Filomat 30:5 (2016), 1375–1385
DOI 10.2298/FIL1605375D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fractional in Time Diffusion-Wave Equation and its Numerical Approximation

Aleksandra Delić[a]

[a]*University of Belgrade, Faculty of Mathematics*

Abstract. In this paper an initial-boundary value problem for fractional in time diffusion-wave equation is considered. A priori estimates in Sobolev spaces are derived. A fully discrete difference scheme approximating the problem is proposed and its stability and convergence are investigated. A numerical example demonstrates the theoretical results.

1. Introduction

Fractional partial differential equations have attracted considerable attetion in recent years due to their various applications in many fields of science and engineering [5, 9, 10]. In many cases fractional-order models are more adequate than integer-order models, because fractional derivatives and integrals enable the description of the memory properties of various materials and processes. The analytical solutions of most fractional differential equations cannot be obtained, and as consequence, approximate and numerical techniques are playing important role in identifying the solutions behavior of such fractional equations.

This article is concerned with a numerical solution of a fractional in time diffusion-wave equation subjected to homogeneous field. Fractional in time diffusion-wave equation is obtained from the classical diffusion equation, by replacing first-order time derivative by fractional derivative of order $\alpha \in (1, 2)$. It has been investigated by many authors. For example, Schneider and Wyss [14] considered time fractional diffusion-wave equation. Mainardy [8] solved a fractional diffusion-wave equation using the Laplace transform in a one-dimensional bounded domain. Fujita [3] discussed an integrodifferential equation which interpolates the heat equation and the wave equation in an unbounded domain. A number of numerical techniques were developed and their stability and convergence were investigated. For example, Tadjeran [16] present a second-order approximation for the fractional diffusion equation with RiemmanLiouville derivative in spatial direction based on Crank-Nicolson method. Sun and Wu [15] derived a fully discrete difference scheme for the time fractional diffusion-wave equation. They have proved convergence order $O(\tau^{3-\alpha} + h^2)$ for solutions $u \in C^{4,3}(Q)$.

In this article, we consider an initial-boundary value problem for fractional in time diffusion-wave equation. Its well posedness in the corresponding Sobolev spaces is proved. We prove that the proposed finite difference scheme is unconditionally stable in L^2 norm and that the order of convergence is same as in [15] for even less smooth solutions.

2010 *Mathematics Subject Classification*. Primary 35R11; Secondary 65M06, 65M12, 65M15
Keywords. Fractional in time diffusion-wave equation, Initial-boundary value problem, Finite-differences, Error-bounds
Received: 11 April 2014; Accepted: 09 August 2014
Communicated by Dragan S. Djordjević
Research supported by Ministry of Education, Science and Technological Development of Republic of Serbia through project No. 174015
Email address: adelic@matf.bg.ac.rs (Aleksandra Delić)

2. Fractional Derivatives and Fractional Derivative Spaces

Let u be defined on the interval $I = (0,T)$ and $\alpha > 0$. Then the left and right fractional integral of order α is defined to be, respectively, (see [10, 11])

$$\partial_{0+}^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u(s)(t-s)^{\alpha-1} ds, \quad t > 0,$$

$$\partial_{T-}^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^T u(s)(s-t)^{\alpha-1} ds, \quad t < T$$

where $\Gamma(\cdot)$ denotes Gamma function. For $\alpha = 0$ one sets $\partial_{0+}^0 u = \partial_{T-}^0 u = u$.
The left and right Riemman-Liouville fractional derivative of order α are defined as

$$\partial_{0+}^{\alpha} u(t) := \frac{d^n}{dt^n} \partial_{0+}^{\alpha-n} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t u(s)(t-s)^{n-\alpha-1} ds, \quad t > 0$$

$$\partial_{T-}^{\alpha} u(t) := (-1)^n \frac{d^n}{dt^n} \partial_{T-}^{\alpha-n} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^T u(s)(s-t)^{n-\alpha-1} ds, \quad t < T,$$

respectively, where $n-1 \le \alpha < n$, $n \in \mathbb{N}$. The left Riemann-Liouville fractional derivative of order α acts as a left inverse of the left fractional integral of order α:

$$\partial_{0+}^{\alpha} \partial_{0+}^{-\alpha} u(t) = u(t), \quad \left(\partial_{T-}^{\alpha} \partial_{T-}^{-\alpha} u(t) = u(t)\right),$$

while

$$\partial_{0+}^{-\alpha} \partial_{0+}^{\alpha} u(t) = u(t) - \sum_{k=1}^{n} \partial_{0+}^{\alpha-k} u(t)\Big|_{t=0} \frac{t^{\alpha-k}}{\Gamma(\alpha-k+1)}$$

where $n-1 \le \alpha < n$. Riemann-Liouville fractional derivatives satisfies semigroup property:

$$\partial_{0+}^{\alpha} \partial_{0+}^{\beta} u = \partial_{0+}^{\beta} \partial_{0+}^{\alpha} u = \partial_{0+}^{\alpha+\beta} u$$

under assumptions that u has sufficient number of continuous derivatives and

$$\frac{d^j u}{dt^j}(0) = 0, \quad j = 0, 1, ..., \max\left([\alpha], [\beta]\right), \ [\alpha] \le \alpha < [\alpha]+1, [\beta] \le \beta < [\beta]+1 \quad \text{(see [10])}.$$

The Caputo definitions of fractional derivatives one obtains by commuting $\frac{d^n}{dt^n}$ and $\partial_{0+}^{\alpha-n}(\partial_{T-}^{\alpha-n})$, $n-1 \le \alpha < n$:

$${}^C\partial_{0+}^{\alpha} u(t) := \partial_{0+}^{\alpha-n} \frac{d^n}{dt^n} u(t), \quad {}^C\partial_{T-}^{\alpha} u(t) := (-1)^n \partial_{T-}^{\alpha-n} \frac{d^n}{dt^n} u(t).$$

The left Caputo fractional derivative of order α acts as a left inverse of the left fractional integral of order α:

$${}^C\partial_{0+}^{\alpha} \partial_{0+}^{-\alpha} u(t) = u(t),$$

but

$$\partial_{0+}^{-\alpha} {}^C\partial_{0+}^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{d^k u(t)}{dt^k}\Big|_{t=0} \frac{t^k}{k!}, \quad n-1 \le \alpha < n.$$

By a direct calculation, one can obtain the following a relation between Riemann-Liouville and Caputo fractional derivatives

$${}^C\partial_{0+}^{\alpha} u(t) = \partial_{0+}^{\alpha} u(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} \frac{d^k u(0)}{dt^k}.$$

Thus, for the zero initial conditions Riemann-Liouville and Caputo fractional derivatives coincide.

As usual, we denote by $C^k(0,T)$ and $C^k[0,T]$ the space of k-fold differentiable functions. By $C_0^\infty(0,T)$ we denote the space of infinitely differentiable functions with compact support in $(0,T)$. The inner product and norm in the space of measurable functions whose square is integrable in $(0,T)$, denoted by $L^2(0,T)$, are defined by

$$(u,v)_{L^2(0,T)} = \int_\Omega uvdt, \quad \|u\|_{L^2(0,T)} = (u,u)^{1/2}_{L^2(0,T)}.$$

We also use $H^\alpha(0,T)$ and $H_0^\alpha(0,T)$ to denote the usual Sobolev spaces [?], whose norms are denoted by $\|u\|_{H^\alpha(0,T)}$.

For $\alpha > 0$ let us define semi-norms

$$|u|_{H_+^\alpha(0,T)} := \|\partial^\alpha_{0+}u\|_{L^2(0,T)}, \quad |u|_{H_-^\alpha(0,T)} := \|\partial^\alpha_{T-}u\|_{L^2(0,T)},$$

and norms

$$\|u\|_{H_\pm^\alpha(0,T)} := \left(\|u\|^2_{L^2((0,T)} + |u|^2_{H_\pm^\alpha(0,T)}\right)^{1/2}.$$

We then define the spaces $H_\pm^\alpha(0,T)$ and $H^\alpha_{0,\pm}(0,T)$ as the closure of $C^\infty[0,T]$ and $C_0^\infty(0,T)$, respectively, with respect to the norm $\|u\|_{H_\pm^\alpha(0,T)}$. From Corollary 2 of Theorem 2.4 in [11] it follows:

Lemma 2.1. *Let* $0 < \alpha < 1$, $u \in H_+^\alpha(0,T)$, $v \in H_-^\alpha(0,T)$ *and* $u(0) = v(0) = 0$. *Then*

$$\left(\partial^\alpha_{0+}u, v\right)_{L^2(0,T)} = \left(u, \partial^\alpha_{T-}v\right)_{L^2(0,T)}.$$

Lemma 2.2. *[2] Let* $\alpha > 0$, $u \in C^\infty(\mathbb{R})$ *and* $\operatorname{supp} u \subset (0,T]$. *Then*

$$\left(\partial^\alpha_{0+}u, \partial^\alpha_{T-}u\right)_{L^2(0,T)} = \cos\pi\alpha\, \|\partial^\alpha_{0+}u\|^2_{L^2(0,\infty)}.$$

For $\alpha > 0$, $\alpha \neq n + \frac{1}{2}$, $n \in \mathbb{N}$ we define semi-norm and norm

$$|u|_{H_c^\alpha(0,T)} := \left|\left(\partial^\alpha_{0+}u, \partial^\alpha_{T-}u\right)_{L^2(0,T)}\right|^{1/2}, \quad \|u\|_{H_c^\alpha(0,T)} := \left(\|u\|^2_{L^2(0,T)} + |u|^2_{H_c^\alpha(0,T)}\right)^{1/2}$$

and the space $H^\alpha_{0,c}(0,T)$ as the closure of $C_0^\infty(0,T)$ with respect to the norm $\|\cdot\|_{H_c^\alpha(0,T)}$.

Lemma 2.3. *[2, 6] For* $\alpha > 0$, $\alpha \neq n + \frac{1}{2}$, $n \in \mathbb{N}$, *the spaces* $H^\alpha_{0+}(0,T)$, $H^\alpha_{0-}(0,T)$, $H^\alpha_{0c}(0,T)$ *and* $H_0^\alpha(0,T)$ *are equivalent and their seminorms as well as norms are equivalent.*

For the functions of many variables partial derivatives of fractional order are defined in analogous manner, for example:

$$\partial^\alpha_{t,0+}u(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t (t-s)^{n-\alpha-1}u(x,s)ds, \quad t > 0,$$

$$\partial^\beta_{x,0+}u(x,t) = \frac{1}{\Gamma(m-\beta)}\frac{\partial^m}{\partial x^m}\int_0^t (x-s)^{n-\beta-1}u(s,t)ds, \quad x > 0,$$

where $n-1 \le \alpha < n$, $m-1 \le \beta < m$ and $n, m \in N$.

For $\alpha, \beta \ge 0$ and $Q = (0,1)\times(0,T)$, we introduce anisotropic Sobolev type spaces:

$$H^{\alpha,\beta}(Q) = L^2((0,T), H^\alpha(0,1)) \cap H^\beta((0,T), L^2(0,1))$$

and

$$H_\pm^{\alpha,\beta}(Q) = L^2((0,T), H^\alpha(0,1)) \cap H_\pm^\beta((0,T), L^2(0,1)).$$

Notice that for $0 \le \beta < 1/2$: $H_+^{\alpha,\beta}(Q) = H_-^{\alpha,\beta}(Q) = H^{\alpha,\beta}(Q)$.

Finally, we introduce the Banach spaces

$$C([0,T],\mathcal{U}) \quad \text{and} \quad C_+^{\alpha}([0,T],\mathcal{U}), \quad \alpha > 0$$

of vector valued functions $u : [0,T] \to \mathcal{U}$ equipped, respectively, with the norm

$$\|u\|_{C([0,T],\mathcal{U})} = \max_{t\in[0,T]} \|u(t)\|_{\mathcal{U}} \quad \text{and} \quad \|u\|_{C_+^{\alpha}([0,T],\mathcal{U})} = \max_{t\in[0,T]} \|\partial_{t,0+}^{\alpha} u(t)\|_{\mathcal{U}}.$$

3. Diffusion-Wave Equation

Let $\Omega = (0,1)$ and $I = (0,T)$ be the space and time domain respectively and $Q = \Omega \times I$. We consider the following diffusion-wave equation for $1 < \alpha < 2$:

$$\partial_{t,0+}^{\alpha} u - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in Q, \tag{1}$$

subject to the following boundary and initial conditions:

$$u(0,t) = 0, \quad u(1,t) = 0, \quad \forall t \in I, \tag{2}$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \forall x \in \Omega. \tag{3}$$

Theorem 3.1. *Let $\alpha \in (1,2)$ and $f \in L^2(Q)$. Then the solution of initial-boundary value problem* (1)-(3) *satisfies a priori estimate*

$$\max_{t\in[0,T]} \|\partial_{t,0+}^{\alpha-1} u\|_{L^2(\Omega)} + \left\| \partial_{t,0+}^{\frac{\alpha-1}{2}} \frac{\partial u}{\partial x} \right\|_{L^2(Q)} \le C\|f\|_{L^2(Q)}. \tag{4}$$

Proof. Let $\beta = \alpha - 1$. Taking the $L^2(\Omega)$ inner product of (1) with $2\partial_{t,0+}^{\beta} u$ and using property $\partial_{t,0+}^{\alpha} u = \frac{\partial}{\partial t}\partial_{t,0+}^{\beta} u$ we obtain

$$\frac{\partial}{\partial t}\| \partial_{t,0+}^{\beta} u\|_{L^2(\Omega)}^2 + 2\left(\frac{\partial u}{\partial x}, \partial_{t,0+}^{\beta}\frac{\partial u}{\partial x}\right)_{L^2(\Omega)} = 2(f, \partial_{t,0+}^{\beta} u)_{L^2(\Omega)},$$

Integrating last inequality between 0 and t and using Lemmas 2.1, 2.2 and estimating the right-hand side by Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \|\partial_{t,0+}^{\beta} u(\cdot,t)\|_{L^2(\Omega)}^2 + 2\cos\frac{\pi\beta}{2} \left\| \partial_{t,0+}^{\beta/2} \frac{\partial u}{\partial x} \right\|_{L^2((0,\infty),L^2(\Omega))}^2 &\le \|\partial_{t,0+}^{\beta} u(\cdot,0)\|_{L^2(\Omega)}^2 + 2\|f\|_{L^2(Q_t)} \|\partial_{t,0+}^{\beta} u\|_{L^2(Q_t)} \\ &\le 2\sqrt{T} \max_{s\in[0,T]} \|\partial_{s,0+}^{\beta} u(\cdot,s)\|_{L^2(\Omega)} \|f\|_{L^2(Q)}, \end{aligned} \tag{5}$$

where $Q_t = (0,1) \times (0,t)$. Ommiting the second positive term on the left-hand side we obtain

$$\max_{t\in[0,T]} \|\partial_{t,0+}^{\beta} u\|_{L^2(\Omega)} \le 2\sqrt{T}\|f\|_{L^2(Q)}. \tag{6}$$

Similarly, omitting the first positive term on the left-hand side of inequality (5), taking $t = T$ and using (6) we obtain

$$\left\| \partial_{t,0+}^{\beta/2} \frac{\partial u}{\partial x} \right\|_{L^2(Q)} \le \sqrt{\frac{2T}{\cos\frac{\pi\beta}{2}}} \|f\|_{L^2(Q)}. \tag{7}$$

The result (4) follows from (6) and (7). □

Theorem 3.2. *Let $\alpha \in (1,2)$ and $f \in L^2(Q)$. Then the solution of initial-boundary value problem* (1)-(3) *satisfies a priori estimate*

$$\left\|\partial_{t,0+}^{\frac{\alpha-1}{2}} \frac{\partial u}{\partial t}\right\|_{L^2(Q)}^2 + \max_{t\in[0,T]} \left\|\frac{\partial u}{\partial x}\right\|_{L^2(\Omega)}^2 \le C\|f\|_{L^2(Q)}. \tag{8}$$

Proof. Let $\beta = \alpha - 1$. Taking the inner product of (1) with $2\frac{\partial u}{\partial t}$, using property $\partial_{t,0+}^{\alpha} u = \frac{\partial}{\partial t}\partial_{t,0+}^{\beta} u = \partial_{t,0+}^{\beta} \frac{\partial u}{\partial t}$ and estimating the right-hand side by Cauchy-Schwarz inequality we obtain

$$2\left(\frac{\partial u}{\partial t}, \partial_{t,0+}^{\beta}\frac{\partial u}{\partial t}\right)_{L^2(\Omega)} + \frac{\partial}{\partial t}\left\|\frac{\partial u}{\partial x}\right\|_{L^2(\Omega)}^2 \le 2\varepsilon \left\|\frac{\partial u}{\partial t}\right\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon}\|f\|_{L^2(\Omega)}^2,$$

Integrating last inequality between 0 and t and using Lemmas 2.1, 2.2

$$2\cos\frac{\pi\beta}{2}\left\|\partial_{t,0+}^{\beta/2}\frac{\partial u}{\partial t}\right\|_{L^2((0,\infty),L^2(\Omega))}^2 + \left\|\frac{\partial u(\cdot,t)}{\partial x}\right\|_{L^2(\Omega)}^2 \le \left\|\frac{\partial u(\cdot,0)}{\partial x}\right\|_{L^2(\Omega)}^2 + 2\varepsilon\left\|\frac{\partial u}{\partial t}\right\|_{L^2(Q_t)}^2 + \frac{1}{2\varepsilon}\|f\|_{L^2(Q_t)}^2, \tag{9}$$

where $Q_t = (0,1)\times(0,t)$. From Lemma 2.6 in [2] follows that

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2(Q_t)}^2 = \left\|\partial_{t,0+}^{-\beta/2}\partial_{t,0+}^{\beta/2}\frac{\partial u}{\partial t}\right\|_{L^2(Q_t)}^2 \le \left(\frac{t^{\beta/2}}{\Gamma(\beta/2+1)}\right)^2\left\|\partial_{t,0+}^{\beta/2}\frac{\partial u}{\partial t}\right\|_{L^2(Q_t)}^2 \le \left(\frac{T^{\beta/2}}{\Gamma(\beta/2+1)}\right)^2\left\|\partial_{t,0+}^{\beta/2}\frac{\partial u}{\partial t}\right\|_{L^2(Q_t)}^2.$$

Taking $\varepsilon = \frac{\Gamma^2(\beta/2+1)}{T^\beta}\cos\frac{\pi\beta}{2}$ we obtain

$$\max_{t\in[0,T]}\left\|\frac{\partial u}{\partial x}\right\|_{L^2(\Omega)}^2 \le C\|f\|_{L^2(Q)}, \tag{10}$$

where $C = (2\varepsilon)^{-1/2}$.

Similarly, omitting the second positive term on the left-hand side of inequality (9) and taking $t = T$, we obtain

$$2\cos\frac{\pi\beta}{2}\left\|\partial_{t,0+}^{\beta/2}\frac{\partial u}{\partial t}\right\|_{L^2(Q)}^2 \le 2\cos\frac{\pi\beta}{2}\left\|\partial_{t,0+}^{\beta/2}\frac{\partial u}{\partial t}\right\|_{L^2((0,\infty),L^2(\Omega))}^2 \le \frac{2\varepsilon T^\beta}{\Gamma^2(\beta/2+1)}\left\|\partial_{t,0+}^{\beta/2}\frac{\partial u}{\partial t}\right\|_{L^2(Q)}^2 + \frac{1}{2\varepsilon}\|f\|_{L^2(Q)}.$$

For $\varepsilon = \frac{\Gamma^2(\beta/2+1)}{2T^\beta}\cos\frac{\pi\beta}{2}$ it follows that

$$\left\|\partial_{t,0+}^{\beta/2}\frac{\partial u}{\partial t}\right\|_{L^2(Q)}^2 \le C\|f\|_{L^2(Q)}, \tag{11}$$

where $C = \left(2\varepsilon\cos\frac{\pi\beta}{2}\right)^{-1/2}$. The result (8) follows from (10) and (11). □

Let us also consider equation

$$\partial_{t,0+}^{\alpha} u - \frac{\partial^2 u}{\partial x^2} = \frac{\partial g(x,t)}{\partial x}, \quad (x,t)\in Q, \tag{12}$$

subject to the boundary and initial conditions defined by (2) and (3).

Using Cauchy-Schwarz inequality and Lemmas 2.1-2.3 one obtains the following results

Theorem 3.3. *Let $\alpha \in (1,2)$ and $g \in L^2(Q)$. Then the solution of initial-boundary value problem* (12), (2), (3) *satisfies a priori estimate*

$$\max_{t\in[0,T]}\|u\|_{L^2(\Omega)} \le C\|g\|_{L^2(Q)}.$$

4. Finite Differences

Let $\bar{\omega}_h = \{x_i = ih | i = 0, 1 \dots, N\}$ be a uniform mesh in $[0,1]$ with the step-size $h = 1/N$, $N > 1$. We denote $\omega_h := \bar{\omega}_h \cap (0,1)$, $\omega_h^- := \omega_h \cup \{0\}$. Also, we define a uniform mesh $\bar{\omega}_\tau = \{t_j = j\tau | j = 0, 1, \dots M\}$ in $[0,T]$ with the step size $\tau = T/M$, $M > 1$ and set $\omega_\tau := \bar{\omega}_\tau \cap (0,T)$, $\omega_\tau^+ = \omega_\tau \cup \{T\}$. We will consider a grid function v defined on $\bar{\omega}_h \times \bar{\omega}_\tau$. We introduce the following notations [13]

$$v_x = \frac{v(x+h,t) - v(x,t)}{h} = v_{\bar{x}}(x+h,t), \qquad v_t = \frac{v(x,t+\tau) - v(x,t)}{\tau} = v_{\bar{t}}(x,t+\tau),$$

$$v_{\hat{t}} = \frac{v_t + v_{\bar{t}}}{2} = \frac{v(x,t+\tau) - v(x,t-\tau)}{2\tau}, \qquad v_{\bar{x}x} = \frac{v_x - v_{\bar{x}}}{h}, \qquad v_{\bar{t}t} = \frac{v_t - v_{\bar{t}}}{\tau},$$

$$v^j = v(x,t_j), \qquad \bar{v}^j = \frac{v(x,t_j) + v(x,t_{j-1})}{2}.$$

For $0 < \beta < 1$ fractional derivative operator $\partial_{t,0+}^{\beta}$ we approximate by [15]

$$\Delta_{t,0+}^{\beta} v^j = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \left(a_1 v^j - \sum_{k=1}^{j-1} (a_{j-k+1} - a_{j-k}) v^k - a_j v^0 \right), \quad j = 1, 2, \dots M,$$

where $a_j = j^{1-\beta} - (j-1)^{1-\beta} > 0$. For $j = 0$ we formally take $\Delta_{t,0+}^{\beta} v^0 = 0$.

We define the Steklov averaging operators [4, 12] as follows

$$T_t v(x,t) := \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} v(x,s) ds = T_t^+ v(x, t-\tau/2) = T_t^- v(x, t+\tau/2),$$

$$T_x^+ v(x,t) := \frac{1}{h} \int_x^{x+h} v(s,t) ds = T_x^- v(x+h,t), \quad T_x^2 v(x,t) := \frac{1}{h} \int_{x-h}^{x+h} \left(1 - \left| \frac{x-s}{h} \right| \right) v(s,t) ds,$$

Notice that these operators commute and maps the derivatives of sufficiently smooth function u into finite differences, for example

$$T_t^+ \frac{\partial u}{\partial t} = u_t, \qquad T_x^2 \frac{\partial^2 u}{\partial x^2} = u_{\bar{x}x}. \tag{13}$$

We approximate the fractional derivative of order $1 < \alpha < 2$ in the following manner [15]

$$\begin{aligned}
\partial_{t,0+}^{\alpha} u(x,t_j) &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \frac{\partial^2 u(x,s)}{\partial s^2} (t_j - s)^{1-\alpha} ds \\
&\approx \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \left(\frac{\partial u(x,t_k)}{\partial t} \right)_t \int_{t_k}^{t_{k+1}} (t_j - s)^{1-\alpha} ds \\
&= \frac{\tau^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{k=0}^{j-1} \left(\frac{\partial u(x,t_k)}{\partial t} \right)_t a_{j-k} \\
&= \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left(a_1 \frac{\partial u(x,t_j)}{\partial t} - \sum_{k=1}^{j-1} \frac{\partial u(x,t_k)}{\partial t} (a_{j-k+1} - a_{j-k}) + a_j \frac{\partial u(x,t_0)}{\partial t} \right),
\end{aligned}$$

in which

$$a_{j-k} = (j-k)^{2-\alpha} - (j-k-1)^{2-\alpha} > 0.$$

Using the initial condition (3) and substituting $\alpha = \beta + 1$ we obtain

$$\partial^{\alpha}_{t,0+} u^{j-1/2} \approx \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left(a_1 u^j_{\bar t} - \sum_{k=1}^{j-1} (a_{j-k+1} - a_{j-k}) u^k_{\bar t} \right) = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \left(a_1 u^j_{\bar t} - \sum_{k=1}^{j-1} (a_{j-k+1} - a_{j-k}) u^k_{\bar t} \right) =: \Delta^{\beta}_{t,0+} u^j_{\bar t}. \quad (14)$$

Let us define discrete inner products and associated norms

$$(u,v)_h = (u,v)_{L^2(\omega_h)} = h \sum_{x\in\omega_h} u(x)v(x), \quad \|v\|_h = \|v\|_{L^2(\omega_h)} = (v,v)_h^{1/2},$$

$$(u,v]_h = (u,v]_{L^2(\omega_h^+)} = h \sum_{x\in\omega_h^+} u(x)v(x), \quad \|v]|_h = \|v]|_{L^2(\omega_h^+)} = (v,v]_h^{1/2},$$

$$[u,v)_h = [u,v)_{L^2(\omega_h^-)} = h \sum_{x\in\omega_h^-} u(x)v(x), \quad \|[v\|_h = \|[v\|_{L^2(\omega_h^-)} = [v,v)_h^{1/2},$$

$$|v|_{H^1(\omega_h)} = \|[v_x\|_h, \quad \|v\|_{H^1(\omega_h)} = \left(|v|^2_{H^1(\omega_h)} + \|v\|^2_{L^2(\omega_h)} \right)^{1/2},$$

$$\|v\|_{L^2(Q_{h\tau})} = \left(\tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|_h^2 \right)^{1/2}, \quad \|[v\|_{L^2(Q_{h\tau})} = \left(\tau \sum_{t\in\omega_\tau^+} \|[v(\cdot,t)\|_h^2 \right)^{1/2}.$$

5. Finite Difference Scheme

The initial-boundary problem (1)-(3) we approximate with the following finite difference scheme

$$\Delta^{\beta}_{t,0+} v^j_{\bar t} + A_h \bar v^j = \varphi^j, \qquad j = 1,2,\ldots,M, \quad x \in \omega_h, \tag{15}$$

$$v(0,t) = 0,\ v(1,t) = 0, \quad t \in \bar\omega_\tau, \tag{16}$$

$$v(x,0) = 0, \quad x \in \bar\omega_h, \tag{17}$$

where $\varphi^j_i = T_t T^2_x f(x_i, (t_j + t_{j-1})/2)$, $A_h v = -v_{\bar x x}$, $\beta = \alpha - 1$. Approximation of second initial condition in (3) is already included in definition of $\Delta^{\beta}_{t,0+} v^j_{\bar t}$. That approximation is equivalent with $\Delta^{\beta}_{t,0+} v^0 = 0$.

Theorem 5.1. *(see [1, 15]) Let $0 < \beta < 1$. For every function $v(t)$ defined on the mesh ω_τ which satisfies $v(0) = 0$ the following inequality is valid*

$$\tau \sum_{j=1}^{M} v^j \Delta^{\beta}_{t,0+} v^j \geq \frac{T^{-\beta}}{\Gamma(1-\beta)} \tau \sum_{j=1}^{M} (v^j)^2. \tag{18}$$

Lemma 5.2. *The finite difference scheme* (15)-(17) *is absolutely stable and its solution satisfies a priori estimate:*

$$\|v\|_{L^2(Q_{h\tau})} \leq C \|A_h^{-1/2} \varphi\|_{L^2(Q_{h\tau})}. \tag{19}$$

Proof. Taking the inner product of (15) with $2A_h^{-1} v_{\bar t}$ and using Cauchy-Schwarz inequality we obtain

$$2\tau \left(\Delta^{\beta}_{t,0+} A_h^{-1/2} v^j_{\bar t}, A_h^{-1/2} v^j_{\bar t} \right) + \|v^j\| - \|v^{j-1}\| \leq 2\tau\varepsilon \|A_h^{-1/2} \varphi^j\|^2 + \frac{\tau}{2\varepsilon} \|A_h^{-1/2} v^j_{\bar t}\|^2.$$

Summing for $j = 1,2,\ldots,k$, $1 \leq k \leq M$, using (18) and taking $\varepsilon = \varepsilon_k = (k\tau)^\beta \Gamma(1-\beta)/2$ we obtain

$$\frac{(k\tau)^{-\beta}}{\Gamma(1-\beta)} \tau \sum_{j=1}^{k} \|A_h^{-1/2} v^j_{\bar t}\|^2 + \|v^k\|^2 \leq \Gamma(1-\beta)(k\tau)^\beta \sum_{j=1}^{k} \|A_h^{-1/2} \varphi^j\|^2 \leq \Gamma(1-\beta) T^\beta \|A_h^{-1/2} \varphi^j\|^2_{L^2(Q_{h\tau})}.$$

Omitting the positive sum on the left side and summing again for $k = 1,2,\ldots,M$ we obtain a priori estimate (19) with $C = \sqrt{\frac{T^\alpha}{\Gamma(2-\alpha)}}$. □

6. Convergence of the Finite Difference Scheme

Let u be the solution of the initial-boundary value problem (1)-(3) and v the solution of of the difference problem (15)-(17). The error $z = u - v$ satisfies

$$\Delta^{\beta}_{t,0+} z^{j}_{\bar{t},i} + A_h \bar{z}^j = \psi^{j}_{\bar{t},i} + A_h \mu^j, \quad j = 1, 2, ..., M, \quad x \in \omega_h \tag{20}$$

$$z(0,t) = 0,\ z(1,t) = 0, \quad t \in \bar{\omega}_\tau\,, \tag{21}$$

$$z(x,0) = 0, \quad x \in \bar{\omega}_h, \tag{22}$$

where

$$\psi = \Delta^{\beta}_{t,0+} u - T^2_x \partial^{\beta}_{t,0+} u \qquad \text{and} \qquad \mu = \bar{u} - T^{-}_t u.$$

Lemma 6.1. *The finite difference scheme*

$$\Delta^{\beta}_{t,0+} z^{j}_{\bar{t}} + A_h \bar{z}^j = \psi^{j}_{\bar{t}}, \quad j = 1, 2, ..., M, \quad x \in \omega_h \tag{23}$$

$$z(0,t) = 0,\ z(1,t) = 0, \quad t \in \bar{\omega}_\tau\,,$$

$$z(x,0) = 0 \quad i = 0, 1, ..., N,$$

satisfies the a priori estimate

$$\|\bar{z}\|_{L^2(Q_{h\tau})} \le C \|\psi\|_{L^2(Q_{h\tau})}. \tag{24}$$

Proof. Taking the inner product of (23) with $2\tau A_h^{-1}\left(\Delta^{\beta}_{t,0+}\bar{z}^j - \bar{\psi}^j\right)$, using relation

$$2\tau\left(\Delta^{\beta}_{t,0+} z^j_{\bar{t}} - \psi^j_{\bar{t}}, A_h^{-1}(\Delta^{\beta}_{t,0+}\bar{z}^j - \bar{\psi}^j)\right) = \|A_h^{-1/2}(\Delta^{\beta}_{t,0+} z^j - \psi^j)\|^2 - \|A_h^{-1/2}(\Delta^{\beta}_{t,0+} z^{j-1} - \psi^{j-1})\|^2,$$

and Cauchy-Schwarz inequality we obtain

$$\|A_h^{-1/2}(\Delta^{\beta}_{t,0+} z^j - \psi^j)\|^2 - \|A_h^{-1/2}(\Delta^{\beta}_{t,0+} z^{j-1} - \psi^{j-1})\|^2 + 2\tau(\bar{z}^j, \Delta^{\beta}_{t,0+}\bar{z}^j) = 2\tau(\bar{z}^j, \bar{\psi}^j) \le 2\varepsilon\tau\|\bar{\psi}^j\|^2 + \frac{\tau}{2\varepsilon}\|\bar{z}^j\|^2.$$

Summing for $j = 1, 2, ..., M$ and using (18) we obtain

$$\|A_h^{-1/2}(\Delta^{\beta}_{t,0+} z^M - \psi^M)\|^2 + \frac{2T^{-\beta}}{\Gamma(1-\beta)}\|\bar{z}^j\|^2_{L^2(Q_{h\tau})} \le \|A_h^{-1/2}(\Delta^{\beta}_{t,0+} z^0 - \psi^0)\|^2 + 2\varepsilon\|\bar{\psi}^j\|^2_{L^2(Q_{h\tau})} + \frac{1}{2\varepsilon}\|\bar{z}^j\|^2_{L^2(Q_{h\tau})}.$$

Omitting the positive term on the left side and taking $\varepsilon = T^{\beta}\Gamma(1-\beta)/2$ we obtain a priori estimate (24) with $C = T^{\beta}\Gamma(1-\beta)$. □

Theorem 6.2. *The finite difference scheme* (20)-(22) *satisfies a priori estimate*

$$\|\bar{z}\|_{L^2(Q_{h\tau})} \le C\left(\|\psi\|_{L^2(Q_{h\tau})} + \|[\mu_x\|_{L^2(Q_{h\tau})}\right). \tag{25}$$

Proof. Results follow directly from Lemmas 5.2, 6.1 and inequality $\|\bar{v}\|_{L^2(Q_{h\tau})} \le \|v\|_{L^2(Q_{h\tau})}$. □

Lemma 6.3. *[15] Suppose that* $v \in C^2[0,t]$, $t \in \omega_\tau$ *and* $0 < \beta < 1$. *Then*

$$|\partial^{\beta}_{t,0+} v - \Delta^{\beta}_{t,0+} v| \le \tau^{2-\beta}\frac{1}{1-\beta}\left(\frac{1-\beta}{12} + \frac{2^{2-\beta}}{2-\beta} - (1 + 2^{-\beta})\right)\max_{0\le s\le t}|v''(s)|.$$

Theorem 6.4. *Let* $u \in C^2([0,T], C[0,1]) \cap H^2((0,T), H^1(0,1)) \cap H_+^{\alpha-1}((0,T), H^2(0,1))$. *Then the solution v of finite difference scheme* (15)-(17) *converges to the solution u of initial-boundary-value problem* (1)-(3) *and the following convergence rate estimates holds:*

$$\|\bar{z}\|_{L^2(Q_{h\tau})} \leq C(\tau^{3-\alpha} + h^2). \tag{26}$$

Proof. In order to estimate the rate of convergence of the difference scheme (15)-(17), it is sufficiently to estimate the right hand sides of the inequality (25) decomposed in the following manner

$$\|\bar{z}\|_{L^2(Q_{h\tau})} \leq C\left(\|\Delta_{t,0+}^{\beta} u - \partial_{t,0+}^{\beta} u\|_{L^2(Q_{h\tau})} + \left\|\partial_{t,0+}^{\beta}\left(u - T_x^2 u\right)\right\|_{L^2(Q_{h\tau})} + \|[\mu_x\|_{L^2(Q_{h\tau})}\right).$$

Using (13) we directly obtain the following integral representations:

$$\mu_x = \frac{1}{2h\tau}\int_x^{x+h}\int_{t-\tau}^{t}\int_{t'}^{t}\int_{t'}^{t''}\frac{\partial^3 u(x',t''')}{\partial x\partial t^2}dt'''dt''dt'dx',$$

$$u - T_x^2 u = \frac{1}{h}\int_{x-h}^{x+h}\int_{x'}^{x}\int_{x}^{x''}\left(1 - \frac{|x'-x|}{h}\right)\frac{\partial^2 u}{\partial x^2}(x''',t)\,\mathrm{d}x'''\mathrm{d}x''\mathrm{d}x',$$

Summing over the the nodes of the mesh $\omega_h^- \times \omega_\tau^+$ and $\omega_h \times \omega_\tau^+$, respectively, we get

$$\|[\mu_x\|_{L^2(Q_{h\tau})} \leq C\tau^2\left\|\frac{\partial^3 u}{\partial x\partial t^2}\right\|_{L^2(Q)} \leq C\tau\|u\|_{H^2((0,T),H^1(0,1))}, \tag{27}$$

$$\left\|\partial_{t,0+}^{\beta}\left(u - T_x^2 u\right)\right\|_{L^2(Q_{h\tau})} \leq Ch^2\left\|\partial_{t,0+}^{\beta}u(\cdot,t)\right\|_{H^2(0,1)} \leq Ch^2\|u\|_{H_+^{\alpha-1}((0,T),H^2(0,1))}. \tag{28}$$

Using Lemma 6.3 we obtain

$$\|\Delta_{t,0+}^{\beta} u - \partial_{t,0+}^{\beta} u\|_{L^2(Q_{h\tau})} \leq C\tau^{2-\beta}\max_{t\in[0,T]}\max_{x\in[0,1]}\left|\frac{\partial^2 u}{\partial t^2}\right| \leq C\tau^{3-\alpha}\,\|u\|_{C^2([0,T],C[0,1])}. \tag{29}$$

The result (26) follows from (27),(28) and (29). □

7. Numerical Example

To check the stability and convergence properties of the numerical method we solved the problem (1)-(3) for

$$f(x,t) = \sin(\pi x)t^{5/2}\left(\frac{t^{-\alpha}\Gamma(7/2)}{\Gamma(7/2-\alpha)} - \pi^2.\right)$$

The exact solution of the above problem is $u(x,t) = \sin(\pi x)t^{5/2}$.

Table 1 lists the computational results with different time step sizes τ when space step size is fixed as $h = 2^{-13}$. From the table, we can draw the conclusion that the order of convergence in time direction is $3-\alpha$.

Table 2 gives numerical results for small and fixed $\tau = 2^{-14}$ with different h. The reason why we have used a very small step size τ is to make sure that the dominated error is from space discretization. From the table, we can see that the order of convergence in space direction is 2.

α	τ	$\|z\|_{L^2(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{L^2(Q_{h\tau})}}{\|z\|_{L^2(Q_{h\tau/2})}}$	α	τ	$\|z\|_{L^2(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{L^2(Q_{h\tau})}}{\|z\|_{L^2(Q_{h\tau/2})}}$
1.9	2^{-5}	$2.6306e-3$	1.04	1.7	2^{-5}	$1.0894e-3$	1.21
	2^{-6}	$1.2779e-3$	1.07		2^{-6}	$4.7081e-4$	1.24
	2^{-7}	$6.0833e-4$	1.09		2^{-7}	$1.9895e-4$	1.26
	2^{-8}	$2.8670e-4$	1.09		2^{-8}	$8.3116e-5$	1.27
	2^{-9}	$1.3445e-4$	1.10		2^{-9}	$3.4521e-5$	1.27
	2^{-10}	$6.2888e-5$	no data		2^{-10}	$1.4296e-5$	no data
1.4	2^{-5}	$2.4162e-4$	1.52	1.15	2^{-5}	$1.3848e-4$	1.99
	2^{-6}	$8.4098e-5$	1.51		2^{-6}	$3.4916e-5$	1.97
	2^{-7}	$2.9434e-5$	1.52		2^{-7}	$8.8963e-6$	1.95
	2^{-8}	$1.0279e-5$	1.52		2^{-8}	$2.2982e-6$	1.93
	2^{-9}	$3.5727e-6$	1.53		2^{-9}	$6.0330e-7$	1.91
	2^{-10}	$1.2359e-6$	no data		2^{-10}	$1.6063e-7$	no data

Table 1: The experimental error results and convergence order in time direction (the last column) with $h = 2^{-13}$

α	h	$\|z\|_{L^2(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{L^2(Q_{h\tau})}}{\|z\|_{L^2(Q_{h/2\tau})}}$	α	h	$\|z\|_{L^2(Q_{h\tau})}$	$\log_2 \frac{\|z\|_{L^2(Q_{h\tau})}}{\|z\|_{L^2(Q_{h/2\tau})}}$
1.9	2^{-3}	$2.2355e-3$	2.00	1.7	2^{-3}	$2.5977e-3$	2.00
	2^{-4}	$5.5954e-4$	2.00		2^{-4}	$6.4905e-4$	2.00
	2^{-5}	$1.4019e-4$	1.98		2^{-5}	$1.6220e-4$	2.00
	2^{-6}	$3.5414e-5$	1.89		2^{-6}	$4.0509e-5$	2.00
	2^{-7}	$9.5436e-6$	no data		2^{-7}	$1.0093e-5$	no data
1.4	2^{-3}	$2.9787e-3$	2.00	1.15	2^{-3}	$3.1844e-3$	2.00
	2^{-4}	$7.4297e-4$	2.00		2^{-4}	$7.9355e-4$	2.00
	2^{-5}	$1.8563e-4$	2.00		2^{-5}	$1.9823e-4$	2.00
	2^{-6}	$4.6400e-5$	2.00		2^{-6}	$4.9547e-5$	2.00
	2^{-7}	$1.1598e-5$	no data		2^{-7}	$1.2386e-5$	no data

Table 2: The experimental error results and convergence order in space direction (the last column) with $\tau = 2^{-14}$

References

[1] A. Delić, B.S. Jovanović, Numerical approximation of an interface problem for fractional in time diffusion equation, Appl. Math. Comput. 229 (2014) 467–479.

[2] V.J. Ervin, J.P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numer. Meth. P.D.E 23(2006) 558–576.

[3] Y. Fujita, Integrodifferential equation which interpolates the heat equation and wave equation, Osaka J. Math. 27(1990) 309–321,797–804.

[4] B.S. Jovanović, E. Süli, Analysis of Finite Difference Schemes, Springer, 2014.

[5] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science and Technology, Boston, 2006.

[6] X. Li, C. Xu, A Space-Time Method for the Time Fractional Diffusion Equation, SIAM J. Numer. Anal. 47(2009) 2108–2131.

[7] J.L. Lions, E. Magenes, Non homogeneous boundary value problems and applications, Springer-Verlag, Berlin and New York, 1972.
[8] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, Appl. Math. Lett. 9(1996) 23–28.
[9] K.B. Oldham, J. Spanier, The fractional calculus, Academic Press, New York, 1974.
[10] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[11] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives and Some of Their Applications, Minsk, Nauka i Tekhnika, 1987 (in Russian).
[12] A.A. Samarskii, R.D. Lazarov and V.L. Makarov, Difference schemes for differential equations with generalized solutions, Moscow, Vysshaya shkola, 1987 (in Russian).
[13] A.A. Samarskii, The Theory of Difference Schemes, Marcel Dekker, 2001.
[14] W.R. Schneider, W. Wess, Fractional diffusion and wave equations, J. Math. Phys. 213(2006), 205–213.
[15] Z.Z. Sun, X. Wu, A fully discrete difference scheme for a diffusion-wave system, Appl. Numer. Math. 56(2006) 193–209.
[16] C. Tadjeran, M.M. Meerschaert, H.-P. Scheffler, A second-order accurate numerical approximation for the fractional diffusion equation, J. Math. Phys. 213(2006) 205–213.