Filomat 30:5 (2016), 1363–1373
DOI 10.2298/FIL1605363A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On The Fine Spectrum of Generalized Upper Triangular Triple-Band Matrices $\left(\Delta^2_{uvw}\right)^t$ Over the Sequence Space l_1

Selma Altundağ[a,b], Merve Abay[a]

[a] *Sakarya University, Department of Mathematics, 54187, Sakarya, Turkey*
[b] *Faculty of Engineering and Natural Sciences, International University of Sarajevo, Sarajevo, Bosnia and Herzegovina*

Abstract. In this work, we determine the fine spectrum of the matrix operator $(\Delta^2_{uvw})^t$ which is defined generalized upper triangular triple band matrix on l_1 . Also, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $(\Delta^2_{uvw})^t$ on l_1.

1. Introduction

In functional analysis, the spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, which are point spectrum, the continuous spectrum and residual spectrum. The calculation of these three parts of the spectrum of an operator is called calculating the fine spectrum of the operator.

Several authors studied the spectrum and fine spectrum of linear operators defined by some triangle matrices over some sequence spaces. We introduce knowledge in the existing literature concerning the spectrum and the fine spectrum. The fine spectrum of the Cesàro operator on the sequence space l_p for $(1 < p < \infty)$ was studied by Gonzalez [8]. Reade [15] studied the spectrum of the Cesàro operator over the sequence space c_0 . The fine spectrum of the difference operator Δ over the sequence spaces c_o and c has been studied by Altay and Başar [1]. The same authors have studied the fine spectrum of the generalized difference operator $B(r,s)$ over c_o and c, in [2]. The fine spectra of Δ over l_1 and bv have been studied by Kayaduman and Furkan [12]. The fine spectrum of generalized difference operator $B(r,s)$ over the sequence spaces l_1 and bv has been studied by Furkan, Bilgiç and Kayaduman [5]. Recently, the fine spectrum of $B(r,s,t)$ over the sequence spaces c_o and c has been studied by Furkan et al. [6]. Vatan Karakaya and Muhammed Altun have studied $U(r,s)$ which is upper triangular double-band matrices over the sequences c_o and c [11]. In 2012, Srivastava and Kumar have studied fine spectrum of generalized difference operator Δ_v on l_1 [16]. Fine spectrum of the generalized difference operator Δ_{uv} on sequence space l_1 has been studied by Srivastava and Kumar [17]. Ali Karaisa has studied fine spectrum of upper triangular double-band matrices over sequence space l_p, $(1 < p < \infty)$ [9]. Fathi and Lashkaripour have studied on the fine

2010 *Mathematics Subject Classification*. 47A10, 47B37, 47A39
Keywords. Spectrum of an operator, fine spectrum, Goldberg's classification, approximate point spectrum, defect spectrum,compression spectrum
Received: 15 April 2014; Accepted: 17 September 2014
Communicated by Eberhard Malkowsky
The corresponding author: Selma Altundağ
Email addresses: scaylan@sakarya.edu.tr (Selma Altundağ), abaymerve@hotmail.com.tr (Merve Abay)

spectrum of generalized upper double-band matrices Δ^{uv} which is transpose of the Δ_{uv} over the sequence space on l_1 [4]. Quite recently, Karaisa studied fine spectra of upper triangular triple-band matrices over the sequence space l_p, $(0 < p < \infty)$ [10]. Panigrahi and Srivastava have studied spectrum and fine spectrum of generalized second order forward difference operator Δ^2_{uvw} on sequence space l_1 [14].

In this paper, we study the fine spectrum of the transpose of matrix operator Δ^2_{uvw} on the sequence space l_1. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $\left(\Delta^2_{uvw}\right)^t$ on l_1.

2. Preliminaries and Notations

Let X and Y be the Banach spaces and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T , i.e.,

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\varphi)(x) = \varphi(Tx)$ for all $\varphi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subset X$. By T , associate the operator

$$T_\alpha = T - \alpha I,$$

where α is a complex number and I is the identity operator on $D(T)$. If T_α has an inverse, which is linear, we denote it by ${T_\alpha}^{-1}$, that is

$${T_\alpha}^{-1} = (T - \alpha I)^{-1},$$

and it is called to be the resolvent operator of T. The name is appropriate, since ${T_\alpha}^{-1}$ helps to solve the equation $T_\alpha x = y$. Thus, $x = {T_\alpha}^{-1} y$ provided ${T_\alpha}^{-1}$ exist. More important, the investigation of properties of ${T_\alpha}^{-1}$ will be basic for an understanding of the operator T itself. Naturally, many properties of T_α and ${T_\alpha}^{-1}$ depend on α, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all α in the complex plane such that ${T_\alpha}^{-1}$ exist. Boundedness of ${T_\alpha}^{-1}$ is another property that will be essential. We shall also ask for what α 's the domain of ${T_\alpha}^{-1}$ is dense in X , to name just a few aspects. For our investigaton of T , T_α and ${T_\alpha}^{-1}$, we need some basic concepts in spectral theory which are given as follows (see[10 pp. 370-371]).

Definition 2.1. *Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ also be a linear operator with domain $D(T) \subset X$. A reguler value of α of T is a complex number such that*
$(R1)$ ${T_\alpha}^{-1}$ *exist*
$(R2)$ ${T_\alpha}^{-1}$ *is bounded*
$(R3)$ ${T_\alpha}^{-1}$ *is defined on a set which is dense in X .*

The resolvent set $\rho(T)$ of T is the set of all reguler values α of T . Its complement $\sigma(T) = \mathbb{C}\backslash\rho(T)$ in the complex plane $\mathbb{C}$ is called the spectrum of T. Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows:
The point spectrum $\sigma_p(T)$ is the set such that ${T_\alpha}^{-1}$ does not exist. A $\alpha \in \sigma_p(T)$ is called an eigenvalue of T .
The continuous spectrum $\sigma_c(T)$ is the set such that ${T_\alpha}^{-1}$ exist and satisfies $(R3)$ but not $(R2)$.

The residual spectrum $\sigma_r(T)$ is the set such that ${T_\alpha}^{-1}$ exist (and may be bounded or not) but not satisfy ($R3$). Therefore, these three subspectras from disjoint subdivisions

$$\sigma(T, X) = \sigma_p(T, X) \cup \sigma_c(T, X) \cup \sigma_r(T, X). \tag{1}$$

In this section, following Appell et al. [3], we call the three more subdivisions of the spectrum called the approximate point spectrum, defect spectrum, and compression spectrum.
Given a bounded linear operator T in a Banach space X , we call a sequence (x_k) in X as a Wely sequence for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.
In what follows, we call the set

$$\sigma_{ap}(T, X) := \{\alpha \in \mathbb{C} : \text{there exists a Wely sequence for } \alpha I - T\} \tag{2}$$

the approximate point spectrum of T . Moreover, the subspectrum

$$\sigma_\delta(T, X) := \{\alpha \in \mathbb{C} : \alpha I - T \text{ is not surjective}\} \tag{3}$$

is called defect spectrum of T .
The two subspectra given by (2) and (3) from a (not necessarily disjoint) subdivisions

$$\sigma(T, X) = \sigma_{\mathrm{ap}}(T, X) \cup \sigma_\delta(T, X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{\mathrm{co}}(T, X) := \left\{\alpha \in : \overline{R(\alpha I - T)} \neq X\right\}$$

which is often called compression spectrum in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T, X) = \sigma_{\mathrm{ap}}(T, X) \cup \sigma_{\mathrm{co}}(T, X)$$

of spectrum. Clearly, $\sigma_{\mathrm{p}}(T, X) \subseteq \sigma_{\mathrm{ap}}(T, X)$ and $\sigma_{\mathrm{co}}(T, X) \subseteq \sigma_\delta(T, X)$. Moreover, comparing these subspectra with those in (1) we note that

$$\begin{aligned}\sigma_{\mathrm{r}}(T, X) &= \sigma_{\mathrm{co}}(T, X) / \sigma_{\mathrm{p}}(T, X)\\ \sigma_c(T, X) &= \sigma(T, X) / \left[\sigma_{\mathrm{p}}(T, X) \cup \sigma_{\mathrm{co}}(T, X)\right].\end{aligned}$$

Proposition 2.2. *Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by following relations:*
(a)$\sigma(T^*, X^*) = \sigma(T, X)$
(b)$\sigma_{\mathrm{c}}(T^*, X^*) \subseteq \sigma_{\mathrm{ap}}(T, X)$
(c)$\sigma_{\mathrm{ap}}(T^*, X^*) = \sigma_\delta(T, X)$
(d)$\sigma_\delta(T^*, X^*) = \sigma_{\mathrm{ap}}(T, X)$
(e)$\sigma_p(T^*, X^*) = \sigma_{\mathrm{co}}(T, X)$
(f)$\sigma_{\mathrm{co}}(T^*, X^*) \supseteq \sigma_{\mathrm{p}}(T, X)$
(g)$\sigma(T, X) = \sigma_{\mathrm{ap}}(T, X) \cup \sigma_{\mathrm{p}}(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{\mathrm{ap}}(T^*, X^*)$.

Relation (c)-(f) show that the approximate point spectrum is in a certain sense dual to the defect spectrum, and the point spectrum dual to compression spectrum.

The equality (g) implies, in particular, that $\sigma(T,X) = \sigma_{\mathrm{ap}}(T,X)$ if T is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operator on Hilbert space are most similar to matrices in finite dimensional spaces (see [3]).

From Goldberg [7], If $T \in B(X)$, X a Banach space, then there are three possibilities for $R(T)$, the range of T
(A)$R(T) = X$,
(B)$R(T) \neq \overline{R(T)} = X$,
(C)$\overline{R(T)} \neq X$,
and
(1)T^{-1} exists an is continuous,
(2) T^{-1} exists but is discontinuous,
(3)T^{-1} does not exist.

If these possibilities are combined in all ways, nine different states are created. These are labelled by: A_1, A_2 , A_3 , B_1 , B_2 ,B_3 ,C_1 ,C_2 ,C_3 . If an operator is in state C_2 , for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see [7]).

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} , where $n,k \in \mathbb{N} = \{1,2,3,\ldots\}$. Then, we say that A defines a matrix mapping from X into Y , and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}_{n\in\mathbb{N}}$, the A-transform of x , is in Y , where

$$(Ax)_n = \sum_k a_{nk}x_k \;\; (n \in \mathbb{N}) \tag{4}$$

By $(X : Y)$, we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X : Y)$ if and only if the series on the right side of (4) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in Y$ for all $x \in X$.

Lemma 2.3. *The adjoint operator T^* of T is onto if and only if T has a bounded inverse.*

Lemma 2.4. *T has a dense range if and only if T^* is one to one.*

Lemma 2.5. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_1)$ from l_1 to itself if and only if the supremum of l_1 norms of the columns of A is bounded.*

Corollary 2.6. $\sigma_r(T,X) \subseteq \sigma_p(T^*,X^*) \subseteq \sigma_r(T,X) \cup \sigma_p(T,X)$.

3. Main Results

In this section, we prove that operator $\left(\Delta^2_{uvw}\right)^t : l_1 \to l_1$ is bounded linear operator and we compute spectrum, point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the operator $\left(\Delta^2_{uvw}\right)^t$ over space l_1 .
Let $u = (u_k)$ is either a constant sequence or sequence of distinct positive real numbers with $U = \lim_{k\to\infty} u_k$ so that $u_k \neq 0$ for each $k \in \mathbb{N}_0$, $v = (v_k)$ is a sequence of positive real numbers such that $v_k \neq 0$ for each $k \in \mathbb{N}_0$ with $V = \lim_{k\to\infty} v_k$ and $w = (w_k)$ is a sequence of positive real numbers such that $w_k \neq 0$ for each $k \in {}_0$ with

$W = \lim_{k\to\infty} w_k$ and $\sup_k u_k < U + V$. We define the operator $\left(\Delta^2_{uvw}\right)^t$ on sequence space l_1 as

$\left(\Delta^2_{uvw}\right)^t x = (u_k x_k + v_k x_{k+1} + w_k x_{k+2})_{k=0}^{\infty}$ with $x_{-1} = 0$, $x_{-2} = 0$ where $x = (x_n) \in l_1$.

It is easy to verify that the operator $\left(\Delta^2_{uvw}\right)^t$ can be represented by the matrix

$$\left(\Delta^2_{uvw}\right)^t = \begin{bmatrix} u_0 & v_0 & w_0 & 0 & 0 & \cdots \\ 0 & u_1 & v_1 & w_1 & 0 & \cdots \\ 0 & 0 & u_2 & v_2 & w_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

If we take $u = (r)$, $v = (s)$, $w = (t)$, then the operator $\left(\Delta^2_{uvw}\right)^t$ reduces to $A(r,s,t)$ which is determined in [10]. Thus, the results of this paper is generalized condition results of many operator whose matrix respresentation is a upper trianguler triple-band matrix.

In this work, if z is a complex number than by $\sqrt{z}$ we always mean the square root of z with non-negative real part. If $\mathrm{Re}\left(\sqrt{z}\right) = 0$ then $\sqrt{z}$ represents the square root of z with $\mathrm{Im}\left(\sqrt{z}\right) \geq 0$. The same results are obtained if $\sqrt{z}$ represents the other square root.

Theorem 3.1. *The operator* $\left(\Delta^2_{uvw}\right)^t : l_1 \to l_1$ *is a bounded linear operator and*

$$\left\|\left(\Delta^2_{uvw}\right)^t\right\|_{(l_1,l_1)} = \sup_k \left(|w_k| + |v_{k+1}| + |u_{k+2}|\right).$$

Proof. Proof is simple. Hence we omit. □

Theorem 3.2. $\sigma_p\left(\left(\left(\Delta^2_{uvw}\right)^t\right)^*, l_1^*\right) = \emptyset.$

Proof. Let $\left(\left(\Delta^2_{uvw}\right)^t\right)^* f = \alpha f$ for $\theta \neq f \in l_\infty$. Then, by solving system of linear equation

$$\begin{aligned} & u_0 f_0 = \alpha f_0 \\ & v_0 f_0 + u_1 f_1 = \alpha f_1 \\ & w_0 f_0 + v_1 f_1 + u_2 f_2 = \alpha f_2 \\ & w_1 f_1 + v_2 f_2 + u_3 f_3 = \alpha f_3 \\ & \vdots \\ & w_{k-2} f_{k-2} + v_{k-1} f_{k-1} + u_k f_k = \alpha f_k. \\ & \vdots \end{aligned} \tag{5}$$

Part 1. Suppose (u_k) is a constant sequence, say $u_k = U$ for each $k \in \mathbb{N}_0$. We consider that (5). Let f_m be the first non-zero entry of the sequence (f_n) . So we get $f_m = 0$, which is a contradiction to our assumption. For this reason,

$$\sigma_p\left(\left(\left(\Delta^2_{uvw}\right)^t\right)^*, l_1^*\right) = \emptyset.$$

Part 2.Assume that (u_k) is a sequence of distinct positive real numbers. Consider $\left(\left(\Delta^2_{uvw}\right)^t\right)^* f = \alpha f$, for $\theta \neq f \in l_\infty$, which gives (5) system of equations.
For all $\alpha \notin \{u_0, u_1, u_2, \ldots\}$, we have $f_k = 0$ for all $k \in \mathbb{N}_0$, which is a contradiction.
Assume that $\alpha = u_m$ for some m . Then $f_0 = f_1 = \ldots = f_{m-1} = 0$.

$w_{m-1}f_{m-1} + v_m f_m + (u_{m+1} - \alpha)\, f_{m+1} = 0$ If $f_m = 0$, then $f_k = 0$ for all $k \in \mathbb{N}$, which is a contradiction.
$\vdots$
If $f_m \neq 0$, then

$$f_{k+1} = \frac{-v_k}{u_{k+1} - u_m} f_k, \textit{for all } k \geq m,$$

and so

$$\lim_{k\to\infty} \left| \frac{f_{k+1}}{f_k} \right| = \left| \frac{V}{u_m - U} \right| > 1 \textit{for all } k \geq m.$$

Because $u_m < V + U$. So,$f \notin {l_1}^*$. Consequently

$$\sigma_p \left(\left(\left(\Delta_{uvw}^2 \right)^t \right)^*, l_1^* \right) = \emptyset.$$

□

Theorem 3.3. $\sigma_r \left(\left(\left(\Delta_{uvw}^2 \right)^t \right)^*, {l_1}^* \right) = \left\{ \alpha \in \mathbb{C} : \left| \frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}} \right| < 1 \right\} = S_1$, *where* $\left(\left(\Delta_{uvw}^2 \right)^t \right)^* = \Delta_{uvw}^2$.

Proof. $\Delta_{uvw}^2 - \alpha I$ is one to one, by Theorem 3.2.
Suppose $\left(\Delta_{uvw}^2 \right)^* y = \alpha y$, for $\theta \neq y \in l_1$. This gives

$$\begin{aligned} &u_o y_0 + v_o y_1 + w_0 y_2 = \alpha y_0 \\ &u_1 y_1 + v_1 y_2 + w_1 y_3 = \alpha y_1 \\ &\vdots \end{aligned} \tag{6}$$

If $y_0 = y_1 = 0$, then $y_k = 0$ for all $k \in \mathbb{N}_0$. So, $y_0 \neq 0,\ y_1 \neq 0$ and solving the system of linear equations (6) in terms of y_0 and y_1 , we get

$y_k = (b_{k-1,0}y_1 - b_{k-1,1}y_0)\, \frac{(u_0-\alpha)(u_1-\alpha)(u_2-\alpha)\cdots(u_{k-1}-\alpha)}{w_0 w_1 \cdots w_{k-2}}$,

where $b_{k-1,0}$ and $b_{k-1,1}$ are defined as in [14].

Let $y_0 = 1$ and $y_1 = \frac{1}{r_1}$.

$\lim\limits_{k\to\infty} \left| \frac{y_{k+1}}{y_k} \right| = \lim\limits_{k\to\infty} \left| \frac{u_k-\alpha}{w_{k-1}} \right| \left| \frac{b_{k,0}y_1 - b_{k,1}y_0}{b_{k-1,0}y_1 - b_{k-1,1}y_0} \right| = \frac{1}{|r_1|} < 1$

provided $\left| \frac{-V+\sqrt{V^2-4W(U-\alpha)}}{2(U-\alpha)} \right| = |r_1| > 1$.

So, if $|r_1| > 1$, then $y = (y_k) \in l_1$, which shows that $\left(\Delta_{uvw}^2 \right)^* - \alpha I$ is not one to one. Lemma 2.4 gives that $\Delta_{uvw}^2 - \alpha I$ has not dense range. □

Theorem 3.4. $\sigma_p \left(\left(\Delta_{uvw}^2 \right)^t, l_1 \right) = \left\{ \alpha \in \mathbb{C} : \left| \frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}} \right| < 1 \right\} = S_1.$

Proof. This proof is elementary by Corollary 2.6. □

Remark 3.5. $\sigma_p \left(\left(\Delta_{uvw}^2 \right)^t, l_1 \right) = \left\{ \alpha \in \mathbb{C} : \left| \frac{2(U-\alpha)}{-V-\sqrt{V^2-4W(U-\alpha)}} \right| < 1 \right\} = S_2.$

Proof. This proof is made similarly to Theorem 3.4. □

Theorem 3.6. $\sigma_r\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \emptyset$.

Proof. $\left(\left(\Delta^2_{uvw}\right)^t\right)^* - \alpha I$ is one to one for all α , by Theorem 3.2. Hence $\left(\Delta^2_{uvw}\right)^t - \alpha I$ is a dense range for all α , by Lemma 2.4. Accordingly $\sigma_r\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \emptyset$. □

Theorem 3.7. *Assume* $\sqrt{V^2} = -V$ *and define set* S_3 *by* $\left\{\alpha \in \mathbb{C} : \left|\frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}}\right| \leq 1\right\} = S_3$. *Then* $\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = S_3$.

Proof. Let $\alpha \notin S_3$ and $y = (y_k) \in l_\infty$. Then, by solving the equation $\left[\left(\left(\Delta^2_{uvw}\right)^t\right)^* - \alpha I\right] x = y$. We obtain

$$
\begin{aligned}
&(u_0 - \alpha)\, x_0 = y_0 \\
&v_0 x_0 + (u_1 - \alpha)\, x_1 = y_1 \\
&w_0 x_0 + v_1 x_1 + (u_2 - \alpha)\, x_2 = y_2 \\
&\vdots \\
&w_k x_k + v_{k+1} x_{k+1} + (u_{k+2} - \alpha)\, x_{k+2} = y_{k+2} \\
&\vdots
\end{aligned}
$$

and in this way we can get,

$$x_0 = \frac{y_0}{u_0-\alpha} \, ,$$

$$x_1 = \frac{1}{u_1-\alpha} y_1 + \frac{-v_0}{(u_1-\alpha)(u_0-\alpha)} y_0 \, ,$$

$$x_2 = \frac{1}{u_2-\alpha} y_2 + \frac{-v_1}{(u_1-\alpha)(u_2-\alpha)} y_1 + \left(\frac{v_0 v_1}{(u_0-\alpha)(u_1-\alpha)(u_2-\alpha)} - \frac{w_0}{(u_2-\alpha)(u_0-\alpha)}\right) y_0$$

$\vdots$

$\alpha \neq u_k$ for all $k \in \mathbb{N}_0$ and $\alpha \neq U$, by $\alpha \notin S_3$. Thus $\left(\left(\left(\Delta^2_{uvw}\right)^t\right)^* - \alpha I\right)^{-1} = (b_{nk})$ exist and

$$
b_{nk} = \begin{bmatrix}
\frac{1}{u_0-\alpha} & 0 & 0 & \cdots \\
\frac{-v_0}{(u_0-\alpha)(u_1-\alpha)} & \frac{1}{u_1-\alpha} & 0 & \cdots \\
\frac{v_0 v_1}{(u_0-\alpha)(u_1-\alpha)(u_2-\alpha)} - \frac{w_0}{(u_0-\alpha)(u_2-\alpha)} & \frac{-v_1}{(u_1-\alpha)(u_2-\alpha)} & \frac{1}{u_2-\alpha} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

Let $b_{k,k} = \frac{1}{u_k-\alpha}, b_{k+1,k} = \frac{-v_k}{(u_k-\alpha)(u_{k+1}-\alpha)}, \ldots$ for all $k \in \mathbb{N}_0$. We can see

$$x_n = b_{n,0} y_n + b_{n,1} y_{n-1} + \cdots + b_{n,n} y_0 = \sum_{k=0}^{n} b_{n,n-k} y_k.$$

We can observe,

$$\lim_{k\to\infty} \frac{1}{u_k-\alpha} = \frac{1}{U-\alpha} = a_1, \, , \lim_{x\to\infty} \frac{-v_k}{(u_k-\alpha)(u_{k+1}-\alpha)} = \frac{-V}{(U-\alpha)^2} = a_2 \, , \ldots \, .$$

Clearly, $a_n = \frac{(r_1)^n-(r_2)^n}{\sqrt{V^2-4W(U-\alpha)}}$ for $n = 1, 2, 3, \ldots$ where $r_1 = \frac{-V+\sqrt{V^2-4W(U-\alpha)}}{2(U-\alpha)}$ and $r_2 = \frac{-V-\sqrt{V^2-4W(U-\alpha)}}{2(U-\alpha)}$.

We may suppose that $V^2 \neq 4W(U - \alpha)$. Since $\alpha \notin S_3$, $|r_1| < 1$ and thus we have

$$\left|1 + \sqrt{1 - \frac{4W(U-\alpha)}{V^2}}\right| < \left|\frac{2(U-\alpha)}{-V}\right|.$$

Since $\left|1-\sqrt{z}\right| \leq \left|1+\sqrt{z}\right|$ for any $z \in \mathbb{C}$, we must have

$\left|1-\sqrt{1-\frac{4W(U-\alpha)}{V^2}}\right| < \left|\frac{2(U-\alpha)}{-V}\right|$

which leads us to the fact that $|r_2| < 1$ and $|r_2| < |r_1|$.
We can see that

$$|x_n| \leq \sum_{k=0}^{n} \left|b_{n,n-k}\right| \left|y_k\right| \tag{7}$$

for all $n \in \mathbb{N}_0$. Taking limit on the inequality (7) as $n \to \infty$, we get

$\|x\|_\infty \leq \left\|y\right\|_\infty \sum\limits_{k=0}^{\infty} \left|b_{n,n-k}\right|$.

We can see that

$\lim\limits_{k\to\infty} \left|\frac{b_{2k+1,k}}{b_{2k,k}}\right| = \lim\limits_{x\to\infty} \left|\frac{a_{k+2}}{a_{k+1}}\right| = \lim\limits_{x\to\infty} \left|\frac{(r_1)^{k+2}-(r_2)^{k+2}}{(r_1)^{k+1}-(r_2)^{k+1}}\right| = |r_1| < 1.$

This shows that $\left(\left(\left(\Delta^2_{uvw}\right)^t\right)^* - \alpha I\right)$ is onto for $|r_1| < 1$ and $\left(\left(\Delta^2_{uvw}\right)^t - \alpha I\right)$ has a bounded inverse by Lemma 2.4.

If $V^2 = 4W(U-\alpha)$, then $a_n = \left(\frac{2n}{-V}\right)\left(\frac{-V}{2(U-\alpha)}\right)^n$, for all $n \geq 1$. Similarly
$|x_n| \leq \sum\limits_{k=0}^{n} \left|b_{n,n-k}\right| \left|y_k\right|$ for all $n \in \mathbb{N}_0$ and taking limit on the inequality (7) as $n \to \infty$, we get

$\|x\|_\infty \leq \left\|y\right\|_\infty \sum\limits_{k=0}^{\infty} \left|b_{n,n-k}\right|$

and

$\lim\limits_{k\to\infty} \left|\frac{b_{2k+1,k}}{b_{2k,k}}\right| = \lim\limits_{x\to\infty} \left|\frac{a_{k+2}}{a_{k+1}}\right| = \left|\frac{-V}{2(U-\alpha)}\right| < 1.$

This means that $\left(\left(\left(\Delta^2_{uvw}\right)^t\right)^* - \alpha I\right)$ is onto for $|r_1| < 1$ and $\left(\left(\Delta^2_{uvw}\right)^t - \alpha I\right)$ has a bounded inverse by Lemma 2.4.
Thus $\alpha \notin \sigma_c\left(\left(\Delta^2_{uvw}\right)^t, l_1\right)$. In that case

$$\sigma_c\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) \subseteq \sigma\left\{\alpha \in \mathbb{C} : \left|\frac{2\,(U-\alpha)}{-V+\sqrt{V^2-4W\,(U-\alpha)}}\right| \leq 1\right\} = S_3. \tag{8}$$

By Theorem 3.4, we get

$$\sigma_p\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \left\{\alpha \in \mathbb{C} : \left|\frac{2\,(U-\alpha)}{-V+\sqrt{V^2-4W\,(U-\alpha)}}\right| < 1\right\} \subseteq \sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right). \tag{9}$$

Since the spectrum of any bounded operator is closed, we have

$$\left\{\alpha \in \mathbb{C} : \left|\frac{2\,(U-\alpha)}{-V+\sqrt{V^2-4W\,(U-\alpha)}}\right| \leq 1\right\} \subseteq \sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right). \tag{10}$$

and from Theorem 3.4, 3.6 and (3.8),

$$\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) \subseteq \left\{\alpha \in \mathbb{C} : \left|\frac{2\,(U-\alpha)}{-V + \sqrt{V^2 - 4W\,(U-\alpha)}}\right| \leq 1\right\}. \tag{11}$$

Combining (10) and (11), this completes the proof. □

Remark 3.8. *If* $\sqrt{V^2} = V$, *then* $\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \left\{\alpha \in \mathbb{C} : \left|\frac{2(U-\alpha)}{-V - \sqrt{V^2-4W(U-\alpha)}}\right| \leq 1\right\}.$

Theorem 3.9. $\sigma_c\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \left\{\alpha \in \mathbb{C} : \left|\frac{2(U-\alpha)}{-V + \sqrt{V^2-4W(U-\alpha)}}\right| = 1\right\}.$

Proof. This is clear by Theorem 3.4, 3.6, 3.7. □

Theorem 3.10. *Assume that* $\sqrt{V^2} = -V$. *If* $|2\,(U-\alpha)| < \left|-V - \sqrt{V^2 - 4W\,(U-\alpha)}\right|$, *then* $\alpha \in A_3\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right)$.

Proof. By Remark 3.5, $\left(\left(\Delta^2_{uvw}\right)^t - \alpha I\right)^{-1}$ dosen't exist. Let $y = (y_0, y_1, ...) \in l_1$. Solving the linear equation

$$\left(\left(\Delta^2_{uvw}\right)^t - \alpha I\right)x = y,$$

$$\begin{aligned}
&(u_0-\alpha)\,x_0 + v_o x_1 + w_0 x_2 = y_0\\
&(u_1-\alpha)\,x_1 + v_1 x_2 + w_1 x_3 = y_1\\
&(u_2-\alpha)\,x_2 + v_2 x_3 + w_2 x_4 = y_2\\
&\vdots\\
&(u_k-\alpha)\,x_k + v_k x_{k+1} + w_k x_{k+2} = y_k\\
&\vdots
\end{aligned}$$

Let $x_0 = 0$ and $x_1 = 0$. So

$x_2 = \frac{1}{w_0}y_0$, $x_3 = \frac{1}{w_1}y_1 + \frac{-v_1}{w_o w_1}y_0$, $x_4 = \frac{1}{w_2}y_2 + \frac{-v_2}{w_1 w_2}y_1 + \left(\frac{v_2 v_1}{w_0 w_1 w_2} - \frac{(u_2-\alpha)}{w_0 w_2}\right)y_0, \ldots$.

Let,$c_{k,k+2} = \frac{1}{w_k}$, $c_{k,k+3} = \frac{-v_{k+1}}{w_k w_{k+1}}$, $c_{k,k+4} = \frac{v_{k+2}v_{k+1}}{w_k w_{k+1} w_{k+2}} - \frac{(u_{k+2}-\alpha)}{w_k w_{k+2}}, \ldots$.

Hence, we say that

$x_k = c_{0,k}y_0 + c_{1,k}y_1 + \cdots + c_{k-2,k}y_{k-2} = \sum\limits_{n=0}^{k-2} c_{n,k}y_n$. Then,

$\sum\limits_k |x_k| \leq \sup\limits_k(R_k)\sum\limits_k \left|y_k\right|$, where

$R_k = \frac{1}{|w_k|} + \left|\frac{-v_{k+1}}{w_k w_{k+1}}\right| + \left|\frac{v_{k+2}v_{k+1}}{w_k w_{k+1} w_{k+2}} - \frac{u_{k+2}-\alpha}{w_{k+2}w_k}\right| + \cdots$ for all $k \in \mathbb{N}_0$.

Now by letting

$r_1 = \frac{-V+\sqrt{V^2-4W(U-\alpha)}}{2(U-\alpha)}$ and $r_1 = \frac{-V-\sqrt{V^2-4W(U-\alpha)}}{2(U-\alpha)}$,

we can observe,

$\lim\limits_{k\to\infty} c_{k,k+2} = \frac{1}{W} = t_1 = \frac{1}{\sqrt{V^2-4W(U-\alpha)}}(-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$

$\lim\limits_{k\to\infty} c_{k,k+3} = \frac{-V}{W^2} = t_2 = \frac{1}{\sqrt{V^2-4W(U-\alpha)}}(-1)^2\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right),$

$\lim\limits_{k\to\infty} c_{k,k+4} = \frac{V^2}{W^3} = t_3 = \frac{1}{\sqrt{V^2-4W(U-\alpha)}}(-1)^3\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right),$

$\lim\limits_{k\to\infty} c_{k,k+5} = \frac{-V^3}{W^4} = t_4 = \frac{1}{\sqrt{V^2-4W(U-\alpha)}}(-1)^4\left(\frac{1}{r_1^4} - \frac{1}{r_2^4}\right),$

$\vdots$

where $t_n = \frac{1}{\sqrt{V^2-4W(U-\alpha)}}(-1)^n\left(\frac{1}{r_1^n} - \frac{1}{r_2^n}\right) n = 1,2,3,\ldots.$

Since $|r_2| > 1$, we have $|r_1| > 1$.

Let $V^2 \neq 4W(U-\alpha)$.

Since $\lim\limits_{k\to\infty}\left|\frac{c_{k,2k+2}}{c_{k,2k+1}}\right| = \lim\limits_{k\to\infty}\left|\frac{t_{k+1}}{t_k}\right| = \left|\frac{1}{r_2}\right| < 1$, then R_k is convergent for all $k \in \mathbb{N}$.

That is $\lim\limits_{k\to\infty} R_k = \sum\limits_{n=1}^{\infty}|t_n| \leq \frac{1}{\left|\sqrt{V^2-4W(U-\alpha)}\right|}\left(\sum\limits_{n=1}^{\infty}\left|\frac{1}{r_1}\right|^n + \sum\limits_{n=1}^{\infty}\left|\frac{1}{r_2}\right|^n\right) < \infty.$

(R_k) is a convergent sequence of positive real numbers and $\lim\limits_{k\to\infty} R_k < \infty$, hence $\sup\limits_k R_k < \infty$.

This shows $x = (x_k) \in l_1$.

If $V^2 = 4W(U-\alpha)$, then $t_n = \frac{1}{-W}n\left(\frac{2(U-\alpha)}{-V}\right)^{n-1}(-1)^n$. Consequently,

$\lim\limits_{k\to\infty}\left|\frac{c_{k,2k+2}}{c_{k,2k+1}}\right| = \lim\limits_{k\to\infty}\left|\frac{t_{k+1}}{t_k}\right| = \frac{2|U-\alpha|}{|-V|} < 1$. So, R_k is convergent for all $k \in \mathbb{N}$ and

$\lim\limits_{k\to\infty} R_k = \sum\limits_{n=1}^{\infty}|t_n| = \sum\limits_{n=1}^{\infty}\left|\frac{n}{W}\right|\left|\frac{2(U-\alpha)}{-V}\right|^{n-1} < \infty.$

(R_k) is a convergent sequence of positive real numbers and $\lim\limits_{k\to\infty} R_k < \infty$, hence $\sup\limits_k R_k < \infty$. This shows $x = (x_k) \in l_1$. Thus, $\left(\left(\Delta^2_{uvw}\right)^t - \alpha I\right)$ is onto. So we have $\alpha \in A_3\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right)$. □

Theorem 3.11. *Let* $\sqrt{V^2} = -V$. *The following statements hold:*

i. $\sigma_{ap}\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = S_3$.

ii. $\sigma_{co}\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \emptyset$.

Proof. i. Since from Table 1 determined [10],
$\sigma_{ap}\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right)/C_1\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right)$.

We have by Theorem 3.6
$C_1\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = C_2\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \emptyset$.

So, $\sigma_{ap}\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = S_3$.

ii. From Table 1, we have

$\sigma_{co}\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = C_1\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) \cup C_2\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) \cup C_3\sigma\left(\left(\Delta^2_{uvw}\right)^t, l_1\right)$.

By Theorem 3.2, $\sigma_{co}\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \emptyset$.

□

Theorem 3.12. $\sigma_c\left(\left(\left(\Delta^2_{uvw}\right)^t\right)^*, l_1^*\right) = \left\{\alpha \in \mathbb{C} : \frac{2|U-\alpha|}{\left|-V+\sqrt{V^2-4W(U-\alpha)}\right|} = 1\right\}$.

Proof. The proof is obvious, so is ommitted. □

Theorem 3.13. *Let* $\sqrt{V^2} = -V$. *If* $|2(U-\alpha)| < \left|-V + \sqrt{V^2 - 4W(U-\alpha)}\right|$, *then* $\alpha \in C_1\sigma\left(\left(\left(\Delta^2_{uvw}\right)^t\right)^*, l_1^*\right)$.

Proof. By Theorem 3.2, $\left(\left(\left(\Delta^2_{uvw}\right)^t\right)^* - \alpha I\right)^{-1}$ is exist. By Theorem 3.3 and 3.12, proof is completed. □

Theorem 3.14. $\sigma_\delta\left(\left(\Delta^2_{uvw}\right)^t, l_1\right) = \left\{\alpha \in \mathbb{C} : \left|\frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}}\right| = 1\right\}$.

Proof. By Proposition 2.2 (c), it can be seen readily. □

4. Acknowledgement

This work was supported by the Sakarya University Research Fund with Project Number 2013-02-00-004. This article was studied partly in International University of Sarajevo, Faculty of Engineering and Natural Sciences, Sarajevo, BIH. The authors also thank Prof. Dr. Fuat GURCAN (Faculty Dean, International University of Sarajevo, Faculty of Engineering and Natural Sciences, Sarajevo, BIH) for his contributions to this study.

References

[1] B. Altay, F. Başar, *On the fine spectrum of the difference operator* Δ *on* c_0 *and* c , Inform. Sci.,**168** (2004), 217–224.

[2] B. Altay, F. Başar,*On the fine spectrum of the generalized difference operator* $B(r,s)$ *over the sequence spaces* c_0 *and* c , Int. J. Math. Math. Sci., **18** (2005), 3005–3013.

[3] J.Appell, E. Pascale, A. Vingoli, *Nonlinear Spectral Theory*, de Gruyter Series in Nonlinear Analysis and Applications**10**, Walter de Gruyter, Berlin, Germany, (2004).

[4] J. Fathi, R. Lashkaripour, *On the fine spectrum of generalized upper double-band matrices* Δ^{uv} *over the sequence space* l_1 , Mat. Vesnik, **65 (1)** (2013), 64–73.

[5] H. Furkan, H. Bilgiç, K. Kayaduman, *On the fine spectrum of the generalized difference operator* $B(r,s)$ *over the sequence spaces* l_1 *and* bv, Hokkaido Math. J.,**35** (2006), 897–908.

[6] H. Furkan, H. Bilgiç, B. Altay, *On the fine spectrum of the operator* $B(r,s,t)$ *over* c_0 *and* c , Comput. Math. Appl., **53** (2007), 989–998.

[7] S. Goldberg, *Unbounded Linear Operator*, Dover publications, Inc. New York, (1985).

[8] M. Gonzalez, *The fine spectrum of the Cesàro operator in* l_p, $(1 < p < \infty)$, Arch. Math.,**44** (1985), 355–358.

[9] A. Karaisa, *Fine spectrum of upper triangular double-band matrices over the sequence space* l_p , $(1 < p < \infty)$, Discrete Dyn. Nat. Soc., vol. 2012, Article ID 381069, (2012), 19 pages.

[10] A. Karaisa, *Fine spectrum of upper triangular triple-band matrices over the sequence space* l_p , $(0 < p < \infty)$, Abstr. Appl. Anal., **v**ol. 2013, Article ID 34282,(2013), 10 pages.

[11] V. Karakaya, M. Altun, *Fine spectra of upper triangular double-band matrices*, J. Comput. Appl. Math.,**234** (2010), 1387–1394.

[12] K. Kayaduman, H. Furkan, *The fine spectra of difference operator* Δ *over the sequence spaces* l_1 *and* bv , Int. Math. Forum, **24** (2006), 1153–1160.

[13] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley and Sons Inc, New York, (1978).

[14] B. L. Panigrahi, P. D. Sirvastava, *Spectrum and fine spectrum of generalized second order forward difference operator* Δ^2_{UVW} *on sequence space* l_1 , Demonstratio Math., **45 (3)** (2012), 593–609.

[15] J. B. Reade, *On the spectrum of the Cesàro operator*, Bull. Lond. Math. Soc., **17** (1985), 263–267.

[16] P.D. Srivastava and S. Kumar, *Fine spectrum of generalized difference operator* Δ_v *on sequence space* l_1, Thai J. Math., **8 (2)** (2010), 221–233.

[17] P.D. Srivastava, S. Kumar, *Fine spectrum of generalized difference operator* Δ_{uv} *on sequence space* l_1 , Appl. Math. Comput., **218** (2012), 6407–6414.

[18] A. Wilansky, *Summability through Functional Analysis*, North-Holland Mathematics Studies, Amsterdam-New York-Oxford, (1984).