right)^{m} e^{cdyt}}{\left(\beta^{b} e^{dt} - a^{b}\right)^{m+1} \left(\beta^{b} e^{ct} - a^{b}\right)^{m+1}}$$

By using same calculations in Theorem 3.4, we get the desired result. Because this is straightforward calculations of the algebric results. □

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