Filomat 30:4 (2016), 1005–1012
DOI 10.2298/FIL1604005C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Two-Sided Crossed Products of Groups

Esra K. Cetinalp[a], Eylem G. Karpuz[b], Firat Ates[c], A. Sinan Cevik[d]

[a]*Department of Mathematics, Faculty of Science, Karamanoğlu Mehmetbey University, Campus, 70100, Karaman, Turkey*
[b]*Department of Mathematics, Faculty of Science, Karamanoğlu Mehmetbey University, Campus, 70100, Karaman, Turkey*
[c]*Department of Mathematics, Faculty of Science, Balikesir University, Campus, 10145, Balikesir, Turkey*
[d]*Department of Mathematics, Faculty of Science, Selcuk University, Campus, 42075, Konya, Turkey*

Abstract. In this paper, we first define a new version of the crossed product of groups under the name of *two-sided crossed product*. Then we present a generating and relator sets for this new product over cyclic groups. In a separate section, by using the monoid presentation of the two-sided crossed product of cyclic groups, we obtain the complete rewriting system and normal forms of elements of this new group construction.

1. Introduction and Preliminaries

The classification of groups has taken so much interest for ages. For instance, in [3], the authors have recently identified the related tensor degree of finite groups. On the other hand, some other part of the classification is based on the usage of automorphism groups (see, for example, [8]) and this would give an advantage of obtaining some new groups in the meaning of products of groups. As a consequence of that the constructions such as direct and semidirect product of groups are current in mathematics. They are used when new groups are constructed that inherit some properties of initial groups and they are also used for some complex groups are reduced to some simple groups. In this paper, we will follow this idea to get a new classification.

As known crossed product construction appears in different areas of algebra such as Lie algebras, C^*-algebras and group theory. This product has also many applications in other fields of mathematics like group representation theory and topology. Here, by considering crossed product construction from view of group theory, we define a generalization of this product. We call this new generalization as *two-sided crossed product of groups*. This new product is more important than the known group products since it contains direct, semidirect, twisted ([10]), knit ([4]) and crossed products of groups. By considering this new product, its identities and normal form of its elements, in the future works, one can consider the solvability of decision problems, study some algebraic properties and algebraic computations over it. One can also study this new product in many applications of Hopf algebra and C^*-algebra.

2010 *Mathematics Subject Classification*. Primary 16S15; Secondary 20E22, 20M05
Keywords. Crossed product, presentation, rewriting system
Received: 05 July 2015; Accepted: 24 October 2015
Communicated by Gradimir Milovanović and Yilmaz Simsek
Presented in the conference 28th ICJMS-Turkey. This work is supported by the Scientific Research Fund of Karamanoğlu Mehmetbey University Project No: 08-YL-15.
Corresponding author is Esra K. Cetinalp
Email addresses: esrakirmizi@kmu.edu.tr (Esra K. Cetinalp), eylem.guzel@kmu.edu.tr (Eylem G. Karpuz), firat@balikesir.edu.tr (Firat Ates), sinan.cevik@selcuk.edu.tr (A. Sinan Cevik)

Let H and G be two groups. A crossed system of these groups is a quadruple (H, G, α, f), where $\alpha : G \to Aut(H)$ and $f : G \times G \to H$ are two maps such that the following compatibility conditions hold:

$$g_1 \triangleleft_\alpha (g_2 \triangleleft_\alpha h) = f(g_1, g_2)((g_1g_2) \triangleleft_\alpha h)f(g_1, g_2)^{-1}, \tag{1}$$

$$f(g_1, g_2)f(g_1g_2, g_3) = (g_1 \triangleleft_\alpha f(g_2, g_3))f(g_1, g_2g_3), \tag{2}$$

for all $g_1, g_2, g_3 \in G$ and $h \in H$. The crossed system (H, G, α, f) is called *normalized* if $f(1, 1) = 1$. The map $\alpha : G \to Aut(H)$ is called weak action and $f : G \times G \to H$ is called an α-cocycle. (H, G, α, f) is normalized crossed system then $f(1, g) = f(g, 1) = 1$ and $1 \triangleleft_\alpha h = h$, for any $g \in G$ and $h \in H$. As $\alpha(g) \in Aut(H)$ we have $g \triangleleft_\alpha 1 = 1$ and $g \triangleleft_\alpha (h_1h_2) = (g \triangleleft_\alpha h_1)(g \triangleleft_\alpha h_2)$. The crossed product of H and G associated to the crossed system, denoted by $H\#^f_\alpha G$, is the set $H \times G$ with the multiplication

$$(h_1, g_1)(h_2, g_2) = (h_1(g_1 \triangleleft_\alpha h_2)f(g_1, g_2), g_1g_2),$$

for all $h_1, h_2 \in H$ and $g_1, g_2 \in G$. Then $(H\#^f_\alpha G, \cdot)$ is a group with the unit $1_{H\#^f_\alpha G} = (1, 1)$ if and only if (H, G, α, f) is a normalized crossed system. It is easy to see that, for $(h, g) \in H\#^f_\alpha G$, $(h, g)^{-1} = (f(g^{-1}, g)^{-1}g^{-1} \triangleleft_\alpha h^{-1}, g^{-1})$. Then $H\#^f_\alpha G$ is called *the crossed product* of H and G associated to the crossed system (H, G, α, f) (cf. [1]).

The following result is one of the main applications of the crossed product construction which the proof of it can be found in [1].

Proposition 1.1 ([1]). *Let E be a group, H be normal subgroup of E and G be the quotient of E by H. Then there exist maps $\alpha : G \to Aut(H)$ and $f : G \times G \to H$ such that (H, G, α, f) is normalized crossed system and $E \cong (H\#^f_\alpha G, \cdot)$.*

The organization of this paper is as follows: In the first section, we will recall the construction and fundamental properties of crossed product of groups. After that, in Section 2, we will define the two-sided crossed product of groups and also, as an application of the theory, we will obtain a presentation for the two-sided crossed product of two cyclic groups. At the final section, we will present the complete rewriting system for two-sided crossed product of two cyclic groups by using the monoid presentation version, and then we will get the normal forms of elements of this group construction. As a result of this, we will get the solvability of the word problem.

Throughout this paper, we order words in given alphabet in the deg-lex way by comparing two words first with their degrees (lengths), and then lexicographically when the lengths are equal. Additionally, the notation $(i) \cap (j)$ and $(i) \cup (j)$ will denote the intersection and inclusion overlapping words of left hand side of relations (i) and (j), respectively.

2. Two-sided Crossed Product

Let H and G be two groups. Assume that

$$\alpha : G \to Aut(H),\ f : G \times G \to H \quad \text{and} \quad \alpha' : H \to Aut(G),\ f' : H \times H \to G \tag{3}$$

be maps such that (1),(2) and the following compatability conditions hold:

$$h_1 \triangleleft_{\alpha'} (h_2 \triangleleft_{\alpha'} g) = f'(h_1, h_2)((h_1h_2) \triangleleft_{\alpha'} g)f'(h_1, h_2)^{-1}, \tag{4}$$

$$f'(h_1, h_2)f'(h_1h_2, h_3) = (h_1 \triangleleft_{\alpha'} f'(h_2, h_3))f'(h_1, h_2h_3), \tag{5}$$

for all $h_1, h_2, h_3 \in H$ and $g \in G$. Then two-sided crossed product of H and G, denoted by $H\#^{f,f'}_{\alpha,\alpha'}G$, with respect to the actions given above is the set $H \times G$ endowed with the operation

$$(h_1, g_1)(h_2, g_2) = (h_1(g_1 \triangleleft_\alpha h_2)f(g_1, g_2), g_1(h_1 \triangleleft_{\alpha'} g_2)f'(h_1, h_2)), \tag{6}$$

for all $h_1, h_2 \in H$ and $g_1, g_2 \in G$.

Unlikely crossed products of groups, the two-sided crossed product need not always be a group. In fact, the following first main result of this paper identify when this new product defines a group.

Theorem 2.1. *Let H and G be any groups. For all $h_1, h_2, h \in H$ and $g_1, g_2, g \in G$, let us consider again the actions given in* (3) *with the properties*

$$g^{-1}(h_1 \triangleleft_{\alpha'} g)f'(h_1, h_2) \in Ker\alpha\,, \tag{7}$$
$$h^{-1}(g_1 \triangleleft_{\alpha} h)f(g_1, g_2) \in Ker\alpha'\,. \tag{8}$$

Then the two-sided normalized crossed product $H\#^{f,f'}_{\alpha,\alpha'}G$ defines a group.

Proof. We verify the group properties of the two-sided crossed product of groups. Firstly, we show the associative property. To do that, for any $h_1, h_2, h_3 \in H$ and $g_1, g_2, g_3 \in G$, let $(h_1, g_1), (h_2, g_2), (h_3, g_3) \in H\#^{f,f'}_{\alpha,\alpha'}G$. So the left hand side $[(h_1, g_1)(h_2, g_2)](h_3, g_3)$ is equal to

$$\begin{aligned}
&= ((h_1(g_1 \triangleleft_\alpha h_2)f(g_1, g_2))(g_1(h_1 \triangleleft_{\alpha'} g_2)f'(h_1, h_2) \triangleleft_\alpha h_3)f(g_1(h_1 \triangleleft_{\alpha'} g_2)f'(h_1, h_2), g_3),\\
&\quad g_1(h_1 \triangleleft_{\alpha'} g_2)f'(h_1, h_2)(h_1(g_1 \triangleleft_\alpha h_2)f(g_1, g_2) \triangleleft_{\alpha'} g_3)f'(h_1(g_1 \triangleleft_{\alpha'} h_2)f(g_1, g_2), h_3))\\
&= (h_1h_2(g_1g_2 \triangleleft_\alpha h_3)f(g_1g_2, g_3)\,,\ g_1g_2(h_1h_2 \triangleleft_{\alpha'} g_3)f'(h_1h_2, h_3)) \quad \text{(by (7) and (8))}
\end{aligned}$$

and the right hand side $(h_1, g_1)[(h_2, g_2)(h_3, g_3)]$ is equal to

$$\begin{aligned}
&= (h_1(g_1 \triangleleft_\alpha (h_2(g_2 \triangleleft_\alpha h_3)f(g_2, g_3)))f(g_1, g_2(h_2 \triangleleft_{\alpha'} g_3)f'(h_2, h_3)),\\
&\quad g_1(h_1 \triangleleft_{\alpha'} (g_2(h_2 \triangleleft_{\alpha'} g_3)f'(h_2, h_3)))f'(h_1, h_2(g_2 \triangleleft_\alpha h_3)f(g_2, g_3)))\\
&= (h_1(g_1 \triangleleft_\alpha h_2)(g_1 \triangleleft_\alpha (g_2 \triangleleft_\alpha h_3))(g_1 \triangleleft_\alpha f(g_2, g_3))f(g_1, g_2(h_2 \triangleleft_{\alpha'} g_3)f'(h_2, h_3)),\\
&\quad g_1(h_1 \triangleleft_{\alpha'} g_2)(h_1 \triangleleft_{\alpha'} (h_2 \triangleleft_{\alpha'} g_3))(h_1 \triangleleft_{\alpha'} f'(h_2, h_3))f'(h_1, h_2(g_2 \triangleleft_\alpha h_3)f(g_2, g_3)))\\
&= (h_1(g_1 \triangleleft_\alpha h_2)f(g_1, g_2)(g_1g_2 \triangleleft_\alpha h_3)f(g_1, g_2)^{-1}(g_1 \triangleleft_\alpha f(g_2, g_3))f(g_1, g_2(h_2 \triangleleft_\alpha g_3)f'(h_2, h_3)),\\
&\quad g_1(h_1 \triangleleft_{\alpha'} g_2)f'(h_1, h_2)(h_1h_2 \triangleleft_{\alpha'} g_3)f'(h_1, h_2)^{-1}(h_1 \triangleleft_{\alpha'} f'(h_2, h_3))f'(h_1, h_2(g_2 \triangleleft_{\alpha'} h_3)f(g_2, g_3)))\\
&= (h_1h_2(g_1g_2 \triangleleft_\alpha h_3)f(g_1, g_2)^{-1}(g_1 \triangleleft_\alpha f(g_2, g_3))f(g_1, g_2g_3),\\
&\quad g_1g_2(h_1h_2 \triangleleft_{\alpha'} g_3)f'(h_1, h_2)^{-1}(h_1 \triangleleft_{\alpha'} f'(h_2, h_3))f'(h_1, h_2h_3)) \quad \text{(by (7) and (8))}\\
&= (h_1h_2(g_1g_2 \triangleleft_\alpha h_3)f(g_1g_2, g_3)\,,\ g_1g_2(h_1h_2 \triangleleft_{\alpha'} g_3)f'(h_1h_2, h_3)). \quad \text{(by (2) and (5))}
\end{aligned}$$

Now, for the identity elements 1_H and 1_G of groups H and G, respectively, we obtain

$$\begin{aligned}
(h, g)(1_H, 1_G) &= (h(g \triangleleft_\alpha 1_H)f(g, 1_G), g(h \triangleleft_{\alpha'} 1_G)f'(h, 1_H)) = (h1_H, g1_G) = (h, g) \text{ and}\\
(1_H, 1_G)(h, g) &= (1_H(1_G \triangleleft_\alpha h)f(1_G, g), 1_G(1_H \triangleleft_{\alpha'} g)f'(1_H, h)) = (1_Hh, 1_Gg) = (h, g).
\end{aligned}$$

Finally, let us find the inverse element of $(h, g) \in H\#^{f,f'}_{\alpha,\alpha'}G$.

$$\begin{aligned}
(h, g)(h', g') = (e_H, e_G) &\Rightarrow (h(g \triangleleft_\alpha h')f(g, g'), g(h \triangleleft_{\alpha'} g')f'(h, h')) = (e_H, e_G)\\
&\Rightarrow h(g \triangleleft_\alpha h')f(g, g') = e_H \text{ and } g(h \triangleleft_{\alpha'} g')f'(h, h')) = e_G
\end{aligned}$$

Thus, we obtain $g' = h^{-1} \triangleleft_{\alpha'} g^{-1}f'(h, h^{-1})$ and $h' = g^{-1} \triangleleft_\alpha h^{-1}f(g, g^{-1})$. Hence the result. □

Now, as consequences of Theorem 2.1, we can give the following results according to the cases of maps α, α', f and f'.

Corollary 2.2. *Let (H, G, α, f) and (G, H, α', f') be two crossed systems.*

1. *Assume α, α', f and f' are trivial maps. Then $H\#^{f,f'}_{\alpha,\alpha'}G$ is the direct product of H and G.*
2. *Assume f and f' are trivial maps. Then $H\#^{f,f'}_{\alpha,\alpha'}G$ is the knit product $H \bowtie_{\alpha,\alpha'} G$ of H and G.*

Corollary 2.3. *Let (H, G, α, f) and (G, H, α', f') be two crossed systems.*

1. *Let $f, f', \alpha'(\alpha)$ be trivial maps. Then $H\#^{f,f'}_{\alpha,\alpha'}G$ is the semi-direct product of H by G (or of G by H), denoted by $H \rtimes_\alpha G$ (or $G \rtimes_{\alpha'} H$).*
2. *Let $\alpha'(\alpha), f'(f)$ be trivial maps. Then $H\#^{f,f'}_{\alpha,\alpha'}G$ is the crossed product of H by G (or of G by H), denoted by $H\#^f_\alpha G$ (or $G\#^{f'}_{\alpha'}H$).*

3. *Let $f'(f)$ be a trivial map. Then $H\#^{f,f'}_{\alpha,\alpha'}G$ is a mix of semi-direct and crossed products of H by G (or of G by H) and denoted by $H\natural G$ (or $G\natural H$). This new construction is a group with the multiplications $(h_1, g_1)(h_2, g_2) = (h_1(g_1 \triangleleft_\alpha h_2)f(g_1, g_2), g_1(h_1 \triangleleft_{\alpha'} g_2))$ and $(g_1, h_1)(g_2, h_2) = (g_1(h_1 \triangleleft_{\alpha'} g_2)f'(h_1, h_2), h_1(g_1 \triangleleft_\alpha h_2))$, for all $h_1, h_2 \in H$ and $g_1, g_2 \in G$, under the conditions given in Theorem* 2.1.

Corollary 2.4. *Let (H, G, α, f) and (G, H, α', f') be two crossed systems such that for all $h_1, h_2, h_3 \in H$ and $g_1, g_2, g_3 \in G$ the following compatibility conditions hold:*

$$Im(f) \subseteq Z(H), \quad f(g_1, g_2)f(g_1g_2, g_3) = f(g_2, g_3)f(g_1, g_2g_3) \quad \text{and}$$
$$Im(f') \subseteq Z(G), \quad f'(h_1, h_2)f'(h_1h_2, h_3) = f'(h_2, h_3)f'(h_1, h_2h_3).$$

Then we have the following cases.

1. *Let $\alpha, \alpha', f'(f)$ be trivial maps. Then $H\#^{f,f'}_{\alpha,\alpha'}G$ is the twisted product of H by G (or of G by H), denoted by $H \times^f G$ ($G \times^{f'} H$).*
2. *Let α, α' be trivial maps. Then $H\#^{f,f'}_{\alpha,\alpha'}G$ is the two-sided twisted product H and G, denoted by $H \times^{f,f'} G$. This new product is a mix of twisted products $H \times^f G$ and $G \times^{f'} H$. This construction is a group with the multiplication $(h_1, g_1)(h_2, g_2) = (h_1h_2f(g_1, g_2), g_1g_2f'(h_1, h_2))$ under the conditions given in Theorem* 2.1.
3. *Let $\alpha'(\alpha)$ be a trivial map. Then $H\#^{f,f'}_{\alpha,\alpha'}G$ is a mix of twisted and crossed products of H by G (or of G by H) and denoted by $H \not\natural G$ (or $G \not\natural H$). This new construction is a group with the multiplications $(h_1, g_1)(h_2, g_2) = (h_1(g_1 \triangleleft_\alpha h_2)f(g_1, g_2), g_1g_2f'(h_1, h_2))$ and $((g_1, h_1)(g_2, h_2) = (g_1(h_1 \triangleleft_{\alpha'} g_2)f'(h_1, h_2), h_1h_2f(g_1, g_2)))$, under the conditions given in Theorem* 2.1.

Corollary 2.5. *Let (H, G, α, f) and (G, H, α', f') be crossed systems. Then*

$$1 \to H \xrightarrow{i_H} H\#^f_\alpha G \xrightarrow{\pi_G} G \to 1 \quad \text{and} \quad 1 \to G \xrightarrow{i_G} G\#^{f'}_{\alpha'} H \xrightarrow{\pi_H} H \to 1\,,$$

where $i_H(h) := (h, 1)$, $i_G(g) := (g, 1)$, $\pi_G(h, g) := g$ and $\pi_H(g, h) := h$ for all $h \in H$, $g \in G$ are exact sequences of groups, i.e. $(H\#^f_\alpha G, i_H, \pi_G)$ and $(G\#^{f'}_{\alpha'} H, i_G, \pi_H)$ are extensions of H by G and of G by H, respectively.

2.1. Two-Sided Crossed Products of Cyclic Groups

In this subsection, we obtain a presentation for two-sided crossed product of two cyclic groups. To do that, let C_n and C_m be cyclic groups of order n and m generated by a and b, respectively. As a result of Theorem 2.1, we have the following result that the proof can be done easily.

Theorem 2.6. *Two-sided normalized crossed product $C_n\#^{f,f'}_{\alpha,\alpha'}C_m$ is a group such that $\alpha : C_n \to Aut(C_m)$, $a \mapsto a \triangleleft_\alpha b = b^{-1}$, $\alpha' : C_m \to Aut(C_n)$, $b \mapsto b \triangleleft_{\alpha'} a = a^{-1}$, $f : C_n \times C_n \to C_m$, $f(a^{t_1}, 1) = f(1, a^{t_1}) = f(1, 1) = 1$ and $f(a^{t_1}, a^{t_2}) = b$, $f' : C_m \times C_m \to C_n$, $f'(b^{k_1}, 1) = f'(1, b^{k_1}) = f'(1, 1) = 1$ and $f'(b^{k_1}, b^{k_2}) = a$, for all $a^{t_1}, a^{t_2} \in C_n$ $(t_1, t_2 \neq 0)$ and $b^{k_1}, b^{k_2} \in C_m$ $(k_1, k_2 \neq 0)$.*

Theorem 2.7. *A finite group E is isomorphic to a two-sided crossed product $C_n\#^{f,f'}_{\alpha,\alpha'}C_m$ if and only if E is a group generated by two generators a and b subject to the relations*

$$a^n = b^{i_2}, \;\; b^m = a^{i_1}, \;\; ba = a^{j_1}b^{j_2}, \tag{9}$$

where $0 \le i_1 \le n-1$, $0 \le i_2 \le m-1$, $1 \le |j_1| \le n-1$ and $1 \le |j_2| \le m-1$ such that

$$i_1.(j_1 - 1) \equiv 0(mod\, n), \quad j_1^m \equiv 1(mod\, n), \quad i_2.(j_2 - 1) \equiv 0(mod\, m), \quad j_2^n \equiv 1(mod\, m). \tag{10}$$

Proof. Suppose that the groups E_1 and E_2 are isomorphic to crossed products $C_n\#^f_\alpha C_m$ and $C_m\#^{f'}_{\alpha'}C_n$, respectively. So, there exists a normal subgroup C_n of E_1 such that $C_n \trianglelefteq E_1$ and $E_1/C_n \cong C_m$. It follows that $C_n = \langle a^n = 1\rangle \trianglelefteq E_1$ and there exists $b \in E_1$ such that $E_1/C_n = \{C_n, bC_n, \cdots, b^{m-1}C_n\}$ and $b^m \in C_n$. This shows that there exists $0 \le i_1 \le n-1$ such that $b^m = a^{i_1}$. Since $C_n \trianglelefteq E_1$, we obtain that $b^{-1}ab \in C_n$ and so we get

$b^{-1}ab = a^{j_1}$ for $0 \le j_1 \le n-1$. Similarly, since $C_m \trianglelefteq E_2$ and $E_2/C_m \cong C_n$, we obtain that $a^n = b^{i_2}$ and $a^{-1}ba = b^{j_2}$ $(0 \le i_2, j_2 \le m-1)$.

By $b^m = a^{i_1}$ and $b^{-1}ab = a^{j_1}$, we have $b^{-1}a^{i_1}b = b^{-1}b^m b = b^m = a^{i_1}$ and $b^{-1}a^{i_1}b = a^{i_1 j_1}$. It follows that $a^{i_1(j_1-1)} = 1$ and so $i_1(j_1-1) \equiv 0\ (mod\ n)$. By using the similar argument, we obtain $b^{-m}ab^m = a^{-i_1}aa^{i_1} = a$ and $a^{j_1^2} = (b^{-1}ab)^{j_1} = b^{-1}a^{j_1}b = b^{-2}ab^2$. By induction process we get $b^{-m}ab^m = a^{j_1^m}$. Hence, we have $a = a^{j_1^m}$, that is $j_1^m \equiv 0\ (mod\ n)$. Similarly, we obtain that $i_2(j_2-1) \equiv 0\ (mod\ m)$ and $j_2^n \equiv 0\ (mod\ m)$.

Let $\delta_{j_1}(1 \le |j_1| \le n-1)$ and $\psi_{j_2}(1 \le |j_2| \le m-1)$ be automorphisms of C_n and C_m, respectively. Since $a^{j_1^m} = a$ and $b^{j_2^n} = b$, we have mappings $b \mapsto Aut(C_n)$ and $a \mapsto Aut(C_m)$. These induce homomorphisms $\alpha : C_m \to Aut(C_n)$, $b \mapsto \delta_{j_1}^m$ and $\alpha' : C_n \to Aut(C_m)$, $a \mapsto \psi_{j_2}^n$ if and only if $\delta_{j_1}^m = 1_{C_n}$ and $\psi_{j_2}^n = 1_{C_m}$. By the assumption on the generator a, the homomorphisms $\delta_{j_1}^m$ and 1_{C_n} are equal if and only if $\delta_{j_1}^m[a] = [a]$. Similarly, by the assumption on b, $\psi_{j_2}^n$ and 1_{C_m} are equal if and only if $\psi_{j_2}^n[b] = [b]$. These imply that $ba = a^{j_1}b^{j_2}$.

Conversely, let us suppose that the relations in (9) and conditions in (10) hold. We aim to show that $C_n \trianglelefteq E_1$ and $C_m \trianglelefteq E_2$, that is $xa^t x^{-1} \in C_n$ $(0 \le t \le n-1)$ and $yb^l y^{-1} \in C_m$ $(0 \le l \le m-1)$, for every $x \in E_1$ and $y \in E_2$. Since $x \in E_1$ and $y \in E_2$, we can take $x = x_1x_2\cdots x_{k_1}$ and $y = y_1y_2\cdots y_{k_2}$, where $k_1, k_2 \in \mathbb{N}$, $x_{s_1} \in \{a, a^{-1}, b^{i_2}, b^{-i_2}\}$ and $y_{s_2} \in \{b, b^{-1}, a^{i_1}, a^{-i_1}\}$, $0 \le s_1 \le k_1$, $0 \le s_2 \le k_2$, $0 \le i_1 \le n-1$, $0 \le i_2 \le m-1$. This gives that $xa^t x^{-1} = x_1x_2\cdots x_{k_1}a^t x_{k_1}^{-1}\cdots x_2^{-1}x_1^{-1}$ and $yb^l y^{-1} = y_1y_2\cdots y_{k_2}b^l y_{k_2}^{-1}\cdots y_2^{-1}y_1^{-1}$. By a direct computation, we get $xa^t x^{-1} \in C_n$ and $yb^l y^{-1} \in C_m$. Hence $C_n \trianglelefteq E_1$ and $C_m \trianglelefteq E_2$. By a similar way, it can be showed that every element of groups E_1 and E_2 can be written as $a^{p_1}b^{q_1}$ and $b^{p_2}a^{q_2}$ for $p_1, p_2, q_1, q_2 \in \mathbb{Z}$, respectively. Hence $|E_1| = |E_2| = mn$ and so $|E_1/C_n| = m$, $|E_2/C_m| = n$. So thus; $E_1/C_n = \{C_n, bC_n, \cdots, b^{m-1}C_n\}$ and $E_2/C_m = \{C_m, aC_m, \cdots, a^{n-1}C_m\}$, that is, the groups E_1 and E_2 have normal subgroups C_n and C_m, respectively. Therefore by [2, Theorem 1.3], there exists crossed systems (C_n, C_m, α, f) and (C_m, C_n, α', f') such that $E_1 \cong C_n\#_\alpha^f C_m$ and $E_2 \cong C_m\#_{\alpha'}^{f'} C_n$. Hence the result. $\square$

Corollary 2.8. *Let us consider the two-sided crossed product $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ with a presentation*

$$\left\langle a, b\ ;\ a^n = b^{i_2}, b^m = a^{i_1}, ba = a^{j_1}b^{j_2} \right\rangle.$$

Also assume that $i_1 = i_2 = 0$.

1. *If $j_1 = j_2 = 1$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the direct product of C_n and C_m.*
2. *If $j_1 = 1$ and $j_2 > 0$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the semi-direct product of C_m by C_n.*
3. *If $j_2 = 1$ and $j_1 > 0$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the semi-direct product of C_n by C_m.*
4. *If $|j_1|, |j_2| > 0$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the knit product of C_n and C_m.*

Corollary 2.9. *Let us consider the two-sided crossed product $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ with a presentation*

$$\left\langle a, b\ ;\ a^n = b^{i_2}, b^m = a^{i_1}, ba = a^{j_1}b^{j_2} \right\rangle.$$

Assume also that $i_1 = 0$.

1. *If $j_1 = j_2 = 1$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the twisted product of C_m by C_n.*
2. *If $j_1 = 1$ and $j_2 > 1$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the crossed product of C_m by C_n.*
3. *If $i_2 > 0$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the semi-direct crossed product of C_m by C_n.*

Corollary 2.10. *Let us consider the two-sided crossed product $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ with a presentation*

$$\left\langle a, b\ ;\ a^n = b^{i_2}, b^m = a^{i_1}, ba = a^{j_1}b^{j_2} \right\rangle.$$

1. *If $j_1 = j_2 = 1$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the two-sided twisted product of C_n and C_m.*
2. *If $j_1 = 1$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the twisted crossed product of C_m by C_n.*
3. *If $j_2 = 1$, then $C_n\#_{\alpha,\alpha'}^{f,f'}C_m$ becomes the twisted crossed product of C_n by C_m.*

3. Rewriting Systems for $C_n\#^{f,f'}_{\alpha,\alpha'}C_m$

In this section, by considering the monoid presentation version, we will obtain the complete rewriting system for two-sided crossed product of two cyclic groups and thus, we get normal forms of elements of this group construction. To do that, let us recall some fundamental material that will be needed in the proof of Theorem 3.1 below (which is the first main result of this section).

Let X be a set and let X^* be the free monoid consists of all words obtained by the elements of X. A (string) *rewriting system* on X^* is a subset $R \subseteq X^* \times X^*$ and an element $(u, v) \in R$, also can be written as $u \to v$, is called a rule of R. The idea for a rewriting system is an algorithm for substituting the right-hand side of a rule whenever the left-hand side appears in a word. In general, for a given rewriting system R, we write $x \to y$ for $x, y \in X^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$. Also we write $x \to^* y$ if $x = y$ or $x \to x_1 \to x_2 \to \cdots \to y$ for some finite chain of reductions and $\leftrightarrow^*$ is the reflexive, symmetric, and transitive closure of $\to$. Furthermore an element $x \in X^*$ is called irreducible with respect to R if there is no possible rewriting (or reduction) $x \to y$; otherwise x is called *reducible*. The rewriting system R is called

- *Noetherian* if there is no infinite chain of rewritings $x \to x_1 \to x_2 \to \cdots$ for any word $x \in X^*$,
- *Confluent* if whenever $x \to^* y_1$ and $x \to^* y_2$, there is a $z \in X^*$ such that $y_1 \to^* z$ and $y_2 \to^* z$,
- *Complete* if R is both Noetherian and confluent.

A rewriting system is *finite* if both X and R are finite sets. A *critical pair* of a rewriting system R is a pair of overlapping rules if one of the forms **(i)** (r_1r_2, s), $(r_2r_3, t) \in R$ with $r_2 \neq 1$ or **(ii)** $(r_1r_2r_3, s)$ $(r_2, t) \in R$, is satisfied. Also a critical pair is *resolved* in R if there is a word z such that $sr_3 \to^* z$ and $r_1t \to^* z$ in the first case or $s \to^* z$ and $r_1tr_3 \to^* z$ in the second. A Noetherian rewriting system is complete if and only if every critical pair is resolved ([11]). Knuth and Bendix have developed an *algorithm* for creating a complete rewriting system R' which is equivalent to R, so that any word over X has an (unique) irreducible form with respect to R'. By considering overlaps of left-hand sides of rules, this algorithm basicly proceeds forming new rules when two reductions of an overlap word result in two distinct reduced forms.

We note that the reader is referred to [5] and [11] for a detailed survey on (complete) rewriting sytems.

It is not hard to see that the monoid presentation for $C_n\#^{f,f'}_{\alpha,\alpha'}C_m$ is given as

$$\left\langle a, b, a^{-1}, b^{-1} \,;\, a^n = b^{i_2},\ b^m = a^{i_1},\ ba = a^{j_1}b^{j_2},\ aa^{-1} = a^{-1}a = 1,\ bb^{-1} = b^{-1}b = 1 \right\rangle, \tag{11}$$

where $0 \leq i_1 < n$, $0 \leq i_2 < m$, $1 \leq |j_1| < n$, $1 \leq |j_2| < m$ such that $i_1(j_1 - 1) \equiv 0\ (mod\ n)$, $j_1^m \equiv 1\ (mod\ n)$, $i_2(j_2 - 1) \equiv 0\ (mod\ m)$ and $j_2^n \equiv 1\ (mod\ m)$.

Let us order the generators as $a > a^{-1} > b > b^{-1}$. Now we have the following theorem.

Theorem 3.1. *A complete rewriting system for the monoid presentation in* (11) *consists of the following relations:*

Case 1: Let $n \geq m$.

- *For $0 \leq i_1 < m < n$, we obtain*
  1) $a^n \to b^{i_2}$, 2) $b^m \to a^{i_1}$, 3) $a^{j_1}b^{j_2} \to ba$, 4) $ab^{i_2} \to b^{i_2}a$, 5) $a^{i_1}b \to ba^{i_1}$,
  6) $aa^{-1} \to 1$, 7) $a^{-1}a \to 1$, 8) $bb^{-1} \to 1$, 9) $b^{-1}b \to 1$.
- *For $m \leq i_1 < n$, we obtain*
  1) $a^n \to b^{i_2}$, 2) $a^{i_1} \to b^m$, 3) $a^{j_1}b^{j_2} \to ba$, 4) $ab^{i_2} \to b^{i_2}a$, 5) $ab^m \to b^ma$,
  6) $aa^{-1} \to 1$, 7) $a^{-1}a \to 1$, 8) $bb^{-1} \to 1$, 9) $b^{-1}b \to 1$.

Case 2: Let $m > n$.

- *For $0 \leq i_2 \leq n < m$, we obtain*
  1) $a^n \to b^{i_2}$, 2) $b^m \to a^{i_1}$, 3) $a^{j_1}b^{j_2} \to ba$, 4) $ab^{i_2} \to b^{i_2}a$, 5) $a^{i_1}b \to ba^{i_1}$,
  6) $aa^{-1} \to 1$, 7) $a^{-1}a \to 1$, 8) $bb^{-1} \to 1$, 9) $b^{-1}b \to 1$.

- *For $n < i_2 < m$, we obtain*
  1) $b^{i_2} \to a^n$, 2) $b^m \to a^{i_1}$, 3) $a^{j_1}b^{j_2} \to ba$, 4) $a^n b \to ba^n$, 5) $a^{i_1}b \to ba^{i_1}$,
  6) $aa^{-1} \to 1$, 7) $a^{-1}a \to 1$, 8) $bb^{-1} \to 1$, 9) $b^{-1}b \to 1$.

Proof. Since the ordering has chosen $a > a^{-1} > b > b^{-1}$, there are no infinite reduction steps for all overlapping words. Thus the rewriting system is Noetherian for both cases in theorem. Now, to catch up the aim, we need to show that the confluent property holds for each cases separately.

• For $0 \le i_1 < m < n$ in Case 1, we have the following overlapping words and corresponding critical pairs.

$(1)\cap(1)$: $a^{n+1}, (ab^{i_2}, b^{i_2}a)$, $(1)\cap(3)$: $a^n b^{j_2}, (a^{n-j_1}ba, b^{i_2}b^{j_2})$, $(1)\cap(4)$: $a^n b^{i_2}, (a^{n-1}b^{i_2}a, b^{i_2}b^{i_2})$
$(1)\cap(5)$: $a^n b, (a^{n-i_1}ba^{i_1}, b^{i_2}b)$, $(1)\cap(6)$: $a^n a^{-1}, (a^{n-1}, b^{i_2}a^{-1})$, $(2)\cap(2)$: $b^{m+1}, (a^{i_1}b, ba^{i_1})$,
$(2)\cap(8)$: $b^m b^{-1}, (a^{i_1}b^{-1}, b^{m-1})$, $(3)\cap(2)$: $a^{j_1}b^m, (bab^{m-j_2}, a^{j_1}a^{i_1})$, $(3)\cap(4)$: $a^{j_1}b^{i_2}, (bab^{i_2-j_2}, a^{j_1-1}b^{i_2}a)$,
$(3)\cup(4)$: $a^{j_1}b^{j_2}, (a^{j_1-1}b^{i_2}ab^{j_2-i_2}, ba)$, $(3)\cup(5)$: $a^{j_1}b^{j_2}, (a^{j_1-i_1}ba^{i_1}b^{j_2-1}, ba)$, $(3)\cap(8)$: $a^{j_1}b^{i_2}b^{-1}, (bab^{-1}, a^{j_1}b^{j_2-1})$

and

$(4)\cap(2)$: $ab^m, (b^{i_2}ab^{m-i_2}, aa^{i_1})$, $(4)\cap(8)$: $ab^{i_2}b^{-1}, (b^{i_2}ab^{-1}, ab^{i_2-1})$, $(5)\cap(2)$: $a^{i_1}b^m, (ba^{i_1}b^{m-1}, a^{i_1}a^{i_1})$,
$(5)\cap(3)$: $a^{i_1}b^{j_2}, (ba^{i_1}b^{j_2-1}, a^{i_1-j_1}ba)$, $(5)\cap(4)$: $a^{i_1}b^{i_2}, (ba^{i_1}b^{i_2-1}, a^{i_1-1}b^{i_2}a)$, $(5)\cap(8)$: $a^{i_1}bb^{-1}, (ba^{i_1}b^{-1}, a^{i_1}b^{i_2}a)$,
$(6)\cap(7)$: $aa^{-1}a, (a, a)$, $(7)\cap(1)$: $a^{-1}a^n, (a^{n-1}, a^{-1}b^{i_2})$, $(7)\cap(3)$: $a^{-1}a^{j_1}b^{j_2}, (a^{j_1-1}b^{j_2}, a^{-1}ba)$,
$(7)\cap(5)$: $a^{-1}a^{i_1}b, (a^{i_1-1}b, a^{-1}ba^{i_1})$, $(7)\cap(6)$: $a^{-1}aa^{-1}, (a^{-1}, a^{-1})$, $(8)\cap(9)$: $bb^{-1}b, (b, b)$,
$(9)\cap(2)$: $b^{-1}b^m, (b^{m-1}, b^{-1}a^{i_1})$.

In fact, all these above critical pairs are resolved by reduction steps which some of them can be shown as follows.

$(1)\cap(3)$: $a^n b^{j_2}, (a^{n-j_1}ba, b^{i_2}b^{j_2})$,

$$a^n b^{j_2} \longrightarrow \begin{cases} a^{n-j_1}ba \to b^{i_2}a^{-j_1}ba \to a^{-j_1}ba \to ba \\ b^{i_2}b^{j_2} \to b^{j_2} \to a^{j_1}b^{j_2} \to ba \end{cases}$$

$(3)\cup(5)$: $a^{j_1}b^{j_2}, (a^{j_1-1}ba^{i_1}b^{j_2-1}, ba)$

$$a^{i_1}b^m \longrightarrow \begin{cases} ba \to a^{i_1}bab \to baba^{i_1} \\ a^{j_1-i_1}ba^{i_1}b^{j_2-1} \to a^{j_1}ba^{i_1}b^{j_2} \to a^{j_1}b^{j_2}ba^{i_1} \to baba^{i_1} \end{cases}$$

• For $m \le i_1 < n$, the following overlapping words and corresponding criticial pairs are obtained:

$(1)\cap(1)$: $a^{n+1}, (ab^{i_2}, b^{i_2}a)$, $(1)\cap(2)$: $a^n, (a^{n-i_1}b^m, b^{i_2})$, $(1)\cap(3)$: $a^n b^{j_2}, (a^{n-j_1}ba, b^{i_2}b^{j_2})$,
$(1)\cap(4)$: $a^n b^{i_2}, (a^{n-1}b^{i_2}a, b^{i_2}b^{i_2})$, $(1)\cap(5)$: $a^n b^m, (a^{n-1}b^m a, b^{i_2}b^m)$, $(1)\cap(6)$: $a^n a^{-1}, (b^{i_2}a, a^{n-1})$,
$(2)\cap(1)$: $a^n, (b^m a^{n-i_1}, b^{i_2})$, $(2)\cap(2)$: $a^{i_1+1}, (ab^m, b^m a)$, $(2)\cap(3)$: $a^{i_1}b^{j_2}, (a^{i_1-j_1}ba, b^m b^{j_2})$,
$(2)\cap(4)$: $a^{i_1}b^{i_2}, (a^{i_1-1}b^{i_2}a, b^m b^{i_2})$, $(2)\cap(5)$: $a^{i_1}b^m, (a^{i_1-1}b^m a, b^m b^m)$, $(2)\cap(6)$: $a^{i_1}a^{-1}, (b^m a, a^{i_1-1})$,
$(3)\cup(2)$: $a^{j_1}b^{j_2}, (b^m a^{j_1-i_1}b^{j_2}, ba)$, $(3)\cap(4)$: $a^{j_1}b^{i_2}, (a^{j_1-1}b^{i_2}a, bab^{i_2-j_2})$, $(3)\cup(4)$: $a^{j_1}b^{j_2}, (a^{j_1-1}b^{i_2}ab^{j_2-i_2}, ba)$,
$(3)\cap(5)$: $a^{j_1}b^m, (a^{j_1-1}b^m a, bab^{m-j_2})$, $(3)\cap(8)$: $a^{j_1}b^{j_2}b^{-1}, (bab^{-1}, a^{j_1}b^{j_2-1})$, $(4)\cap(8)$: $ab^{i_2}b^{-1}, (b^{i_2}ab^{-1}, ab^{i_2-1})$

and

$(5)\cup(4)$: $ab^m, (b^{i_2}ab^{m-i_2}, b^m a)$, $(5)\cap(8)$: $ab^m b^{-1}, (b^m ab^{-1}, ab^{m-1})$, $(6)\cap(7)$: $aa^{-1}a, (a, a)$,
$(7)\cap(1)$: $a^{-1}a^n, (a^{n-1}, a^{-1}b^{i_2})$, $(7)\cap(2)$: $a^{-1}a^{i_1}, (a^{i_1-1}, a^{-1}b^m)$, $(7)\cap(3)$: $a^{-1}a^{j_1}b^{j_2}, (a^{j_1-1}b^{j_2}, a^{-1}ba)$,
$(7)\cap(4)$: $a^{-1}a^n b, (a^{n-1}b, a^{-1}ba^n)$, $(7)\cap(5)$: $a^{-1}a^{i_1}b, (a^{i_1-1}b, a^{-1}ba^{i_1})$, $(7)\cap(6)$: $a^{-1}aa^{-1}, (a^{-1}, a^{-1})$,
$(8)\cap(9)$: $bb^{-1}b, (b, b)$, $(9)\cap(8)$: $b^{-1}bb^{-1}, (b^{-1}, b^{-1})$.

At this point we note that the overlapping words and corresponding critical pairs for $0 \le i_2 \le n < m$ in Case 2 are the same with overlapping words and critical pairs given for $0 \le i_1 < m < n$ in Case 1.

• Finally, let us check the conditions for $n < i_2 < m$ in Case 2.

$(1) \cap (1)$: $b^{i_2+1}, (a^n b, ba^n)$, $(1) \cap (2) : b^m, (a^n b^{m-i_2}, a^{i_1})$, $(1) \cap (8) : b^{i_2} b^{-1}, (a^n b^{-1}, b^{i_2-1})$,
$(2) \cap (1)$: $b^m, (b^{m-i_2} a^n, a^{i_1})$, $(2) \cap (2) : b^{m+1}, (a^{i_1} b, ba^{i_1})$, $(2) \cap (8) : b^m b^{-1}, (a^{i_1} b^{-1}, b^{m-1})$,
$(3) \cup (1)$: $a^{j_1} b^{j_2}, (a^{j_1} b^{j_2-i_2} a^n, ba)$, $(3) \cap (1) : a^{j_1} b^{i_2}, (bab^{i_2-j_2}, a^{j_1} a^n)$, $(3) \cap (2) : a^{j_1} b^m, (bab^{m-j_2}, a^{j_1} a^{i_1})$,
$(3) \cap (8)$: $a^{j_1} b^{j_2} b^{-1}, (bab^{-1}, a^{j_1} b^{j_2-1})$, $(3) \cup (5) : a^{j_1} b^{j_2}, (a^{j_1-i_1} ba^{i_1} b^{j_2-1}, ba)$, $(4) \cap (1) : a^n b^{i_2}, (ba^n b^{i_2-1}, a^n a^n)$,
$(4) \cap (2)$: $a^n b^m, (ba^n b^{m-1}, a^n a^{i_1})$, $(4) \cap (3) : a^n b^{j_2}, (ba^n b^{j_2-1}, a^{n-j_1} ba)$, $(4) \cup (5) : a^n b, (a^{n-i_1} ba^{i_1}, ba^n)$,
$(4) \cap (8)$: $a^n bb^{-1}, (ba^n b^{-1}, a^n)$, $(5) \cap (1) : a^{i_1} b^{i_2}, (ba^{i_1} b^{i_2-1}, a^{i_1} a^n)$, $(5) \cap (2) : a^{i_1} b^m, (ba^{i_1} b^{m-1}, a^{i_1} a^{i_1})$,

and

$(5) \cap (3)$: $a^{i_1} b^{j_2}, (ba^{i_1} b^{i_2-1}, a^{i_1-j_1} ba)$, $(5) \cap (8) : a^{i_1} bb^{-1}, (ba^{i_1} b^{-1}, a^{i_1})$, $(6) \cap (7) : aa^{-1} a, (a, a)$,
$(7) \cap (3)$: $a^{-1} a^{j_1} b^{j_2}, (a^{j_1-1} b^{j_2}, a^{-1} ba)$, $(7) \cap (4) : a^{-1} a^n b, (a^{n-1} b, a^{-1} ba^n)$, $(7) \cap (5) : a^{-1} a^{i_1} b, (a^{i_1-1} b, a^{-1} ba^{i_1})$,
$(7) \cap (6)$: $a^{-1} aa^{-1}, (a^{-1}, a^{-1})$, $(8) \cap (9) : bb^{-1} b, (b, b)$, $(9) \cap (1) : b^{-1} b^{i_2}, (b^{i_2-1}, b^{-1} a^n)$,
$(9) \cap (2)$: $b^{-1} b^m, (b^{m-1}, b^{-1} a^{i_1})$, $(9) \cap (8) : b^{-1} bb^{-1}, (b^{-1}, b^{-1})$.

After all these above processes, we see that all critical pairs can be resolved (as we applied for some couples after the case $0 \leq i_1 < m < n$). Hence the result. $\square$

As a first consequence of Theorem 3.1, we have the following result.

Corollary 3.2. *Let us consider the words w_1, w_2, w_3, $w_4 \in C_n \#_{\alpha,\alpha'}^{f,f'} C_m$. Thus, for the orderings $0 \leq i_1 < m < n$, $m \leq i_1 < n$, $0 \leq i_2 \leq n < m$ and $n < i_2 < m$, respectively, the normal forms of these words are given as*

- $C(w_1) = b^{k_1} a^{l_1} b^{k_2} a^{l_2} \cdots b^{k_s} a^{l_s}$, $\quad 0 \leq k_1 \leq m-1,\ 0 \leq l_\delta \leq i_1 - 1\ (1 \leq \delta \leq s),\ 0 \leq k_\epsilon \leq i_2 - 1\ (2 \leq \epsilon \leq s)$.
- $C(w_2) = b^{k_1} a^{l_1} b^{k_2} a^{l_2} \cdots b^{k_s} a^{l_s}$, $\quad 0 \leq l_\delta \leq j_1 - 1\ (1 \leq \delta \leq s),\ 0 \leq k_\epsilon \leq i_2 - 1\ (1 \leq \epsilon \leq s)$.
- $C(w_3) = b^{k_1} a^{l_1} b^{k_2} a^{l_2} \cdots b^{k_s} a^{l_s}$, $\quad 0 \leq k_1 \leq m-1,\ 0 \leq l_\delta \leq i_1 - 1\ (1 \leq \delta \leq s),\ 0 \leq k_\epsilon \leq i_2 - 1\ (2 \leq \epsilon \leq s)$.
- $C(w_4) = b^{k_1} a^{l_1} b^{k_2} a^{l_2} \cdots b^{k_s} a^{l_s}$, $\quad 0 \leq l_\delta \leq i_1 - 1\ (1 \leq \delta \leq s-1),\ 0 \leq k_\epsilon \leq i_2 - 1\ (1 \leq \epsilon \leq s),\ l_s \in \mathbb{Z}$.

By Theorem 3.1 and Corollary 3.2, we have the following result.

Corollary 3.3. *Let us consider the product $C_n \#_{\alpha,\alpha'}^{f,f'} C_m$ with a monoid presentation as in* (11). *Then the word problem for it is solvable.*

Conjecture 3.4. *For a future work, one may obtain the general presentation for the two-sided crossed product of arbitrary two groups, and then get the complete rewriting system in the meaning of its monoid presentation. Therefore, the general version of Corollary* 3.3 *is obtained.*

Acknowledgement. The authors would like to thank to the referees for their kind suggestions and valuable comments.

References

[1] A. L. Agore, G. Militaru, Crossed product of groups, applications, Arabian J. Sci. Eng. 33 (2008) 1-17.
[2] A. L. Agore, D. Fratila, Crossed product of cyclic groups, Czechoslovak Mathematical Journal 60 (2010) 889-901.
[3] A. M. Alghamdi, F. G. Russo, Remarks on the Relative Tensor Degree of Finite Groups, Filomat 28-9 (2014) 1929-1933.
[4] F. Ates, A. S. Cevik, Knit products of some groups and their applications, Randiconti del Seminario Mathematics delta Universita di Padova 2 (2009) 1-12.
[5] R. V. Book, F. Otto, String-Rewriting Systems, Springer-Verlag, New York, 1993.
[6] M. G. Brin, On the Zappa-Szep product, Comm. Algebra 33 (2005) 393-424.
[7] A. Emin, F. Ates, S. Ikikardes, I. N. Cangul, A new monoid construction under crossed products, Journal of Inequalities and Applications 244 (2013).
[8] S.-E. Han, Non-Ultra Regular Digital Covering Spaces with Nontrivial Automorphism Groups, Filomat 27-7 (2013) 1205-1218.
[9] P. W. Michor, Knit products of graded Lie algebras and groups, Suppl. Rendiconti Circolo Matematico di Palermo Ser. II, 22 (1989) 171-175.
[10] M. A. Rudkovskii, Twisted product of Lie groups, Siberian Math. Journal, 38 (1997) 1120-1129.
[11] C. C. Sims, Computation for Finitely Presented Groups, Cambridge University Press, 1994.