Filomat 30:6 (2016), 1387–1411
DOI 10.2298/FIL1606387S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Lie Algebras of Infinitesimal CR Automorphisms of Weighted Homogeneous and Homogeneous CR-Generic Submanifolds of $\mathbb{C}^N$

Masoud Sabzevari[a], Amir Hashemi[b], Benyamin M.-Alizadeh[c], Joël Merker[d]

[a] *Department of Pure Mathematics, University of Shahrekord, 88186-34141 Shahrekord, IRAN and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), 19395-5746, Tehran, IRAN*
[b] *Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, IRAN and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, 19395-5746, IRAN*
[c] *School of Mathematics and Computer Sciences, Damghan University, 3671641167, Damghan, IRAN*
[d] *Départment de Mathématiques d'Orsay, Bâtiment 425, Faculté des Sciences, Université Paris XI - Orsay, F-91405 Orsay Cedex, FRANCE*

Abstract. We consider the significant class of homogeneous CR manifolds represented by some weighted homogeneous polynomials and we derive some plain and useful features which enable us to set up a fast effective algorithm to compute homogeneous components of their Lie algebras of infinitesimal CR automorphisms. This algorithm mainly relies upon a natural gradation of the sought Lie algebras, and it also consists in treating *separately* the related graded components. While classical methods are based on constructing and solving some associated PDE systems which become time consuming as soon as the number of variables increases, the new method presented here is based on plain techniques of linear algebra. Furthermore, it benefits from a *divide-and-conquer* strategy to break down the computations into some simpler subcomputations. Also, we consider the new and effective concept of comprehensive Gröbner systems which provides us some powerful tools to treat the computations in the much complicated parametric case. The designed algorithm is also implemented in the Maple software, what required also implementing a recently designed algorithm of Kapur *et al*.

1. Introduction

Let $M \subset \mathbb{C}^{n+k}$ be a Cauchy-Riemann (CR for short) submanifold of CR dimension $n \geqslant 1$ and of codimension $k \geqslant 1$ (*see* §2 for all pertinent definitions used in this introduction) represented in coordinates z_j and $w_l := u_l + iv_l$ for $j = 1, \ldots, n$ and $l = 1, \ldots, k$. As is standard in the terminology of CR geometry ([3, 5, 6, 10]), one may often assign the *weight* $[z_j] := 1$ to all the complex variables z_j and $[w_l] \in \mathbb{N}$ with $1 < [w_1] \leqslant [w_2] \leqslant \cdots [w_k]$ to the variables $w_1, \ldots, w_k$. Accordingly, the weight of the conjugation of each complex variable and of its real and imaginary parts as well are all equal to that of the variable and, moreover, the assigned weight of any constant number $a \in \mathbb{C}$ and of coordinate vector fields are:

$$[a] := 0, \quad [\partial_{z_j}] := -[z_j] = -1, \quad [\partial_{w_l}] := -[w_l], \qquad {\scriptstyle (j=1,\ldots,n, \quad l=1,\ldots,k)}.$$

2010 *Mathematics Subject Classification*. Primary 32V40; Secondary 68W30
Keywords. CR manifolds, CR automorphism algebra, Comprehensive Gröbner system
Received: 10 April 2014; Accepted: 21 April 2015
Communicated by Miodrag Mateljević
The research of the first and second authors was in part supported by grants from IPM (No. 93510061 and No. 93550420). Also this work has been supported by center for international scientific studies and collaboration (CISSC) and French embassy in Tehran.
Email addresses: sabzevari@math.iut.ac.ir (Masoud Sabzevari), amir.hashemi@cc.iut.ac.ir (Amir Hashemi), B.M.Alizadeh@std.du.ac.ir (Benyamin M.-Alizadeh), merker@dma.ens.fr (Joël Merker)

Furthermore, the weight of a monomial F in the $z_j, w_k, \bar{z}_j, \overline{w}_k$ is the sum taken over the weights of all variables of F with regards to their powers and also, the weight of each coordinate vector field of the form $F\,\partial_x$ with $x = z_i, w_l$ is defined as $[F] - [x]$. For instance, we have $[(a\,z\,w_1^2)\,\partial_{w_2}] = [z] + 2[w_1] - [w_2]$. A polynomial or a vector field is called *weighted homogeneous of weight d* whenever each of its terms is of homogeneity d.

As is well-known (*see* [24, Theorem 2.12]), every real analytic generic CR manifold M of CR dimension n and codimension k can be represented locally in a neighborhood of the origin as the graph of k defining equations of the form:

$$\begin{cases} v_1 := \Phi_1(z, \bar{z}, u) + \mathrm{o}([w_1]), \\ \quad\vdots \\ v_k := \Phi_k(z, \bar{z}, u) + \mathrm{o}([w_k]), \end{cases} \tag{1}$$

where the weight of all variables z_j is 1 and the weights of the variables w_l are the Hörmander numbers of M at the origin. Moreover, each function Φ_l is a certain weighted homogeneous polynomial of the weight $[w_l]$ and $\mathrm{o}(t)$ denotes remainder terms having weights $> t$.

For a CR manifold M passing through the origin, the Lie group $\mathsf{Aut}_{CR}(M)$ is the holomorphic symmetry group of M, that is the local Lie group of local biholomorphisms mapping M to itself. The Lie algebra $\mathfrak{aut}_{CR}(M)$, associated to this group is called the *Lie algebra of infinitesimal CR automorphisms* of M and it consists of all holomorphic vector fields — $(1,0)$ fields with holomorphic coefficients — whose real parts are tangent to M. Due to the fact that many geometric features of CR manifolds can be investigated by means of their associated Lie algebras of infinitesimal CR automorphisms and because of central applications in Cartan geometry and in Tanaka theory exist, studying such algebras has gained an increasing interest during the recent decades (*cf.* [5, 26] and in particular the results of §12 of [25]). As is known, the Lie algebra $\mathfrak{aut}_{CR}(M)$ is finite dimensional if and only if M is holomorphically nondegenerate and of finite type ([3, 15, 16, 24, 38]).

Consider the complex space $\mathbb{C}^{n+k}$ equipped with the coordinates $z_1, \ldots, z_n, w_1, \ldots, w_k$, where $w_j := u_j + iv_j$, assume again that certain weights have been assigned, and consider homogeneous (in Lie theory's sense) CR manifolds $M \subset \mathbb{C}^{n+k}$ represented as graphs of k certain polynomials:

$$M := \left\{ (z, w) : \left[\begin{array}{l} \Xi_1(v_1, z, \bar{z}, u) := v_1 - \Phi_1(z, \bar{z}, u) \equiv 0, \\ \Xi_2(v_2, z, \bar{z}, u) := v_2 - \Phi_2(z, \bar{z}, u) \equiv 0, \\ \quad\vdots \\ \Xi_j(v_j, z, \bar{z}, u) := v_j - \Phi_j(z, \bar{z}, u) \equiv 0, \\ \quad\vdots \\ \Xi_k(v_k, z, \bar{z}, u) := v_k - \Phi_k(z, \bar{z}, u) \equiv 0, \end{array} \right. \right\}, \tag{2}$$

with the right-hand sides Φ_j being weighted homogeneous polynomials of weight equal to $[v_j] = [w_j]$ of the left-hand sides. Recall that a CR manifold M is called *homogenous* whenever its associated automorphism group $\mathrm{Aut}_{CR}(M)$ is locally transitive near the origin. Since it may arise some confusion with the 'homogeneous spaces' terminology, let us stress that we will always use *weighted homogeneous* about functions, and plainly *homogeneous* about $\mathrm{Aut}_{CR}(M)$.

The so introduced class of CR-generic manifolds is already an extremely wide class in CR geometry which includes of course well known quadric CR models such as those of Poincaré [30] or of Chern-Moser [11], and also, there is nowadays an extensive literature dealing with constructing a great number of such weighted homogeneous CR manifolds (*see* for example [3, 9, 20, 37] and [5]–[8]), their associated Lie groups being far from being completely understood.

Indeed, in a series of recent papers (for instance [5]–[8]), Valerii Beloshapka studied extensively the subject of *model surfaces* and found some considerable results in this respect. Specifically in [5], he introduced and established the structure of some nondegenerate models associated (uniquely) to totally nondegenerate germs having arbitrary CR dimensions and codimensions. Each of Beloshapka's models $M \subset \mathbb{C}^{n+k}$ of certain CR dimension and codimension n and k enjoys some *nice* properties ([5], page 484, Theorem 14) which have been encouraging enough to merit further investigation. In particular, computing their Lie algebras of infinitesimal CR automorphisms and studying their structures may reveal some interesting features of these CR models, and also of all totally nondegenerate CR manifolds corresponding to them (*cf.* [5, 25]).

It is worth noting that, traditionally the subject of computing Lie algebras of infinitesimal CR automorphisms is concerned with expensive computations and the cost of calculations increases as much as the number of the variables — namely the dimension or codimension of the CR manifolds — increases. Indeed, solving the PDE systems arising during these computations (*see* subsection 2.1) forms the most complicated part of the procedure. That may be the reason why, in contrast to the importance of the subject, the number of the relevant computational works is still limited (one finds some of them in [8, 20, 26, 37]).

Very recently, the authors provided in [33] a new general algorithm to compute the desired algebras by means of the effective techniques of differential algebra. It enables one to use conveniently the ability of computer algebra for managing the associated computations of the concerning PDE systems. Although this (general) algorithm is able to decrease a lot the complexity of the computations and in particular to utilize systematically the ability of computer algebra, but *because of dealing with the* PDE *systems* — which are complicated in their spirit — the computations are still expensive in essence.

In the present paper we aim to study, by means of a *weight analysis approach and with an algorithmic treatment*, the intrinsic properties of the under consideration CR manifolds in order to provide an effective algorithm — entirely different and more powerful than that of [33] — to compute the associated Lie algebras of infinitesimal CR automorphisms. The obtained results are quite simple but actually enable us to bypass constructing and solving the arising systems of PDEs (which is the classical method of [8, 20, 25, 26, 33, 37]) and to reach the sought algebras by employing just simple techniques of linear algebra. This decreases considerably the cost of computations, hence simultaneously increases the performance of the algorithm.

We see that for a weighted homogeneous CR manifold M represented as (2), the sought algebra $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$ takes the graded (in the sense of Tanaka) form:

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0 \oplus \underbrace{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\varrho \oplus \cdots}_{\mathfrak{g}_+} \qquad \rho,\ \varrho \in \mathbb{N} \tag{3}$$

where each component $\mathfrak{g}_t$ is the Lie subalgebra of all weighted homogeneous vector fields having the weight t. Summarizing, the results provide us with the ways of:

- using a *divide-and-conquer* method to break down the computations of the sought (graded) algebra $\mathfrak{g}$ into some simpler sub-computations of its (homogeneous) components $\mathfrak{g}_t$;

- employing some simple techniques of linear algebra for computing these components $\mathfrak{g}_t$ of infinitesimal CR automorphisms *without* relying on solving the PDE systems;

We notice that the first assumption of weighted homogeneity for the generic CR manifolds as (2) provides us the opportunity of computing, separately, each of the above homogeneous components $\mathfrak{g}_t$ of the desired graded Lie algebras of infinitesimal CR automorphisms. On the other hand, we assumed that such manifolds are also homogeneous since this assumption enables us to find a simple criterion for how long it is necessary to compute the mentioned homogeneous components. In other words, it supplies us to recognize — in an algorithmic point of view — where the *maximum homogeneity* ϱ in (3) is, while the minimum homogeneity ρ is easy to determine. But of particular interest is the Beloshapka's universal models, where the first author in his recent paper [31] has showed that their graded infinitesimal CR automorphisms have no any nontrivial positive part $\mathfrak{g}_+$, *i.e.* $\varrho = 0$. Consequently, one observes that at least in this interesting case, our designed algorithm provides an appropriate fast algorithm to compute the desired Lie algebras.

One of the main — somehow hidden — obstacles appearing among the computations arises when the set of defining equations includes some certain *parametric* polynomials. This case is quite usual as one observes in [5, 8, 20, 37]. To treat such cases, we suggest the modern and effective concept of *comprehensive*

Gröbner systems [17, 18, 27, 28, 41] which enables us to consider and solve (linear) parametric systems appearing among the computations. Besides all advantages of computing the desired algebras in the general case, providing this algorithm in the (much complicated) parametric case can give us the opportunity of studying the modern concept of *moduli spaces of model CR manifolds*, introduced recently by Beloshapka (*cf.* [32]).

This paper is organized as follows. Section 2 presents a brief description of required very basic definitions, notations and terminology. In Section 3, we consider the under consideration manifolds and obtain some simple results which are, in fact, the key entrance to the desired algorithm. In Section 4, we employ the results of the previous section to provide the strategy of computing separately the homogeneous components of the sought algebras. We also provide the necessary criterion for terminating such computations. Finally, in section 5 we introduce briefly the modern concept of comprehensive Gröbner systems and show how it provides some effective tools to consider and solve appearing (linear) parametric systems.

The algorithm designed in this paper is implemented in Maple 15 as the library CRAut, accessible online as [34]. To do this, at first we needed to implement the algorithm PGB introduced in the recently published paper [17] which enables us to consider the parametric defining equations in CRAut.

2. Basic Preliminaries and Definitions

On an arbitrary even-dimensional real vector space V, a *complex structure map* $J : V \to V$ is an $\mathbb{R}$-linear map satisfying $J \circ J = -\mathrm{Id}_V$. For example, in the simple case $V := T_p\mathbb{R}^{2N} = T_p\mathbb{C}^N$, $N \in \mathbb{N}$ with the local coordinates $(z_1 := x_1 + i\, y_1, \ldots, z_N := x_N + i\, y_N)$ and with $p \in \mathbb{C}^N$, one defines the (standard) complex structure by:

$$J\Big(\frac{\partial}{\partial x_j}\Big) = \frac{\partial}{\partial y_j}, \qquad J\Big(\frac{\partial}{\partial y_j}\Big) = -\frac{\partial}{\partial x_j}, \qquad j = 1, \ldots, N.$$

One should notice that in general, an arbitrary subspace H_p of $T_p\mathbb{C}^N$ is not invariant under the complex structure map J. Thus, one may give special designation to the largest J-invariant subspace of H_p as:

$$H_p \cap J(H_p) =: H_p^c,$$

which is called the *complex tangent subspace of* H_p. Due to the equality $J \circ J = -\mathrm{Id}$, this space is even-dimensional, too.

Similarly, one also introduces the smallest J-invariant real subspace of $T_p\mathbb{C}^N$ which *contains* H_p:

$$H_p^{i_c} := H_p + J(H_p)$$

and one calls it the *intrinsic complexification* of H_p.

As an application, consider the linear subspace $H_p := T_pM$ of $\mathbb{C}^N$, for some arbitrary connected differentiable submanifold M of $\mathbb{C}^N$. In general, it is not at all true that the complex-tangent planes:

$$T_p^cM = T_pM \cap J(T_pM)$$

have constant dimensions as p varies in M.

Definition 2.1. *Let M be a real analytic submanifold of $\mathbb{C}^N$. Then M is called* Cauchy-Riemann *(CR for short), if the complex dimensions of T_p^cM are* constant *as p varies on M. Furthermore, M is called* generic *whenever:*

$$T_p^{i_c}M := T_pM + J(T_pM) = T_p\mathbb{C}^N$$

for each $p \in M$, which implies that M is CR thanks to elementary linear algebra.

One knows ([3, 10, 24]) that any CR real analytic $M \subset \mathbb{C}^N$ is contained in a thin strip-like complex submanifold $C' \cong \mathbb{C}^{N'}$ with $N' \leqslant N$ in which $M \subset \mathbb{C}^{N'}$ is CR-generic, hence there is no restriction to assume that M is CR-generic, as we will always do here, since we are interested in local CR geometry.

For such a CR-generic manifold M, we call the complex dimension of T_p^cM by the *CR-dimension* of M. Moreover, the subtraction of the real dimension of T_p^cM from that of T_pM, which is in fact the real dimension of the so-called *totally real part* T_pM/T_p^cM of T_pM, coincides then with the real *codimension* of $M \subset \mathbb{C}^N$.

A tangent vector field:

$$\mathsf{X} := \sum_{j=1}^{N} \left(a_j \frac{\partial}{\partial Z_j} + b_j \frac{\partial}{\partial \overline{Z}_j} \right)$$

of the complexified space $\mathbb{C}T_p\mathbb{C}^N := \mathbb{C} \otimes T_p\mathbb{C}^N$ is called of the type $(1,0)$ whenever all $b_j \equiv 0$ and is called of the type $(0,1)$ whenever all $a_j \equiv 0$. One denotes by $T_p^{1,0}\mathbb{C}^N$ and $T_p^{0,1}\mathbb{C}^N$ the corresponding subspaces of $\mathbb{C}T_p\mathbb{C}^N$. Accordingly, for a CR manifold M and an arbitrary point p of it, we denote:

$$T_p^{1,0}M := T_p^{1,0}\mathbb{C}^N \cap \mathbb{C}T_pM, \quad \text{and} \quad T_p^{1,0}M := T_p^{0,1}\mathbb{C}^N \cap \mathbb{C}T_pM.$$

One easily verifies the equality $T_p^{0,1}M = \overline{T_p^{1,0}M}$. It is proved that (*see* [3], Proposition 1.2.8) the complex tangent space T_p^cM is the real part of $T_p^{1,0}M$, *i.e.* $T_p^cM = \{\mathsf{X} + \overline{\mathsf{X}} : \mathsf{X} \in T_p^{1,0}M\}$. Moreover, the complexified space $\mathbb{C}T_p^cM$ is equal to the direct sum $T_p^{1,0}M \oplus T_p^{0,1}M$.

If n and k are the CR dimension and codimension of a real analytic CR-generic submanifold $M \subset \mathbb{C}^N$, then of course $N = n + k$, and also M can be represented (locally) by k real analytic graphed equations:

$$\operatorname{Im} w_j := \Phi_j(z, \overline{z}, \operatorname{Re} w), \qquad {\scriptstyle (j=1,\dots,k)} \tag{4}$$

with some real-valued defining functions $\Phi_\bullet$ enjoying the *no-pluriharmonic term* condition:

$$0 \equiv \Phi_\bullet(z, 0, \operatorname{Re} w) = \Phi_\bullet(0, \overline{z}, \operatorname{Re} w).$$

Solving the above real-valued defining functions in w or in $\overline{w}$, one also reformulates the defining functions of M as complex defining equations of the kind:

$$w_j + \overline{w}_j = \overline{\Xi}_j(z, \overline{z}, w), \qquad {\scriptstyle (j=1,\dots,k)}. \tag{5}$$

For a real analytic CR-generic manifold M represented by the above defining functions as (5) and for each $p \in M$, it is well-known ([25]) that the space $T_p^{0,1}M$ is generated by the following holomorphic vector fields, tangent to M:

$$\mathcal{L}_j := \frac{\partial}{\partial \overline{z}_j} + \sum_{l=1}^{k} \frac{\partial \overline{\Xi}_l}{\partial \overline{z}_j}(z, \overline{z}, w) \frac{\partial}{\partial \overline{w}_l} \qquad {\scriptstyle (j=1,\dots,n)}.$$

Definition 2.2. *A CR-generic submanifold $M \subset \mathbb{C}^{n+k}$ of CR dimension n and of codimension k is called* of finite type *at a point $p \in M$ whenever the above generators $\mathcal{L}_1, \dots, \mathcal{L}_n, \overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_n$ together with all of their Lie brackets of any length span the complexified tangent space $\mathbb{C}T_pM$ at the point p.*

Of course, finite-typeness at a point is an open condition. In [9], Bloom and Graham introduced an effective method to construct homogeneous CR manifolds that we now explain briefly. Consider the complex space $\mathbb{C}^{n+k}$ equipped with the variables $z_1, \dots, z_n, w_1 := u_1 + iv_1, \dots, w_k := u_k + iv_k$, assign weight 1 to the variables z_i and assign some arbitrary weights ℓ_j to each w_j for $j = 1, \dots, k$. Then a CR manifold

$M \subset \mathbb{C}^{n+k}$ is called *represented in Bloom-Graham normal form* [9, 10] whenever it is defined as the graph of some real-valued functions[1)]:

$$v_j = \Phi_j(z, \overline{z}, u_1, \ldots, u_{j-1}) + \mathrm{o}(\ell_j) \qquad {\scriptstyle (j=1,\ldots,k)}, \tag{6}$$

where each Φ_j is a weighted homogeneous polynomial of the weight ℓ_j enjoying the following two statements:

(*i*) there are no *pure* terms $z^\alpha u^\beta$ or $\overline{z}^\alpha u^\beta$ among the polynomials $\Phi_\bullet$ for some integers α and β;

(*ii*) for each $1 \leqslant j < i$ and for any nonnegative integers $\alpha_1, \ldots, \alpha_j$, the polynomial Φ_i does not include any term of the form $u_1^{\alpha_1} \cdots u_j^{\alpha_j} \Phi_j$.

Every CR manifold represented in this form is of finite type (*see* [10] page 181). Bloom and Graham also showed that every CR manifold represented by the above expressions (6) can be transformed to such a normal form by means of some algebraic changes of coordinates (*see* [9], Theorem 6.2).

Definition 2.3. *A real analytic CR-generic manifold $M \subset \mathbb{C}^N$ with coordinates $(Z_1, \ldots, Z_N)$ is called* holomorphically nondegenerate *at $p \in M$ if there is no local* nonzero *vector field of type $(1, 0)$:*

$$\mathsf{X} := \sum_{j=1}^{N} f_j(Z_1, \ldots, Z_N) \frac{\partial}{\partial Z_j}$$

having coefficients f_j holomorphic in a neighborhood of p such that $\mathsf{X}|_M$ is tangent to M near p.

Every *connected* real analytic generic CR manifold is either holomorphically nondegenerate at every point or at no point ([24]).

Definition 2.4. *([1, 3, 8, 26]) A (local)* infinitesimal CR automorphism *of M, when understood extrinsically, is a local holomorphic vector field:*

$$\mathsf{X} = \sum_{i=1}^{n} Z^i(z, w) \frac{\partial}{\partial z_i} + \sum_{j=1}^{k} W^j(z, w) \frac{\partial}{\partial w_j} \tag{7}$$

whose real part $\mathrm{Re}\, \mathsf{X} = \frac{1}{2}(\mathsf{X} + \overline{\mathsf{X}})$ *is tangent to M.*

The collection of all infinitesimal CR automorphisms of M constitutes a Lie algebra which is called the *Lie algebra of infinitesimal CR automorphisms* of M and is denoted by $\mathfrak{aut}_{CR}(M)$.

The notion of holomorphically nondegeneracy was raised by Nancy Stanton in [38] where she proved that for a *hypersurface* $M \subset \mathbb{C}^{n+1}$ (always generic), $\mathfrak{aut}_{CR}(M)$ is finite-dimensional if and only if M is holomorphically nondegenerate. Amazingly enough, one realizes that the concept of tangent vector fields completely independent of $\overline{Z}_1, \ldots, \overline{Z}_N$ which points out a strong degeneracy can in fact be traced back at least to Sophus Lie's works (*cf.* pp. 13–14 of [22]). In general codimension $k \geqslant 1$, the Lie algebra $\mathfrak{aut}_{CR}(M)$ of infinitesimal CR automorphisms of a CR-generic real analytic $M \subset \mathbb{C}^{n+k}$ is finite-dimensional if and only if M is holomorphically nondegenerate *and* of finite type ([15]).

Determining such Lie algebras $\mathfrak{aut}_{CR}(M)$ is the same as knowing the *CR-symmetries* of M, a question which lies at the heart of the (open) problem of classifying all local analytic CR manifolds up to biholomorphisms. In the groundbreaking works of Sophus Lie and his followers (Friedrich Engel, Georg Scheffers, Gerhard Kowalewski, Ugo Amaldi and others), the most fundamental question in concern was to draw up lists of possible Lie algebras $\mathfrak{aut}_{CR}(M)$ which would classify all possible M's according to their CR symmetries.

1) There is also a more general definition of Bloom-Graham normal form which one can find it for example in [4]

2.1. Infinitesimal CR automorphisms

Now, let us explain briefly the common strategy for computing the Lie algebra $\mathfrak{aut}_{CR}(M)$ associated to an arbitrary real analytic generic CR manifold $M \subset \mathbb{C}^{n+k}$, represented as the graph of the k complex defining equations as (5) (*cf.* [8, 20, 25, 33, 37]).

According to definition, a holomorphic vector field:

$$\mathsf{X} = \sum_{j=1}^{n} Z^j(z,w)\,\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\,\partial_{w_l}$$

belongs to $\mathfrak{aut}_{CR}(M)$ whenever it enjoys the tangency equations:

$$\begin{aligned}
0 \equiv (\overline{\mathsf{X}} + \mathsf{X})\Big[\overline{w}_j + w_j - \overline{\Xi}_j(\overline{z},z,w)\Big] \\
&= \overline{\mathsf{X}}\Big[\overline{w}_j + w_j - \overline{\Xi}_j(\overline{z},z,w)\Big] + \mathsf{X}\Big[\overline{w}_j + w_j - \overline{\Xi}_j(\overline{z},z,w)\Big] \\
&= \overline{W}^j(\overline{z},\overline{w}) - \sum_{i=1}^{n} \overline{Z}^i(\overline{z},\overline{w})\,\frac{\partial\overline{\Xi}_j}{\partial\overline{z}_i}(\overline{z},z,w) \\
&\quad + W^j(z,w) - \sum_{i=1}^{n} Z^i(z,w)\,\frac{\partial\overline{\Xi}_j}{\partial z_i}(\overline{z},z,w) - \sum_{l=1}^{k} W^l(z,w)\,\frac{\partial\overline{\Xi}_j}{\partial w_l}(\overline{z},z,w) \\
&\qquad (j = 1 \cdots k).
\end{aligned} \tag{8}$$

For each $j = 1,\dots,k$, let us refer to the above equality as the *j-th tangency equation* of M. Now, the Taylor series formulas:

$$Z^i(z,w) = \sum_{\alpha\in\mathbb{N}^n} z^\alpha\, Z^i_\alpha(w) \quad \text{and} \quad W^l(z,w) = \sum_{\alpha\in\mathbb{N}^n} z^\alpha\, W^l_\alpha(w), \tag{9}$$

bring the tangency equations into the form:

$$\begin{aligned}
0 \equiv \sum_{\alpha\in\mathbb{N}^n} \overline{z}^\alpha\, \overline{W}^j_\alpha\Big(-w+\overline{\Xi}\Big) - \sum_{k=1}^{n}\sum_{\alpha\in\mathbb{N}^n} \overline{z}^\alpha\, \overline{Z}^k_\alpha\Big(-w+\overline{\Xi}\Big)\,\frac{\partial\overline{\Xi}_j}{\partial\overline{z}_k}(\overline{z},z,w) \\
+ \sum_{\beta\in\mathbb{N}^n} z^\beta\, W^j_\beta(w) - \sum_{k=1}^{n}\sum_{\beta\in\mathbb{N}^n} z^\beta\, Z^k_\beta(w)\,\frac{\partial\overline{\Xi}_j}{\partial z_k}(\overline{z},z,w) - \sum_{l=1}^{k}\sum_{\beta\in\mathbb{N}^n} z^\beta\, W^l_\beta(w)\,\frac{\partial\overline{\Xi}_j}{\partial w_l}(\overline{z},z,w) \\
(j = 1 \cdots k).
\end{aligned} \tag{10}$$

After this, one replaces the appearing functions $\overline{W}^\bullet_\bullet(-w+\overline{\Xi})$ and $\overline{Z}^\bullet_\bullet(-w+\overline{\Xi})$ according to the following slightly artificial expansions:

$$\begin{aligned}
\overline{A}\Big(-w+\overline{\Xi}\Big) &= \overline{A}\Big(w + (-2w+\overline{\Xi})\Big) \\
&= \sum_{\gamma\in\mathbb{N}^d} \frac{\partial^{|\gamma|}\overline{A}}{\partial w_\gamma}(w)\,\frac{1}{\gamma!}\Big(-2w+\overline{\Xi}(\overline{z},z,w)\Big)^\gamma,
\end{aligned}$$

and one next substitutes each $-2w_j + \overline{\Xi}_j$ with:

$$-2w_j + \overline{\Xi}_j(\overline{z},z,w) = \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \overline{z}^\alpha\, z^\beta\, \overline{\Xi}_{j,\alpha,\beta}(w) \qquad (j = 1 \cdots k).$$

Modifying the equations (10) by the two already presented formulaes, we reach k homogenous equations so that their right hand sides are some certain combinations of the (yet unknown) functions $Z^i_\alpha(w)$ and $W^l_\beta(w)$, their derivations and also the variables $w_1, \ldots, w_k$ with the coefficients of the form $z^\mu \overline{z}^\nu$ for $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$ (here we set $z_\mu := z_1^{\mu_1} \ldots z_n^{\mu_n}$). To satisfy these equations, one should extract the coefficients of the various monomials $z^\mu \overline{z}^\nu$ and equate them to zero. Then, one attains a (usually lengthy and complicated) homogeneous linear system of complex partial differential equations with the unknowns $Z^i_\alpha(z, w)$ and $W^l_\alpha(z, w)$ — namely a linear system of differential polynomials of the differential algebra $R := \mathbb{C}(w, \overline{w})[Z^i_\alpha, W^l_\beta]$. The solution of this system yields the desired coefficients $Z^i(z, w)$ and $W^l(z, w)$ in (9). To solve this already constructed PDE system, we employed in [33] the effective tools of *differential algebra*, equipped with some additional operators such as *bar-reduction*. The provided algorithm manages to compute *systematically* the desired algebras associated to *arbitrary* CR manifolds $M \subset \mathbb{C}^{n+k}$. Moreover, it employs the powerful techniques of differential algebra and the ability of computer algebra to provide a more effective method. In fact, it handles more appropriately the most complicated part of the computations, namely solving the associated PDE systems (*cf.* [8, 20, 25, 26, 37]). Specifically, for the significant class of *rigid* CR manifolds — those whose defining equations Ξ are independent of the variables w — the computations are considerably eased up. *However*, the main obstacle we encounter is that, because of the complexity of the PDE systems, as much as the number of variables z_i and w_l increases, then the cost of the associated computation grows extensively and the implementation of the algorithm rapidly reveals limits concerning the capacity of computer systems.

3. Useful Results

Consider the complex space $\mathbb{C}^{n+k}$ equipped with n complex variables $z_1, \ldots, z_n$ of weight one, identically, and k complex variables $w_1 := u_1 + iv_1, \ldots, w_k := u_k + iv_k$ of certain weights $1 < [w_1] \leqslant [w_2] \leqslant \ldots \leqslant [w_k]$ and define the homogeneous manifold M of CR dimension n and codimension k as:

$$M := \left\{ (z, w) : \begin{bmatrix} \Xi_1(v_1, z, \overline{z}, u) := v_1 - \Phi_1(z, \overline{z}, u) \equiv 0, \\ \Xi_2(v_2, z, \overline{z}, u) := v_2 - \Phi_2(z, \overline{z}, u) \equiv 0, \\ \vdots \\ \Xi_j(v_j, z, \overline{z}, u) := v_j - \Phi_j(z, \overline{z}, u) \equiv 0, \\ \vdots \\ \Xi_k(v_k, z, \overline{z}, u) := v_k - \Phi_k(z, \overline{z}, u) \equiv 0, \end{bmatrix} \right\}, \tag{11}$$

with the weighted homogeneous polynomials Φ_j of the certain weight $[w_j]$ for $j = 1, \ldots, k$. Additionally, we also assume that these manifolds are holomorphically nondegenerate and of finite type, which guarantees the finite dimensionality of their associated algebras of infinitesimal CR automorphisms. Notice passim that these two assumptions are made for convenience and even without these assumptions one can still employ the designed algorithm to compute the homogeneous components of the under consideration CR manifolds.

Before proceeding to design our desired algorithm, we first need some plain observations and facts. These simple results lead us to discover the key entrance to the algorithm.

3.1. Gradation and polynomiality

At first, let us show two intrinsic features of the already mentioned weighted homogeneous CR manifolds, namely gradation (in the sense of Tanaka) and polynomiality of their associated algebras of infinitesimal CR automorphisms.

For a CR manifold M as above, consider the holomorphic vector field:

$$\mathsf{X} := \sum_{j=1}^{n} Z^j(z, w)\partial_{z_j} + \sum_{l=1}^{k} W^l(z, w)\partial_{w_l}$$

as an element of $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$. Since the above coefficients Z^j and W^l are all holomorphic, then one can expand them as their Taylor series and thus decompose X into its weighted homogeneous components as follows:

$$\mathsf{X} := \mathsf{X}_{-\rho} + \cdots + \mathsf{X}_{-1} + \mathsf{X}_0 + \mathsf{X}_1 + \cdots + \mathsf{X}_t + \cdots, \qquad {\scriptstyle (\rho,\, t \in \mathbb{N})}. \tag{12}$$

We need the following facts:

Lemma 3.1. *The minimum homogeneous component $\mathsf{X}_{-\rho}$ in the above decomposition of X is of the weight $-\rho = -[w_k]$, where w_k has the maximum homogeneity among the complex variables appearing in* (11)*.*

Proof. It is just enough to observe that the tangent space of holomorphic vector fields can be generated by the standard fields $\partial_{z_j}, \partial_{w_l}$ of the certain weights $-[z_j]$ and $-[w_l]$ for $j = 1, \ldots, n$ and $l = 1, \ldots, k$. Among these standard fields, the minimum homogeneity belongs to $\frac{\partial}{\partial w_k}$. This completes the proof. □

Lemma 3.2. *Each of the above weighted homogeneous components X_t, $t \geqslant -\rho$ is again an infinitesimal CR automorphism, namely belongs to $\mathfrak{g}$.*

Proof. Since X is an infinitesimal CR automorphism then we have:

$$0 \equiv \big(\mathsf{X} + \overline{\mathsf{X}}\big)|_{\Xi_j}, \qquad {\scriptstyle (j=1,\ldots,k)}.$$

Now, for each component X_t of homogeneity t and since each defining function Ξ_j is homogeneous of weight $\ell_j := [w_j]$ then one verifies that the polynomial $(\mathsf{X}_t + \overline{\mathsf{X}}_t)|_{\Xi_j}$ is either zero or a homogeneous polynomial of weight $t + \ell_j$. Hence we have:

$$0 \equiv \big(\mathsf{X} + \overline{\mathsf{X}}\big)|_{\Xi_j} = \underbrace{\big(\mathsf{X}_{-\rho} + \overline{\mathsf{X}}_{-\rho}\big)|_{\Xi_j}}_{P_{\rho,j}} + \ldots + \underbrace{\big(\mathsf{X}_t + \overline{\mathsf{X}}_t\big)|_{\Xi_j}}_{P_{t,j}} + \cdots \qquad {\scriptstyle (j=1,\ldots,k)}, \tag{13}$$

in which each polynomial function $P_{t,j}$ is either zero or a homogeneous polynomial of the weight $t + \ell_j$. Hence, we have some certain weighted homogeneous polynomials $P_\bullet$ with distinct homogeneities and consequently with linear independency. Hence, one obtains from (13) that:

$$0 \equiv P_{t,j}(z, w) = \big(\mathsf{X}_t + \overline{\mathsf{X}}_t\big)|_{\Xi_j}, \quad {\scriptstyle t \geqslant -\rho, \quad (j=1,\ldots,k)}.$$

This equivalently means that each component X_t of X is an infinitesimal CR automorphism. □

Now, we can prove the polynomiality of the sought algebras:

Corollary 3.3. *If:*

$$\mathsf{X}_t := \sum_{j=1}^{n} Z_t^j(z, w)\partial_{z_j} + \sum_{l=1}^{k} W_t^l(z, w)\partial_{w_l}$$

is a weighted homogeneous CR automorphism of weight $t \geqslant -\rho$ then, its coefficients $Z_t^j(z, w)$ and $W_t^l(z, w)$ are weighted homogeneous polynomials of the weights $t + 1$ and $t + [w_l]$.

Proof. It is a straightforward consequence of the decomposition (12). □

Now, let us consider the gradation of $\mathfrak{g}$. First we need the following definition:

Definition 3.4. *The Lie algebra $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$ of an arbitrary CR manifold is called* graded in the sense of Tanaka, *whenever it can be expressed in the form:*

$$\mathfrak{g} := \mathfrak{g}_{-\rho} \oplus \mathfrak{g}_{-\rho+1} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{\varrho}, \qquad {\scriptstyle \rho,\varrho\in\mathbb{N}}$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Furthermore, we say that M has rigidity *if the positive subalgebra:*

$$\mathfrak{g}_+ := \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{\varrho}$$

of $\mathfrak{aut}_{CR}(M)$ is trivial.

Proposition 3.5. *For a finite type holomorphically nondegenerate CR manifold $M \subset \mathbb{C}^{n+k}$ represented by the above defining function* (11)*, the associated Lie algebra $\mathfrak{g}$ of infinitesimal CR automorphisms is a graded Lie algebra, in the sense of Tanaka, of the form:*

$$\mathfrak{g} := \mathfrak{g}_{-\rho} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{\varrho}, \quad {\scriptstyle \rho,\varrho\in\mathbb{N}}.$$

Proof. According to the above two lemmas, if $\mathfrak{g}_t$ is the collection of infinitesimal CR automorphisms of weight t, then $\mathfrak{g}$ admits a gradation like:

$$\mathfrak{g} := \mathfrak{g}_{-\alpha} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t \oplus \cdots .$$

Furthermore, holomorphically nondegeneracy and finite typness of M guarantees that $\mathfrak{g}$ is finite dimensional and hence there exists an integer ϱ in which $g_\beta \equiv 0$ for each $\beta > \varrho$. Now, it remains to show that this gradation is in the sense of Tanaka. Namely, we shall prove that for each two homogeneous infinitesimal CR automorphisms X_1 and X_2 of certain homogeneities t_1 and t_2, the Lie bracket $[\mathsf{X}_1, \mathsf{X}_2]$ belongs to $\mathfrak{g}_{t_1+t_2}$. For this, it is enough to show this statement for vector fields of the forms $\mathsf{X}_i := F_i(z,w)\frac{\partial}{\partial x_i}, i = 1, 2$ where x_i is a complex variable $z_\bullet$ or $w_\bullet$. According to the homogeneities of these vector fields, F_1 and F_2 are two polynomials of the weights $t_1 + [x_1]$ and $t_2 + [x_2]$, respectively. Now, we have:

$$[\mathsf{X}_1, \mathsf{X}_2] = F_1 \frac{\partial F_2}{\partial x_1} \frac{\partial}{\partial x_2} - F_2 \frac{\partial F_1}{\partial x_2} \frac{\partial}{\partial x_1}.$$

In this expression, the derivations $\frac{\partial F_2}{\partial x_1}$ and $\frac{\partial F_1}{\partial x_2}$ are either zero or homogeneous polynomials of the weights $t_2 + [x_2] - [x_1]$ and $t_2 + [x_1] - [x_2]$, respectively. Now, simple simplifications yields that the above Lie bracket is a homogeneous vector field of the weight $t_1 + t_2$, as desired. □

For a CR manifold $M \subset \mathbb{C}^{n+k}$, of *fixed* CR-dimension n and codimension k represented in coordinates $z_1, \ldots, z_n, w_1, \ldots, w_k$ as (11), consider the infinitesimal CR automorphism:

$$\mathsf{X} := \sum_{j=1}^{n} Z^j(z,w)\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\partial_{w_l}.$$

Finding the basis elements of the Lie algebra $\mathfrak{aut}_{CR}(M)$, one should naturally find the explicit expressions of the appearing coefficient functions $Z(z,w)$ and $W(z,w)$. After necessary computations, these coefficients will be found as certain polynomials with some arbitrary coefficients, what we call them in this paper by the free parameters. For example, according to [25] (*see* also the case $K = 3$ of [37], Theorem 1 for another presentation), an infinitesimal CR automorphism for the CR manifold $M \subset \mathbb{C}^{1+3}$, represented in coordinates $\mathbb{C}\{z, w_1, w_2, w_3\}$ as the graph of the defining functions:

$$\begin{cases} w_1 - \overline{w}_1 = 2i\, z\overline{z} \\ w_2 - \overline{w}_2 = 2i\, z\overline{z}(z + \overline{z}) \\ w_3 - \overline{w}_3 = 2\, z\overline{z}(z - \overline{z}) \end{cases} \tag{14}$$

is a holomorphic vector field:

$$\mathsf{X} := Z(z,w)\,\partial_z + \sum_{l=1}^{3} W^l(z,w)\,\partial_{w_l},$$

with the desired coefficients:

$$\begin{aligned}
Z(z,w) &= \mathsf{a} + i\,\mathsf{b} + (\mathsf{d} + i\,\mathsf{e})\,z,\\
W^1(z,w) &= \mathsf{c}_1 + 2\,(\mathsf{b} + i\,\mathsf{a})\,z + 2\,\mathsf{d}\,w_1,\\
W^2(z,w) &= \mathsf{c}_2 + 2\,(\mathsf{b} + i\,\mathsf{a})\,z^2 + 4\,\mathsf{a}\,w_1 + 3\,\mathsf{d}\,w_2 - \mathsf{e}\,w_3,\\
W^3(z,w) &= \mathsf{c}_3 + 2\,(\mathsf{a} - i\,\mathsf{b})\,z^2 + 4\,\mathsf{b}\,w_1 + \mathsf{e}\,w_2 + 3\,\mathsf{d}\,w_3,
\end{aligned} \tag{15}$$

for some seven real integers, namely free parameters $\mathsf{a}, \mathsf{b}, \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3, \mathsf{d}, \mathsf{e}$. Then, the Lie algebra $\mathfrak{aut}_{CR}(M)$ is seven dimensional, with the basis elements extracted as the coefficients of these parameters in the above general form of X:

$$\begin{aligned}
\boxed{\mathsf{a}}\ &: \quad \mathsf{X}_1 := \partial_z + 2\,i\,z\partial_{w_1} + (2\,i\,z^2 + 4\,w_1)\,\partial_{w_2} + 2\,z^2\,\partial_{w_3},\\
\boxed{\mathsf{b}}\ &: \quad \mathsf{X}_2 := i\,\partial_z + 2\,z\,\partial_{w_1} + 2\,z^2\,\partial_{w_2} + (-2\,i\,z^2 + 4\,w_1)\,\partial_{w_3},\\
\boxed{\mathsf{c}_1}\ &: \quad \mathsf{X}_3 := \partial_{w_1},\\
\boxed{\mathsf{c}_2}\ &: \quad \mathsf{X}_4 := \partial_{w_2},\\
\boxed{\mathsf{c}_3}\ &: \quad \mathsf{X}_5 := \partial_{w_3},\\
\boxed{\mathsf{d}}\ &: \quad \mathsf{X}_6 := z\,\partial_z + 2\,w_1\,\partial_{w_1} + 3\,w_2\,\partial_{w_2} + 3\,w_3\,\partial_{w_3},\\
\boxed{\mathsf{e}}\ &: \quad \mathsf{X}_7 := i\,z\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}.
\end{aligned} \tag{16}$$

The weights associated to the appearing complex variables are:

$$[z] = 1, \quad [w_1] = 2, \quad [w_2] = [w_3] = 3.$$

Hence, a glance at the obtained basis holomorphic vector fields $\mathsf{X}_1, \ldots, \mathsf{X}_7$ gives the following weighted homogeneities for them:

X	X_1	X_2	X_3	X_4	X_5	X_6	X_7
Hom	-1	-1	-2	-3	-3	0	0

Therefore, the Lie algebra $\mathfrak{aut}_{CR}(M)$ can be represented as:

$$\mathfrak{aut}_{CR}(M) := \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

with $\mathfrak{g}_{-3} := \langle \mathsf{X}_4, \mathsf{X}_5 \rangle$, with $\mathfrak{g}_{-2} := \langle \mathsf{X}_3 \rangle$, with $\mathfrak{g}_{-1} := \langle \mathsf{X}_1, \mathsf{X}_2 \rangle$ and with $\mathfrak{g}_0 := \langle \mathsf{X}_6, \mathsf{X}_7 \rangle$.

3.2. Homogeneous components

Now, let us inspect the structure of homogeneous components of the desired algebras of infinitesimal CR automorphisms. First, let us consider those CR manifolds which have rigidity.

Proposition 3.6. *Let M be a holomorphically nondegenerate CR manifold of CR-dimension n and codimension k represented as* (11)*. Then, M has rigidity if and only if for any weighted homogeneous infinitesimal CR automorphism:*

$$\mathsf{X} := \sum_{j=1}^{n} Z^j(z,w)\,\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\,\partial_{w_l}$$

of M, each weighted homogeneous polynomial $Z^j(z,w)$ (respectively $W^l(z,w)$) is of weight at most $[z_i]$ (respectively of weight at most $[w_l]$). In particular, $Z^j(z,w)$ (respectively $W^l(z,w)$) is independent of the variables of the weights $\gneqq [z_j]$ (respectively of the weights $\gneqq [w_l]$).

Proof. If X is of weight homogeneity d, then since the standard fields ∂_{z_j} and ∂_{w_l} have the constant weights $-[z_i]$ and $-[w_l]$, respectively, then the weighted homogeneous polynomials Z^j and W^l have the constant degrees $d + [z_i]$ and $d + [w_l]$, respectively. Now, assume that M has rigidity. The dimension of the Lie algebra $\mathfrak{aut}_{CR}(M)$ is equal to the number of free parameters involved in the expressions of the functions $Z^j(z,w), i = 1,\ldots,n$ and $W^l(z,w), l = 1,\ldots,k$. Each generator is extracted from one of such free parameters as its coefficient in the general form $\mathsf{X} := \sum_{j=1}^n Z^j(z,w)\,\partial_{z_j} + \sum_{l=1}^k W^l(z,w)\,\partial_{w_l}$. In such expression, coefficients of the standard fields ∂_{z_j} come from the found functions $Z^j(z,w)$. Now, rigidity of M means to have no any (homogeneous) basis element belonging to $\mathfrak{aut}_{CR}(M)$ of the positive weighted homogeneity. Hence, when we have standard field ∂_{z_j} of the weight $-[z_j]$, then no term of weight bigger than $[z_j]$ appears in its coefficient. Consequently, $Z^j(z,w)$ — which provides the coefficients of ∂_{z_j} in the basis elements — is independent of the variables of the weights bigger than $[z_j]$. Similar fact holds when one considers the coefficients $W^l(z,w)$ of the standard fields ∂_{w_l}.
For the converse, if none of the (weighted homogeneous) holomorphic coefficients $Z^j(z,w), j = 1,\ldots,n$ admits the terms of weight larger than $[z_j]$, then the weight $[Z^j(z,w)] - [z_j]$ of each term $Z^j(z,w)\partial_{z_j}$ of X is non-positive. Similar fact holds for the terms $W^l(z,w)\partial_{w_l}, l = 1,\ldots,k$. Consequently, $\mathfrak{aut}_{CR}(M)$ does not contain any (weighted homogeneous) basis element X of the positive weight. In other words, M has rigidity. □

For a graded Lie algebra:

$$\mathfrak{g} = \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\varrho,$$

let us denote by $\mathfrak{g}^{(t)}$ the graded subspace:

$$\mathfrak{g}^{(t)} := \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_t, \quad {\scriptstyle (t=-\rho,\ldots,\varrho)}.$$

According to definition of the graded algebras, one easily convinces oneself that for $t = -\rho,\ldots,0$ each subspace $\mathfrak{g}^{(t)}$ is in fact a Lie subalgebra of $\mathfrak{g}$. The idea behind the proof of Proposition 3.6 can lead one to obtain the following more general conclusion.

Proposition 3.7. *Let M be a homogeneous CR manifold of CR-dimension n and codimension k represented as* (11). *Let $\mathfrak{g} = \mathfrak{aut}_{CR}(M)$ be of the graded form:*

$$\mathfrak{g} = \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\varrho \tag{17}$$

and let the (weighted homogeneous) infinitesimal CR automorphism:

$$\mathsf{X} = \sum_{j=1}^{n} Z^j(z,w)\,\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\,\partial_{w_l}$$

belongs to $\mathfrak{g}$. Then,

(*i*) X *belongs to $\mathfrak{g}_t$ for $t = -\rho,\ldots,\varrho$, if and only if each coefficient $Z^j(z,w)$ (respectively $W^l(z,w)$) is homogeneous of the precise weight $[z_j] + t$ (respectively $[w_l] + t$). In particular, each $Z^j(z,w)$ (respectively $W^l(z,w)$) is independent of the variables of weights $\gneqq [z_j] + t$ (respectively of weights $\gneqq [w_l] + t$).*

(*ii*) X *belongs to $\mathfrak{g}^{(t)}$ if and only if each coefficient $Z^j(z,w)$ (respectively $W^l(z,w)$) is homogeneous of weight at most $[z_j] + t$ (respectively of weights at most $[w_l] + t$). Specifically, each function $Z^j(z,w)$ (respectively, $W^l(z,w)$) is independent of the variables of weights $\gneqq [z_j] + t$) (respectively of weights $\gneqq [w_l] + t$)*

(*iii*) *In particular, the negative part $\mathfrak{g}_- = \mathfrak{g}^{(-1)}$ of the CR manifold M is generated by the elements X of $\mathfrak{g}$ with the coefficients $Z^j(z,w)$ (respectively $W^l(z,w)$) independent of the variables of weights $\gneqq [z_j] - 1$ (respectively of weights $\gneqq [w_l] - 1$).*

Proof. The proof is similar to that of Proposition 3.6. Here, we prove the first item (i). If a holomorphic vector field X belongs to $\mathfrak{g}_t$, then it is homogeneous of the weight t. In the expression of X, each standard field ∂_{z_j} is of the fixed weight $-[z_j]$ and hence having the field $Z^j(z,w)\,\partial_{z_j}$ of the precise homogeneity t, then the coefficient $Z^j(z,w)$ must be of the weigh $[z_j]+t$. Similar conclusion holds for the functions $W^l(z,w)$. The converse of the assertion can be concluded in a very similar way. In particular, since all the variables in $Z^j(z,w)$ have the positive weight then, each function $Z^j(z,w)$ — which must be of the weight $[z_j]+t$ — is independent of the variables of the weights bigger than $[z_j]+t$. □

4. Computing the Homogeneous Components

According to Proposition 3.7, to compute the holomorphic coefficients $Z^j(z,w)$ and $W^l(z,w)$ of the vector fields $\mathsf{X}\in\mathfrak{g}^{(t)}$, one can assume these functions independent of the variables with the associated weights larger than $[z_j]+t$ and $[w_l]+t$, respectively. In this section, we aim to develop this result for constructing a very convenient method of computing each subspace $\mathfrak{g}^{(t)}$ and each component $\mathfrak{g}_t$ associated to our weighted homogeneous CR manifolds.

4.1. Computing each component $\mathfrak{g}_t$

For each element X of the t-th component $\mathfrak{g}_t$, Proposition 3.7 enables one to attain an upper bound for the weight degree of each of its desired (polynomial) coefficients $Z^j(z,w)$ and $W^l(z,w)$. Hence, we can predict the expression of these polynomials as the elements of the polynomial ring $\mathbb{C}[z,w]$. Then, finding these expressions, it is necessary and sufficient to seek their constant coefficients. One can pick the following convenient strategy for computing the t-th component $\mathfrak{g}_t$ of the desired algebra $\mathfrak{g}$ as (17), for a fixed integer $t=-\rho,\ldots,\varrho$.

(s1) First, we construct the tangency equations (8) of M corresponding to (in general form) holomorphic vector fields:

$$\mathsf{X}=\sum_{j=1}^{n} Z^j(z,w)\,\partial_{z_j}+\sum_{l=1}^{k} W^l(z,w)\,\partial_{w_l}$$

of $\mathfrak{g}_t\subset\mathfrak{g}$.

(s2) Now, to compute the coefficients $Z^j(z,w)$ and $W^l(z,w)$, it is no longer necessary to use the Taylor series (9) and construct and solve the arising PDE systems as is the classical method of [8, 20, 26, 33, 37]. Here there is another, entirely different and much simpler way to proceed the computation. Namely by Corollary 3.3, all desired functions $Z^j(z,w)$ and $W^l(z,w)$ are polynomials with bounded degrees. Then, according to Proposition 3.7(i), the desired coefficients $Z^j(z,w)$ and $W^l(z,w)$ are weighted homogeneous polynomials of the precise weights $[z_j]+t$ and $[w_l]+t$, respectively. Hence, we can assume the following expressions:

$$\boxed{\begin{aligned} Z^j(z,w) &:= \sum_{\substack{\alpha\in\mathbb{N}^n\\ \beta\in\mathbb{N}^k\\ [z^\alpha]+[w^\beta]=[z_j]+t}} \mathsf{c}_{\alpha,\beta}\,.\,z^\alpha\,w^\beta, \\ W^l(z,w) &:= \sum_{\substack{\alpha\in\mathbb{N}^n\\ \beta\in\mathbb{N}^k\\ [z^\alpha]+[w^\beta]=[w_l]+t}} \mathsf{d}_{\alpha,\beta}\,.\,z^\alpha\,w^\beta, \qquad {\scriptstyle (i=1,\ldots,n,\quad l=1,\ldots,k)}, \end{aligned}} \tag{18}$$

for some (unknown yet) complex *free parameters* $\mathsf{c}_{\alpha,\beta}$ and $\mathsf{d}_{\alpha,\beta}$. Here, by z^α we mean $z_1^{\alpha_1}\,z_2^{\alpha_2}\ldots z_n^{\alpha_n}$ for $\alpha:=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$. Furthermore, we have:

$$[z^\alpha]=\sum_{j=1}^{n}[z_j^{\alpha_j}]=\sum_{j=1}^{n}\alpha_j\,[z_j].$$

Similar notations hold for w^β and $[w^\beta]$.

(s3) Determining the parameters $\mathsf{c}_{\alpha,\beta}$ and $\mathsf{d}_{\alpha,\beta}$ is in fact equivalent to find the explicit expressions of the CR automorphisms X of $\mathfrak{g}_t$. For this, we should just put the already assumed expressions (18) in the k tangency equations (8) and next solve the extracted (*not* PDE) homogeneous linear system of equations with the unknowns $\mathsf{c}_{\alpha,\beta}$ and $\mathsf{d}_{\alpha,\beta}$, and with the equations obtained as the coefficients of the various powers $z^\alpha w^\beta$ of the variables z and w.

Let us denote by $\mathsf{Sys}^{t,j}$ the system of equations, mentioned in the step (**s3**), associated to a weighted homogeneous CR manifold M of CR dimension and codimension n and k, extracted from the j-th tangency equation for $j = 1, \dots, k$ along the way of computing the t-th component $\mathfrak{g}_t$ of $\mathfrak{g} = \mathfrak{aut}_{CR}(M)$. Furthermore, we denote by Sys^t the general system of equations:

$$\mathsf{Sys}^t := \bigcup_{j=1}^{k} \mathsf{Sys}^{t,j}.$$

Definition 4.1. *A graded Lie algebra of the form:*

$$\mathfrak{g}_- := \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_{-1}, \quad \rho \in \mathbb{N}$$

is called fundamental *whenever it can be generated by* $\mathfrak{g}_{-1}$.

In the case that the negative part $\mathfrak{g}_-$ of the desired algebra $\mathfrak{g}$ is fundamental, it is even not necessary to compute the homogeneous components $\mathfrak{g}_{-\rho}, \dots, \mathfrak{g}_{-2}$. In this case, one can easily compute the $(-t)$-th components $\mathfrak{g}_{-t}$ inductively as the length t iterated Lie brackets $\mathfrak{g}_{-t} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-t+1}]$. There are many situations where this occurs. For example, all Beloshapka's models enjoy it (see [5], Proposition 4).

In the case that $\mathfrak{g}_-$ is fundamental, one plainly can add the following item to the already presented strategy **(s1)-(s3)**.

(ib) After computing the basis elements of $\mathfrak{g}_{-1}$, to achieve each component $\mathfrak{g}_{-m}$, $m = 2, \dots, \rho$ one should just compute all the length m **i**terated **b**rackets like:

$$[\mathsf{x}_{i_1}, [\mathsf{x}_{i_2}, [\mathsf{x}_{i_3}, \dots, [\mathsf{x}_{i_m}]]] \dots], \qquad {}_{i_1 < i_2 < \cdots < i_m}$$

of the basis elements $\mathsf{x}_\bullet$ of $\mathfrak{g}_{-1}$.

Example 4.2. *Consider again the CR-submanifold $M \subset \mathbb{C}^{1+3}$ of CR-dimension $n = 1$ and codimension $k = 3$ represented as the graph of the defining polynomials* (14). *The Lie algebra $\mathfrak{aut}_{CR}(M)$ is computed in [37] and [25], using the classical method of constructing and solving the arising* PDE *system. Here, we use our new and simple method. According to* (8), *the three fundamental tangency equations are:*

$$\begin{aligned}
0 &\equiv \Big[W^1 - \overline{W}^1 - 2i\bar{z}Z - 2iz\overline{Z}\Big]_{w=\overline{w}+\Xi(z,\bar{z},\overline{w})} \qquad (19)\\
0 &\equiv \Big[W^2 - \overline{W}^2 - 4iz\bar{z}Z - 2i\bar{z}^2Z - 2iz^2\overline{Z} - 4iz\bar{z}\overline{Z}\Big]_{w=\overline{w}+\Xi(z,\bar{z},\overline{w})}\\
0 &\equiv \Big[W^3 - \overline{W}^3 - 4z\bar{z}Z - 2z^2\overline{Z} + 2\bar{z}^2Z + 4z\bar{z}\overline{Z}\Big]_{w=\overline{w}+\Xi(z,\bar{z},\overline{w})}.
\end{aligned}$$

First let us compute the subalgebra $\mathfrak{g}_{-1}$. For this aim, we may set the following expressions for the unknown functions $Z(z, w_1, w_2, w_3)$, $W^l(z, w_1, w_2, w_3)$ for $l = 1, 2, 3$ with their homogeneities at their left hand sides (cf. (18)*):*

$\boxed{1 + (-1)}$: $Z(z, w) := \mathsf{p}$,

$\boxed{2 + (-1)}$: $W^1(z, w) := \mathsf{q}\, z$,

$\boxed{3 + (-1)}$: $W^2(z, w) := \mathsf{r}\, w_1 + \mathsf{s}\, z^2$,

$\boxed{3 + (-1)}$: $W^3(z, w) := \mathsf{t}\, w_1 + \mathsf{u}\, z^2$, *for some six complex functions* $\mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{s}, \mathsf{t}, \mathsf{u}$. *Putting these expressions into the tangency equations and equating to zero the coefficients of the appeared polynomials of* $\mathbb{C}[z, w_1, w_2, w_3]$, *we obtain the following three systems of linear homogeneous equations:*

$$\mathsf{Sys}^{-1,1} = \Big\{ -2\,i\,\overline{\mathsf{p}} + \mathsf{q} = 0, \quad -2\,i\,\mathsf{p} - \overline{\mathsf{q}} = 0 \Big\},$$
$$\mathsf{Sys}^{-1,2} = \Big\{ \mathsf{s} - 2\,i\,\overline{\mathsf{p}} = 0, \quad -2\,\mathsf{p} - 2\,\overline{\mathsf{p}} + \mathsf{r} = 0, \quad 2\,\mathsf{p} + 2\,\overline{\mathsf{p}} - \mathsf{r} = 0 \Big\},$$
$$\mathsf{Sys}^{-1,3} = \Big\{ \mathsf{u} - 2\,\overline{\mathsf{p}} = 0, \quad -2\,i\,\overline{\mathsf{p}} + 2\,i\,\mathsf{p} + \mathsf{t} = 0 \Big\}.$$

Solving the homogeneous linear system Sys^{-1} *of all the above equations, we can write:*

$$\mathsf{p} := \mathsf{a} + i\,\mathsf{b}, \quad \mathsf{q} = \mathsf{s} := 2(\mathsf{b} + i\,\mathsf{a}), \quad \mathsf{r} := 4\,\mathsf{a}, \quad \mathsf{t} := 4\,\mathsf{b}, \quad \mathsf{u} := 2(\mathsf{a} - i\,\mathsf{b}),$$

for two real constants a *and* b *which brings the following general expressions for the desired holomorphic coefficients of the elements* $\mathsf{X} \in \mathfrak{g}_{-1}$ *(compare with* (15)*):*

$$\begin{aligned}
Z(z,w) &= \mathsf{a} + i\,\mathsf{b}, \\
W^1(z,w) &= 2\,(\mathsf{b} + i\,\mathsf{a})\,z, \\
W^2(z,w) &= 2\,(\mathsf{b} + i\,\mathsf{a})\,z^2 + 4\,\mathsf{a}\,w_1, \\
W^3(z,w) &= 2\,(\mathsf{a} - i\,\mathsf{b})\,z^2 + 4\,\mathsf{b}\,w_1.
\end{aligned}$$

Thanks to the two free parameters a, b *appeared in the above expressions, we will have two infinitesimal CR automorphisms as the generators of* $\mathfrak{g}_{-1}$*:*

$$\begin{aligned}
\mathsf{X}_1 &:= \partial_z + 2\,i\,z\partial_{w_1} + (2\,i\,z^2 + +4\,w_1)\,\partial_{w_2} + 2\,z^2\,\partial_{w_3}, \\
\mathsf{X}_2 &:= i\,\partial_z + 2\,z\,\partial_{w_1} + 2\,z^2\,\partial_{w_2} + (-2\,i\,z^2 + 4\,w_1)\,\partial_{w_3}.
\end{aligned}$$

Now, we can follow the step (**ib**) *to seek the elements of the homogeneous components* $\mathfrak{g}_{-2}$ *and* $\mathfrak{g}_{-3}$ *by computing the iterated Lie brackets of* X_1 *and* X_2 *(here notice that* M *is one of the Beloshapka's models and hence* $\mathfrak{g}$ *is fundamental. Moreover, since the maximum homogeneity of the appearing variables is* 3 *then, the minimum homogeneity in* $\mathfrak{g}$ *is* -3*). For* $\mathfrak{g}_{-2}$ *we have only one generator:*

$$\mathsf{X}_3 := [\mathsf{X}_1, \mathsf{X}_2] = 4\,\partial_{w_1}.$$

Then, $\mathfrak{g}_{-3}$ *includes two basis elements*

$$\begin{aligned}
\mathsf{X}_4 &:= [\mathsf{X}_1, \mathsf{X}_3] = -4\,\partial_{w_2}, \\
\mathsf{X}_5 &:= [\mathsf{X}_2, \mathsf{X}_3] = -4\,\partial_{w_3}.
\end{aligned}$$

At present we found $5 = 2 \times 1 + 3$ *basis elements for the subalgebra* $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. *Similarly, one achieves the zeroth component* $\mathfrak{g}_0$ *of* $\mathfrak{g}$. *For this, we may assume the following expressions for the functions* $Z(z, w_1, w_2, w_3)$, $W^l(z, w_1, w_2, w_3)$ *for* $l = 1, 2, 3$ *with their homogeneities at their left hand sides:*

$$\begin{aligned}
\boxed{1+0}: &\quad Z(z,w) := \mathsf{p}_1\,z, \\
\boxed{2+0}: &\quad W^1(z,w) := \mathsf{q}_1\,w_1 + \mathsf{q}_2\,z^2, \\
\boxed{3+0}: &\quad W^2(z,w) := \mathsf{r}_1\,w_2 + \mathsf{r}_2\,w_3 + \mathsf{r}_3\,z^3 + \mathsf{r}_4\,zw_1, \\
\boxed{3+0}: &\quad W^3(z,w) := \mathsf{s}_1\,w_2 + \mathsf{s}_2\,w_3 + \mathsf{s}_3\,z^3 + \mathsf{s}_4\,zw_1.
\end{aligned}$$

Again, substituting the recent expressions in the tangency equations (19) *and equating to zero the coefficients of the appeared polynomials of* $\mathbb{C}[z, w_1, w_2, w_3]$*, we get the following systems of equations:*

$$\mathsf{Sys}^{0,1} = \Big\{\mathsf{q}_2 = 0, \quad -\mathsf{p}_1 - \overline{\mathsf{p}_1} + \mathsf{q}_1 = 0\Big\},$$

$$\mathsf{Sys}^{0,2} = \Big\{\mathsf{r}_3 = 0, \quad -\mathsf{p}_1 + \overline{\mathsf{p}_1} + \mathsf{r}_4 - \overline{\mathsf{r}_4} = 0, \quad \overline{\mathsf{r}_4} = 0, \quad \mathsf{r}_1 - \mathsf{p}_1 - 2\,\overline{\mathsf{p}_1} + \overline{\mathsf{r}_4} = 0, \quad \mathsf{r}_2 = 0\Big\},$$

$$\mathsf{Sys}^{0,3} = \Big\{\mathsf{s}_3 = 0, \quad \mathsf{s}_4 - \overline{\mathsf{s}_4} = 0, \quad \overline{\mathsf{s}_4} = 0, \quad \overline{\mathsf{s}_4} + \mathsf{s}_1 + \frac{i}{2}\,\mathsf{p}_1 - \frac{i}{2}\,\overline{\mathsf{p}_1} = 0, \quad -\overline{\mathsf{s}_4} - \overline{\mathsf{s}_1} - \frac{i}{2}\,\mathsf{p}_1 + \frac{i}{2}\,\overline{\mathsf{p}_1} = 0, \quad -\frac{3}{2}\,\mathsf{p}_1 - \frac{3}{2}\,\overline{\mathsf{p}_1} + \mathsf{s}_2 = 0\Big\}.$$

Solving the linear homogeneous system Sys^0 *of all the above equations, we receive the solution:*

$$\mathsf{p}_1 = \mathsf{d} + i\mathsf{e}, \quad \mathsf{q}_1 = 2\,\mathsf{d}, \quad \mathsf{r}_1 = \mathsf{s}_2 = 3\,\mathsf{d}, \quad \mathsf{r}_2 = -\mathsf{e}, \quad \mathsf{s}_1 = \mathsf{e},$$
$$\mathsf{q}_2 = \mathsf{r}_3 = \mathsf{r}_4 = \mathsf{s}_3 = 0,$$

which implies the following expressions for the desired functions:

$$\begin{aligned} Z(z,w) &= (\mathsf{d} + i\,\mathsf{e})\,z, \\ W^1(z,w) &= 2\,\mathsf{d}\,w_1, \\ W^2(z,w) &= 3\,\mathsf{d}\,w_2 - \mathsf{e}\,w_3, \\ W^3(z,w) &= \mathsf{e}\,w_2 + 3\,\mathsf{d}\,w_3, \end{aligned}$$

where d *and* e *are two real constants. Extracting the coefficients of these two integers brings the following two tangent vector fields, belonging to* $\mathfrak{g}_0$*:*

$$\begin{aligned} \mathsf{X}_6 &:= z\,\partial_z + 2\,w_1\,\partial_{w_1} + 3\,w_2\,\partial_{w_2} + 3\,w_3\,\partial_{w_3}, \\ \mathsf{X}_7 &:= i\,z\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}. \end{aligned}$$

Remark 4.3. *It is worth noting that for computing each subspace* $\mathfrak{g}^{(t)}$*, although one can achieve its basis elements by computing the corresponding components* $\mathfrak{g}_s$*,* $t = -\rho, \ldots, t$*, it is also possible to adopt the above strategy* **(s1)**-**(s3)** *by modifying the assumed expressions of the functions* $Z^j(z,w)$ *and* $W^l(z,w)$ *as follows (cf. Proposition 3.7(ii)):*

$$\begin{aligned} Z^j(z,w) &:= \sum_{\substack{\alpha\in\mathbb{N}^n \\ \beta\in\mathbb{N}^k \\ [z^\alpha]+[w^\beta]\leqslant[z_j]+t}} \mathsf{c}_{\alpha,\beta}\,.\,z^\alpha\,w^\beta, \\ W^l(z,w) &:= \sum_{\substack{\alpha\in\mathbb{N}^n \\ \beta\in\mathbb{N}^k \\ [z^\alpha]+[w^\beta]\leqslant[w_l]+t}} \mathsf{d}_{\alpha,\beta}\,.\,z^\alpha\,w^\beta, \qquad {\scriptstyle (i=1,\ldots,n, \quad l=1,\ldots,k)}. \end{aligned} \tag{20}$$

4.2. Finding the maximum homogeneity

For an arbitrary homogeneous CR manifold, represented as (2), so far we have provided an effective way to compute homogeneous components $\mathfrak{g}_t$ of the graded desired algebra $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$ of the form:

$$\mathfrak{g} := \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\varrho.$$

We also know that the value of ρ in this gradation is equal to the maximum weight $[w_k]$ appearing among the complex variables. The only *not-yet-fixed* problem is to answer how much we have to compute the homogeneous components $\mathfrak{g}_t$ to arrive at the last one $\mathfrak{g}_\varrho$, in the case that $\mathfrak{g}$ is finitely generated or equivalently where M is holomorphically nondegenerate of finite type. Here, we do not aim to find the precise value of ϱ but — in an algorithmic point of view — it suffices to find a criterion to stop the computations. For this aim, we employ the transitivity of the Lie algebra $\mathfrak{g}$. In fact, it is for this reason we assume in this paper that the under consideration CR manifolds are homogeneous, in Lie theory's sense.

Definition 4.4. *A graded Lie algebra $\mathfrak{g}$ as above is called* transitive *whenever for each element $\mathsf{x} \in \mathfrak{g}_t$ with $t \geqslant 0$, the equality $[\mathsf{x}, \mathfrak{g}_-] = 0$ implies that $\mathsf{x} = 0$. In the case that $\mathfrak{g}_-$ is fundamental then, the transitivity means that for any x as above, the equality $[\mathsf{x}, \mathfrak{g}_{-1}] = 0$ implies that $\mathsf{x} = 0$.*

Proposition 4.5. *Consider a transitive graded algebra $\mathfrak{g}$ as above. For each integer $t \geqslant 0$, if $\mathfrak{g}_t = \mathfrak{g}_{t+1} = \cdots = \mathfrak{g}_{t+\rho-1} \equiv 0$ then we have $\mathfrak{g}_{t+\rho} = 0$. Moreover, if $\mathfrak{g}$ is also fundamental then the equality $\mathfrak{g}_t = 0$ implies independently that $\mathfrak{g}_{t+1} = 0$.*

Proof. Assume the following gradation for the transitive algebra $\mathfrak{g}$:

$$\mathfrak{g} := \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{t-1} \oplus \underbrace{0}_{\mathfrak{g}_t} \oplus \underbrace{0}_{\mathfrak{g}_{t+1}} \oplus \ldots \oplus \underbrace{0}_{\mathfrak{g}_{t+\rho-1}} \oplus \mathfrak{g}_{t+\rho} \oplus \ldots$$

and let $\mathsf{x} \in \mathfrak{g}_{t+\rho}$. According to the inequality $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, we have:

$$\begin{aligned}
&[\mathsf{x}, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{t+\rho-1} = 0, \\
&[\mathsf{x}, \mathfrak{g}_{-2}] \subset \mathfrak{g}_{t+\rho-2} = 0, \\
&\qquad \vdots \\
&[\mathsf{x}, \mathfrak{g}_{-\rho}] \subset \mathfrak{g}_{t+\rho-\rho} = 0,
\end{aligned}$$

which implies that $[\mathsf{x}, \mathfrak{g}_-] = 0$. Now, the transitivity of $\mathfrak{g}$ immediately implies that $\mathsf{x} = 0$. For the second part of the assertion, similarly assume the following gradation for $\mathfrak{g}$:

$$\mathfrak{g} := \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{t-1} \oplus \underbrace{0}_{\mathfrak{g}_t} \oplus \mathfrak{g}_{t+1} \oplus \ldots,$$

and let $\mathsf{x} \in \mathfrak{g}_{t+1}$. Consequently we have:

$$[\mathsf{x}, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{t+1-1} = \mathfrak{g}_t = 0.$$

Again, the definition of fundamental transitive algebras immediately implies that $\mathsf{x} = 0$. This completes the proof. □

Accordingly, for a homogeneous CR manifold M and to realize how much we have to compute the homogeneous components of $\mathfrak{g}$ to arrive at the last weighted homogeneous component $\mathfrak{g}_\varrho$ we can apply the following plain strategy:

- **When $\mathfrak{g}_-$ is fundamental.** Compute the homogeneous components $\mathfrak{g}_t$ of $\mathfrak{g}$ as much as it appears the first trivial component.
- **When $\mathfrak{g}_-$ is not fundamental.** Compute the homogeneous components $\mathfrak{g}_t$ of $\mathfrak{g}$ as much as they appear ρ successive trivial components.

Example 4.6. *In Example 4.2, we computed the negative part $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and also zeroth component $\mathfrak{g}_0$ of the graded algebra $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$ associated to the presented homogeneous CR manifold $M \subset \mathbb{C}^{1+3}$ (it is known from [14] that M is homogeneous). Here, let us finalize computation of the desired algebra. Proceeding further in this direction, now let us compute the next component $\mathfrak{g}_1$. In this case, we can set the following expressions for four desired coefficients $Z(z,w)$ and $W^l(z,w)$ $l = 1,2,3$, with their weighted homogeneities at their left hand sides (cf.* (18)*)*

$$\begin{aligned}
\boxed{1+1}: &\quad Z(z,w) := \mathsf{a}_1\, w_1 + \mathsf{a}_2\, z^2, \\
\boxed{2+1}: &\quad W^1(z,w) := \mathsf{a}_3\, w_2 + \mathsf{a}_4\, w_3 + \mathsf{a}_5\, z^3 + \mathsf{a}_6\, z w_1, \\
\boxed{3+1}: &\quad W^2(z,w) := \mathsf{a}_7\, w_1^2 + \mathsf{a}_8\, z^2 w_1 + \mathsf{a}_9\, z^4 + \mathsf{a}_{10}\, z w_2 + \mathsf{a}_{11}\, z w_3, \\
\boxed{3+1}: &\quad W^3(z,w) := \mathsf{a}_{12}\, w_1^2 + \mathsf{a}_{13}\, z^2 w_1 + \mathsf{a}_{14}\, z^4 + \mathsf{a}_{15}\, z w_2 + \mathsf{a}_{16}\, z w_3.
\end{aligned}$$

Now, we should check these predefined expressions in the tangency equations (19). *This gives us the total system* $\mathsf{Sys}^1 = \bigcup_{j=1}^{3} \mathsf{Sys}^{1,j}$ *as follows:*

$$
\mathsf{Sys}^1 = \left\{
\begin{array}{l}
\mathsf{a}_5 = 0, \quad i\Big(\mathsf{a}_6 + \mathsf{a}_3 - \mathsf{a}_2 = 0\Big) + \mathsf{a}_4 = 0, \quad i\Big(\mathsf{a}_3 - \overline{\mathsf{a}}_2\Big) - \mathsf{a}_4 + 2\,\mathsf{a}_1 = 0, \quad -2i\,\overline{\mathsf{a}}_1 + \mathsf{a}_6 = 0, \\
\overline{\mathsf{a}}_5 = 0, \quad -2i\,\mathsf{a}_1 - \overline{\mathsf{a}}_6 = 0, \quad \mathsf{a}_3 - \overline{\mathsf{a}}_3 = 0, \quad \mathsf{a}_4 - \overline{\mathsf{a}}_4 = 0, \quad \mathsf{a}_9 = 0, \quad i\Big(\mathsf{a}_{10} - 2\,\mathsf{a}_2 + \mathsf{a}_8\Big) + \mathsf{a}_{11} = 0, \\
i\Big(\mathsf{a}_{10} - \mathsf{a}_2 - \overline{\mathsf{a}}_2\Big) + 8\,\mathsf{a}_1 - 2\,\mathsf{a}_{11} - 4\,\mathsf{a}_7 = 0, \quad -2i\,\overline{\mathsf{a}}_1 + \mathsf{a}_8 = 0, \quad \mathsf{a}_1 - i\,\overline{\mathsf{a}}_2 = 0, \quad \mathsf{a}_{10} = 0, \\
\mathsf{a}_{11} = 0, \quad i\Big(\mathsf{a}_7 - \mathsf{a}_1 - \overline{\mathsf{a}}_1 = 0\Big) = 0, \quad \overline{\mathsf{a}}_9 = 0, \quad -\overline{\mathsf{a}}_8 - 2i\,\mathsf{a}_1 = 0, \quad \overline{\mathsf{a}}_{10} = 0, \quad \overline{\mathsf{a}}_{11} = 0, \quad \mathsf{a}_{14} = 0, \\
\mathsf{a}_7 - \overline{\mathsf{a}}_7 = 0, \quad i\Big(\mathsf{a}_{13} + \mathsf{a}_{15}\Big) + \mathsf{a}_{16} - 2\,\mathsf{a}_2 = 0, \quad -2\,\mathsf{a}_{12} + \mathsf{a}_2 - \mathsf{a}_{16} - \overline{\mathsf{a}}_2 + i\Big(\mathsf{a}_{15} - 4\,\mathsf{a}_1\Big) = 0, \\
\mathsf{a}_{13} - 2\,\overline{\mathsf{a}}_1 = 0, \quad \overline{\mathsf{a}}_2 + i\,\mathsf{a}_1 = 0, \quad i\,\mathsf{a}_{12} - \mathsf{a}_1 + \overline{\mathsf{a}}_1 = 0, \quad \mathsf{a}_{15} = 0, \quad \mathsf{a}_{16} = 0, \quad \overline{\mathsf{a}}_{14} = 0, \\
-\overline{\mathsf{a}}_{13} + 2\,\mathsf{a}_1 = 0, \quad -\overline{\mathsf{a}}_{15} = 0, \quad -\overline{\mathsf{a}}_{16} = 0, \quad \mathsf{a}_{12} - \overline{\mathsf{a}}_{12} = 0
\end{array}
\right\}.
$$

This system has only the trivial solution $\mathsf{a}_1 = \cdots = \mathsf{a}_{16} = 0$ *which means that we have* $\mathfrak{g}_1 = 0$. *Furthermore as we know, the desired algebra* $\mathfrak{g}$ *is fundamental which guarantees that the next components are trivial, too. Summing up the results of this example with those of Example 4.2, one finds the sought 7-dimensional Lie algebra of infinitesimal CR automorphisms associated to M as the gradation:*

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

with $\mathfrak{g}_{-3} = \langle \mathsf{X}_4, \mathsf{X}_5 \rangle$, *with* $\mathfrak{g}_{-2} = \langle \mathsf{X}_3 \rangle$, *with* $\mathfrak{g}_{-1} = \langle \mathsf{X}_1, \mathsf{X}_2 \rangle$, *with* $\mathfrak{g}_0 = \langle \mathsf{X}_6, \mathsf{X}_7 \rangle$ *and with the Lie commutators displayed in the following table:*

	X_5	X_4	X_3	X_2	X_1	X_6	X_7
X_5	0	0	0	0	0	$3\mathsf{X}_5$	$-\mathsf{X}_4$
X_4	*	0	0	0	0	$3\mathsf{X}_4$	X_5
X_3	*	*	0	$4\mathsf{X}_5$	$4\mathsf{X}_4$	$2\mathsf{X}_3$	0
X_2	*	*	*	0	$-4\mathsf{X}_3$	X_2	$-\mathsf{X}_1$
X_1	*	*	*	*	0	X_1	X_2
X_6	*	*	*	*	*	0	0
X_7	*	*	*	*	*	*	0

One observes that the achieved algebra is exactly that of (16).

One can find another computation of the Lie algebra achieved in the above example via the classical method of solving the arisen PDE system in [25]. Comparing the above process with that of [25] clarifies the effectiveness of the prepared algorithm.

4.3. Summing up the results

Here, let us gather the results obtained so far to provide an algorithm for computing the sought Lie algebras of infinitesimal CR automorphisms associated to the holomorphically nondegenerate homogeneous CR manifolds, represented as (11). The strategy introduced in subsection 4.1 enabled one to compute separately the homogeneous components $\mathfrak{g}_t$, $t = -\rho, \ldots, \varrho$ of the graded algebra $\mathfrak{g}$ of infinitesimal CR automorphisms of such CR manifolds as:

$$\mathfrak{g} := \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{\varrho}. \tag{21}$$

One may follow the following two points for computing the desired algebras associated to the under consideration class of CR manifolds:

Point 1 Executing three steps (**s1**) – (**s2**) – (**s3**) introduced in subsection 4.1 and finding homogeneous components $\mathfrak{g}_t$ successively. In particular if $\mathfrak{g}_-$ is fundamental then one can execute the step **(ib)**.

Point 2 In the above gradation, the minimum homogeneity $-\rho$ is equal to $[w_k]$ where w_k has the maximum homogeneity among the complex variables appearing in (11). Moreover, it suffices to compute successively the homogeneous components $\mathfrak{g}_0, \mathfrak{g}_1, \ldots$ as much as we find ρ successive trivial algebras. In particular if $\mathfrak{g}_-$ is fundamental, we can terminate the computations as much as we find first trivial component.

Let us conclude this section by computing Lie algebras of infinitesimal CR automorphisms associated to the following CR manifold:

$$\mathcal{M} := \begin{cases} w_1 - \overline{w}_1 = 2i\, z\overline{z}, \\ w_2 - \overline{w}_2 = 2i\, z\overline{z}\big(z + \overline{z}\big), \\ w_3 - \overline{w}_3 = 2i\, z\overline{z}\big(z^2 + \frac{3}{2}\, z\overline{z} + \overline{z}^2\big). \end{cases} \tag{22}$$

It may be worth noting to state that this CR manifold — which admits some interesting features [23, 25] — does not belong to the class of CR models introduced by Beloshapka [5].

Example 4.7. *To compute the Lie algebra $\mathfrak{g} := \mathfrak{aut}_{CR}(\mathcal{M})$ associated to the CR manifold $\mathcal{M}$ defined as* (22)*, first one should notice that we have the following assigned weights to the complex variables:*

$$[z] = 1, \quad [w_1] = 2, \quad [w_2] = 3, \quad [w_3] = 4.$$

Hence the minimum homogeneity of the homogeneous components will be $-\rho = -4$. Here, an infinitesimal CR automorphism is of the form:

$$\mathsf{X} := Z(z,w)\,\partial_z + W^1(z,w)\,\partial_{w_1} + W^2(z,w)\,\partial_{w_2} + W^3(z,w)\,\partial_{w_3},$$

enjoying the following three fundamental tangency equations:

$$\begin{aligned} 0 &\equiv \Big[W^1 - \overline{W}^1 - 2i\,\overline{z}Z - 2i\,z\overline{Z}\Big]_{\mathcal{M}}, \\ 0 &\equiv \Big[W^2 - \overline{W}^2 - 4i\,z\overline{z}Z - 2i\,\overline{z}^2 Z - 2i\,z^2\overline{Z} - 4i\,z\overline{z}\overline{Z}\Big]_{\mathcal{M}}, \\ 0 &\equiv \Big[W^3 - \overline{W}^3 - 6i\,z^2\overline{z}Z - 6i\,z\overline{z}^2 Z - 2i\,\overline{z}^3 Z - 2i\,z^3\overline{Z} - 6i\,z^2\overline{z}\overline{Z} - 6i\,z\overline{z}^2\overline{Z}\Big]_{\mathcal{M}}. \end{aligned} \tag{23}$$

Let us start by computing the negative part $\mathfrak{g}_-$. For the (-1)-th component $\mathfrak{g}_{-1}$, the sought coefficients are of the forms:

$$\begin{bmatrix} Z(z,w) := \mathsf{a}_1, \\ W^1(z,w) := \mathsf{a}_2\, z, \\ W^3(z,w) := \mathsf{a}_3\, z^2 + \mathsf{a}_4\, w_1, \\ W^4(z,w) := \mathsf{a}_5\, z^3 + \mathsf{a}_6\, zw_1 + \mathsf{a}_7\, w_2. \end{bmatrix}$$

Checking these predefined polynomials into the tangency equations (23) *gives the following system:*

$$\mathsf{Sys}^{-1} := \left\{ \begin{array}{l} \mathsf{a}_2 - 2i\,\overline{\mathsf{a}}_1 = 0 \quad -4i\,\mathsf{a}_1 - 4i\,\overline{\mathsf{a}}_1 + 2i\,\overline{\mathsf{a}}_4 = 0 \quad \mathsf{a}_3 - 2i\,\overline{\mathsf{a}}_1 = 0 \\ \mathsf{a}_4 - \overline{\mathsf{a}}_4 = 0 \quad \mathsf{a}_5 - 2i\,\overline{\mathsf{a}}_1 = 0 \quad \mathsf{a}_6 = 0 \quad \mathsf{a}_7 - \overline{\mathsf{a}}_7 = 0 \end{array} \right\}$$

which has the solution:

$$\begin{aligned} &\mathsf{a}_1 := \mathsf{a} + i\,\mathsf{b}, \quad \mathsf{a}_2 = 2\,\mathsf{b} + 2i\,\mathsf{a}, \quad \mathsf{a}_3 = 2\,\mathsf{b} + 2i\,\mathsf{a}, \quad \mathsf{a}_4 = 4\,\mathsf{a}, \\ &\mathsf{a}_5 = 2\,\mathsf{b} + 2i\,\mathsf{a}, \quad \mathsf{a}_6 = 0, \quad \mathsf{a}_7 = 6\,\mathsf{a}, \qquad (\mathsf{a},\mathsf{b} \in \mathbb{R}). \end{aligned}$$

Consequently, the sought homogeneous component $\mathfrak{g}_{-1}$ is 2-dimensional with the generators:

$$\begin{aligned} \mathsf{X}_1 &= \partial_z + 2iz\,\partial_{w_1} + 2iz^2\,\partial_{w_2} + 4w_1\,\partial_{w_2} + 2iz^3\,\partial_{w_3} + 6w_2\,\partial_{w_3}, \\ \mathsf{X}_2 &= i\,\partial_z + 2z\,\partial_{w_1} + 2z^2\,\partial_{w_2} + 2z^3\,\partial_{w_3}. \end{aligned}$$

Similar (and even simpler) computations give three vector fields:

$$\begin{aligned} \mathsf{X}_3 &:= \partial_{w_1}, \\ \mathsf{X}_4 &:= \partial_{w_2}, \\ \mathsf{X}_5 &:= \partial_{w_3}, \end{aligned}$$

of homogeneities $-2, -3, -4$*, respectively. Then, we have the negative part of the sought algebra as:*

$$\mathfrak{g}_- := \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

with $\mathfrak{g}_{-1} := \langle \mathsf{X}_1, \mathsf{X}_2 \rangle$*, with* $\mathfrak{g}_{-2} := \langle \mathsf{X}_3 \rangle$*, with* $\mathfrak{g}_{-3} := \langle \mathsf{X}_4 \rangle$ *and with* $\mathfrak{g}_{-4} := \langle \mathsf{X}_5 \rangle$*. One can see all the possible Lie commutators of these generators in the table presented below. From this table, one easily verifies that the negative part* $\mathfrak{g}_-$ *of* $\mathfrak{g}$ *is in fact fundamental. Hence, we have to compute nonnegative components as much as we encounter first trivial one. Now, we have to continue with computing* $\mathfrak{g}_0$*. In this case, the sought coefficients are of the form:*

$$\left[\begin{array}{l} Z := \mathsf{a}_1 z \\ W_1 := \mathsf{a}_2 w_1 + \mathsf{a}_3 z^2 \\ W_2 := \mathsf{a}_4 w_2 + \mathsf{a}_5 z^3 + \mathsf{a}_6 z w_1 \\ W_3 := \mathsf{a}_7 w_3 + \mathsf{a}_8 w_1^2 + \mathsf{a}_9 z^2 w_1 + \mathsf{a}_{10} z^4 + \mathsf{a}_{11} z w_2. \end{array}\right.$$

Checking these predefined functions in the tangency equations (19) *gives the following complex system:*

$$\mathsf{Sys}^0 = \left\{\begin{array}{l} \mathsf{a}_3 = 0, \quad -2i\,\mathsf{a}_1 - 2i\,\overline{\mathsf{a}}_1 + 2i\,\mathsf{a}_2 = 0, \quad -\overline{\mathsf{a}}_3 = 0, \quad -\overline{\mathsf{a}}_2 + \mathsf{a}_2 = 0, \quad \mathsf{a}_5 = 0, \\ -4i\,\mathsf{a}_1 - 2i\,\overline{\mathsf{a}}_1 + 2i\,\mathsf{a}_4 + 2i\,\mathsf{a}_6 = 0, \quad -4i\,\overline{\mathsf{a}}_1 - 2i\,\mathsf{a}_1 + 2i\,\mathsf{a}_4 = 0, \quad \mathsf{a}_6 - \overline{\mathsf{a}}_5 = 0, \quad -\overline{\mathsf{a}}_6 = 0, \\ \mathsf{a}_{10} - 2i\,\overline{\mathsf{a}}_1 + 2i\,\mathsf{a}_9 + 2i\,\mathsf{a}_7 + 2i\,\mathsf{a}_{11} - 6i\,\mathsf{a}_1 = 0, \quad 3i\,\mathsf{a}_7 + 2i\,\mathsf{a}_{11} - 4\mathsf{a}_8 - 6i\,\mathsf{a}_1 - 6i\,\overline{\mathsf{a}}_1 = 0, \\ \mathsf{a}_4 - \overline{\mathsf{a}}_4 = 0, \quad \mathsf{a}_9 = 0, \quad 2i\,\mathsf{a}_7 - 6i\,\overline{\mathsf{a}}_1 - 2i\,\mathsf{a}_1 = 0, \quad 4i\,\mathsf{a}_8 = 0, \quad \mathsf{a}_{11} = 0, \quad -\overline{\mathsf{a}}_{10} = 0, \\ -\overline{\mathsf{a}}_9 = 0, \quad -\overline{\mathsf{a}}_{11} = 0, \quad \mathsf{a}_8 - \overline{\mathsf{a}}_8 = 0, \quad \mathsf{a}_7 - \overline{\mathsf{a}}_7 = 0 \end{array}\right\}.$$

This system has the solution:

$$\mathsf{a}_1 = \mathsf{a}, \quad \mathsf{a}_2 = 2\,\mathsf{a}, \quad \mathsf{a}_4 = 3\,\mathsf{a}, \quad \mathsf{a}_7 = 4\,\mathsf{a}, \quad \mathsf{a}_3 = \mathsf{a}_5 = \mathsf{a}_6 = \mathsf{a}_8 = \mathsf{a}_9 = \mathsf{a}_{10} = \mathsf{a}_{11} \equiv 0$$

for some real number a*. Therefore,* $\mathfrak{g}_0$ *is 1-dimensional with the generator:*

$$\mathsf{X}_0 = z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 4w_3\,\partial_{w_3}.$$

This nonnegative component was not trivial; hence we have to proceed by computing the next component $\mathfrak{g}_1$*. Similar computations that we do not present them for saving space shows that this component is trivial. Then, according to the fundamentality of* $\mathfrak{g}_-$ *we can terminate the computations. Consequently, the sought graded algebra* $\mathfrak{g}$ *is of the form:*

$$\mathfrak{g} := \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

with the negative components as above, with $\mathfrak{g}_0 = \langle \mathsf{X}_0 \rangle$ *and with the table of commutators displayed as follows (see also subsection 3 of §6 of [29] for another computation of this Lie algebra):*

	X_0	X_1	X_2	X_3	X_4	X_5
X_0	0	$-\mathsf{X}_1$	$-\mathsf{X}_2$	$-2\mathsf{X}_3$	$-3\mathsf{X}_4$	$-4\mathsf{X}_5$
X_1	*	0	$-4\mathsf{X}_3$	$-4\mathsf{X}_4$	$6\mathsf{X}_5$	0
X_2	*	*	0	0	0	0
X_3	*	*	*	0	0	0
X_4	*	*	*	*	0	0
X_5	*	*	*	*	*	0

5. Parametric Defining Equations and Gröbner Systems

Actually, one of the main — somehow hidden — obstacles appearing among our computations arises when the set of defining equations includes some certain *parametric* polynomials. This case is quite usual as one observes in [5, 8, 20, 37]. To treat such cases, we suggest the modern and effective concept of *comprehensive Gröbner systems* [17, 18, 27, 28, 41] which enables us to consider and solve the arisen (linear) parametric systems.

To begin with, let $\mathbb{K}$ be a field and let $\mathbf{a} := a_1, \ldots, a_t$ and $\mathbf{x} := x_1, \ldots, x_n$ be two certain sequences of parameters and variables, respectively. Naturally, we call:

$$\mathbb{K}[\mathbf{a}][\mathbf{x}] := \left\{ \sum_{i=1}^{m} p_{\alpha_i} x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} \,|\, p_{\alpha_i} \in \mathbb{K}[\mathbf{a}],\ \alpha_{ij} \in \mathbb{N} \cup \{0\} \right\}$$

the *parametric polynomial ring* over $\mathbb{K}$ with parameters $\mathbf{a}$ and variables $\mathbf{x}$. Let P be a set of parametric polynomials which generates the parametric ideal $\mathcal{I}$. Obviously, the solutions of the *parametric system* $P = 0$ depend on the extant parameters. The main idea behind the modern concept of *comprehensive Gröbner systems* is to treat such solutions by discussing the values of the parameters of the system defined by $\mathcal{I}$. This concept provides some effective and powerful tools in which enables one to divide the space of parameters into a finite number of partitions for which the *general form* of solutions arising from each partition is unique.

Definition 5.1. *Let $\mathcal{I} \subset \mathbb{K}[\mathbf{a}][\mathbf{x}]$ be a parametric ideal, $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$ and $\prec$ be a monomial ordering on $\mathbf{x}$. Then the set:*

$$\mathbf{G}(\mathcal{I}) = \{(E_i, N_i, G_i) \mid i = 1, \ldots, \ell\} \subset \mathbb{K}[\mathbf{a}] \times \mathbb{K}[\mathbf{a}] \times \mathbb{K}[\mathbf{a}][\mathbf{x}]$$

is called a comprehensive Gröbner system for $\mathcal{I}$ if for each homomorphism $\sigma_{(\lambda_1,\ldots,\lambda_t)} : \mathbb{K}[\mathbf{a}][\mathbf{x}] \longrightarrow \overline{\mathbb{K}}[\mathbf{x}]$, associated to a t-tuple $(\lambda_1, \ldots, \lambda_t) \in \overline{\mathbb{K}}^t$ and defined by:

$$\sum_{i=1}^{m} p_{\alpha_i}(a_1, \cdots, a_t)\, x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} \quad \mapsto \quad \sum_{i=1}^{m} p_{\alpha_i}(\lambda_1, \ldots, \lambda_t)\, x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}},$$

there exists a pair (E_i, N_i) with $(\lambda_1, \ldots, \lambda_t) \in V(E_i) \setminus V(N_i)$ such that $\sigma(G_i)$ is a Gröbner basis for $\sigma(I)$ with respect to $\prec$. Here by $V(E_i)$ and $V(N_i)$ we mean the algebraic varieties associated to the polynomial sets E_i and N_i. In this case, E_i and N_i are called null *and* non-null conditions*, respectively.*

Remark that, by [41, Theorem 2.7], every parametric ideal possesses a (*finite*) comprehensive Gröbner system, however, by Definition 5.1, we can observe that such a system may be not unique. The concept of Gröbner systems was introduced first by Weispfenning in 1992 [41]. Later on, Montes [28] proposed DisPGB algorithm for computing Gröbner systems. In 2006, Sato and Suzuki [36] provided an important improvement for computing Gröbner systems by doing only computation of the reduced Gröbner bases in polynomial rings over ground fields. Furthermore, Montes and Wibmer in [27] presented the GröbnerCover algorithm which computes a finite partition of the parameter space into locally closed subsets together with polynomial data from which the *reduced* Gröbner basis for a given values of parameters can immediately be determined. Kapur, Sun and Wang [17, 18] in 2010 and 2013 suggested two new algorithms for computing Gröbner systems by combining Weispfenning's algorithm with Sato and Suzuki's.

It is worth noting that if $V(E_i) \setminus V(N_i) = \emptyset$, for some i, then the triple (E_i, N_i, G_i) is useless and it must be omitted from the comprehensive Gröbner system. In this case, the pair (E_i, N_i) is called *inconsistent*. It is known that inconsistency occurs if and only if $N_i \subset \sqrt{\langle E_i \rangle}$ and thus we need to an efficient radical membership test to determine it.

In the recently published paper [17], Kapur, Sun and Wang introduced an effective algorithm to compute comprehensive Gröbner system of a parametric polynomial ideal. This algorithm which is called by PGB uses a new and efficient radical membership criterion based on linear algebra methods. To the best of our knowledge, it is the most powerful algorithm of computing comprehensive Gröbner systems introduced so far and it is for this reason that we prefer to employ this algorithm in our computations.

Besides the deep theory encompassing this subject, the concept of comprehensive Gröbner bases provides some effective tools to consider and to solve parametric systems by decomposing the space of the extant parameters. To illustrate this ability let us borrow the following example from [17].

Example 5.2. *Consider the following parametric polynomial system in* $\mathbb{C}[a,b,c][x,y]$*:*

$$\Sigma : \begin{cases} ax - b &= 0 \\ by - a &= 0 \\ cx^2 - y &= 0 \\ cy^2 - x &= 0. \end{cases}$$

Choosing the graded reverse lexicographical ordering $y \prec x$ and computing the sought comprehensive Gröbner system using the algorithm PGB *give the results displayed in the following table:*

E_i	N_i	G_i
$\{a,b,c\}$	$\{\ \}$	$\{x,y\}$
$\{a,b\}$	$\{c\}$	$\{cx^2-y, cy^2-x\}$
$\{a^6-b^6, a^3c-b^3, b^3c-a^3, ac^2-a, bc^2-b\}$	$\{b\}$	$\{bx-acy, by-a\}$
$\{\ \}$	$\{a^6-b^6, a^3c-b^3, b^3c-a^3, ac^2-a, bc^2-b\}$	$\{1\}$

Accordingly, the algorithm divides the solution set of the system Σ into four partitions, each of them corresponds to one of the above rows. Let us explain what each of these rows means. For the first row, we have the null conditions $E_1 = \{a,b,c\}$ and there is no any non-null condition. This means that if the elements of E_1 are null, namely if $a = b = c = 0$ then the system Σ reduces to the system $G_1 = \{x = 0, y = 0\}$ which obviously has the single solution $(0,0)$. For the second row, we have null conditions $E_2 = \{a,b\}$ and non-null condition $N_2 = \{c\}$. This means that if $a = b = 0$ and $c \neq 0$ then the system Σ reduces to $G_2 = \{cx^2 - y = 0, cy^2 - x = 0\}$ which has the solution set $\{(\frac{1}{c}, \frac{1}{c}), c \in \mathbb{C}\} \cup \{(0,0), (\frac{-1+\sqrt{3}i}{2c}, \frac{-1-\sqrt{3}i}{2c}), (\frac{-1-\sqrt{3}i}{2c}, \frac{-1+\sqrt{3}i}{2c})\}$. Similar interpretation holds for the third row. Finally, the last row means that if none of the previous null conditions holds, namely if $E_1, E_2, E_3 \neq 0$, then the under consideration system Σ reduces to $G_4 = \{1 = 0\}$ which of course its solution set is empty.

5.1. Implementation

In [34], we have implemented the designed algorithm in Maple 15 which is available online as a library entitled CRAut (together with a sample file relevant to the next example). To do it, at first we implemented the recent algorithm PGB of Kapur, Sun and Wang [17]. By means of this auxiliary algorithm, we have employed techniques of comprehensive Gröbner systems to consider and solve the appearing parametric *linear* systems Sys in the parametric case.

Example 5.3. *Consider the following weighted homogeneous rigid defining equations:*

$$\begin{aligned}
w_1 - \overline{w}_1 &= 2i\, z\overline{z}, \\
w_2 - \overline{w}_2 &= 2i\left(z^2\overline{z} + \overline{z}^2 z\right), \quad w_3 - \overline{w}_3 = 2\left(z^2\overline{z} - \overline{z}^2\, z\right), \\
w_4 - \overline{w}_4 &= 2i\left(z^3\overline{z} + \overline{z}^3 z\right) + 2i\mathbf{a}\, z^2\overline{z}^2, \quad w_5 - \overline{w}_5 = 2\left(z^3\overline{z} - \overline{z}^3 z\right) + 2i\mathbf{b}\, z^2\overline{z}^2, \quad w_6 - \overline{w}_6 = 2i\, z^2\overline{z}^2, \qquad a \in \mathbb{R}, \\
w_7 - \overline{w}_7 &= 2i\left(z^4\overline{z} + z\overline{z}^4\right) + \mathbf{c}i\left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right), \quad w_8 - \overline{w}_8 = 2\left(z^4\overline{z} - z\overline{z}^4\right) + \mathbf{d}i\left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right), \\
w_9 - \overline{w}_9 &= 2i\left(z^3\overline{z}^2 + z^2\overline{z}^3\right) + \mathbf{e}i\left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right), \\
w_{10} - \overline{w}_{10} &= 2\left(z^3\overline{z}^2 - z^2\overline{z}^3\right) + \mathbf{f}i\left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right), \\
w_{11} - \overline{w}_{11} &= \left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} - z\overline{z}^2\right), \quad w_{12} - \overline{w}_{12} = i\left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right), \qquad \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathbb{R},
\end{aligned} \tag{24}$$

and let M_k be the Beloshapka's CR-model of CR-dimension 1 and codimension $\mathsf{k} = 1, \ldots, 12$, represented in coordinates $(z, w_1, \ldots, w_\mathsf{k})$ in $\mathbb{C}^{\mathsf{k}+1}$. For $\mathsf{k} = 1, \ldots, 5$, M_k is represented as the graph of the above first k equations. For $\mathsf{k} = 6, \ldots, 11$, it is represented again by the first k equations but with the assumption $\mathbf{a}, \mathbf{b} = 0$. Finally, M_{12} is represented as the graph of the above 12 equations with the assumption that all the appearing six parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ are vanished — namely a non-parametric model as M_1, M_2, M_3 and M_6 are. These twelve CR manifolds encompass all the Beloshapka's models up to the length five and are constructed by Shananina and Mamai [20, 37]. They also computed the associated Lie algebras of infinitesimal CR automorphisms, thought Mamai did not present the outputs, perhaps because of the length of them. It is also known that these models are all homogeneous ([5]). By means of our implementation, we have computed the associated Lie algebras of infinitesimal CR automorphisms. The following table displays some properties of the obtained results, where the timings were conducted on a personal laptop with Intel(R) Core(TM) i7 CPU@2.80 GHz and 6.00 GB of RAM:

<table>
<tr><td>Model</td><td>M_1</td><td>M_2</td><td>M_3</td><td>M_4</td><td colspan="2">M_5</td><td>M_6</td><td colspan="2">M_7</td><td colspan="2">M_8</td><td colspan="2">M_9</td><td colspan="2">M_{10}</td><td>M_{11}</td><td>M_{12}</td></tr>
<tr><td>Time (sec.)</td><td>0.5</td><td>0.5</td><td>1</td><td>3</td><td colspan="2">6.2</td><td>5.8</td><td colspan="2">17.6</td><td colspan="2">52.2</td><td colspan="2">131.5</td><td colspan="2">340</td><td>198</td><td>22</td></tr>
<tr><td>ρ</td><td>2</td><td>3</td><td>3</td><td>4</td><td colspan="2">4</td><td>4</td><td colspan="2">5</td><td colspan="2">5</td><td colspan="2">5</td><td colspan="2">5</td><td>5</td><td>5</td></tr>
<tr><td>ϱ</td><td>2</td><td>0</td><td>0</td><td>0</td><td colspan="2">0</td><td>0</td><td colspan="2">0</td><td colspan="2">0</td><td colspan="2">0</td><td colspan="2">0</td><td>-1</td><td>-1</td></tr>
<tr><td>dim.</td><td>8</td><td>5</td><td>7</td><td>7</td><td>9</td><td>8</td><td>10</td><td>10</td><td>9</td><td>12</td><td>10</td><td>12</td><td>11</td><td>14</td><td>12</td><td>13</td><td>14</td></tr>
</table>

The last row of the above table needs some explanation. In fact for some models, the dimension of the associated algebra is not unique and depends on the values of the extant parameters. Being more precise, in this table we observe two different values for the dimensions associated to some models. In such cases, the left number is the dimension of the desired Lie algebra associated to the model whereas all the appearing parameters vanish identically; otherwise the dimension is equal to the number at the right hand side. For example for M_8, we have $\dim\big(\mathfrak{aut}_{CR}(M_8)\big) = 12$ if $\mathbf{c}, \mathbf{d} = 0$ and $= 10$, otherwise. More precisely, in the case that $\mathbf{c}, \mathbf{d} \neq 0$, we have the basis elements of the components of $\mathfrak{aut}_{CR}(M_8)$ as:

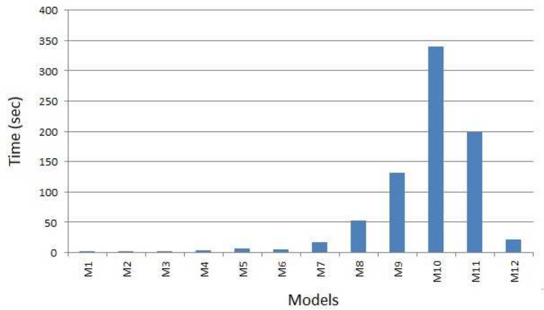
$$
\begin{aligned}
&\mathfrak{g}_{-5} = \langle \partial_{w_7}, \partial_{w_8} \rangle, \qquad \mathfrak{g}_{-4} = \langle \partial_{w_6}, \partial_{w_5}, \partial_{w_4} \rangle, \qquad \mathfrak{g}_{-3} = \langle \partial_{w_3}, \partial_{w_2} \rangle, \\
&\mathfrak{g}_{-2} = \langle \partial_{w_1} + \mathbf{a} w_2 \partial_{w_7} + \mathbf{b} w_2 \partial_{w_8} \rangle, \\
&\mathfrak{g}_{-1} = \Big\langle 2w_4\, \partial_{w_8} - 2w_5 \partial_{w_7} + z\partial_{w_1} + w_3 \partial_{w_6} + z^3 \partial_{w_4} + 2w_1 \partial_{w_3} + z^2 \partial_{w_2} + z^4 \partial_{w_7} - \frac{3}{2} w_3 \partial_{w_4} + \frac{3}{2} w_2 \partial_{w_5} + \\
&\qquad + \mathbf{c} z^2 w_1 \partial_{w_7} + \mathbf{c} w_6 \partial_{w_7} + \mathbf{d} z^2 w_1 \partial_{w_8} + \mathbf{d} w_6 \partial_{w_8} + \frac{i}{2}\Big(\partial_z - 2z^2 \partial_{w_3} - 2z^3 \partial_{w_5} - 2z^4 \partial_{w_8}\Big), \\
&\qquad - i\mathbf{c} z^2 w_1 \partial_{w_7} - iz^3 \partial_{w_4} - i\mathbf{d} z^2 w_1 \partial_{w_8} - iz\partial_{w_1} - iz^4 \partial_{w_7} - iz^2 \partial_{w_2} - \mathbf{c} w_1^2 \partial_{w_7} - \frac{3}{2} \mathbf{d} w_3 \partial_{w_5} - \frac{3}{2} w_2 \partial_{w_4} \\
&\qquad - w_2 \partial_{w_6} - z^4 \partial_{w_8} - z^3 \partial_{w_5} - z^2 \partial_{w_3} - \frac{1}{2} \partial_z - 2w_4 \partial_{w_7} - 2w_1 \partial_{w_2} - 2w_5 \partial_{w_8} - \mathbf{d} w_1^2 \partial_{w_8} \Big\rangle, \\
&\mathfrak{g}_0 = \langle\, \rangle.
\end{aligned}
\tag{25}
$$

In the case that $\mathbf{c}, \mathbf{d} = 0$, the basis elements of $\mathfrak{g}_i$, $i = -1, \ldots, -5$ are as above with of course $\mathbf{c}, \mathbf{d} = 0$ while in this case $\mathfrak{g}_0$ has two basis elements as follows:

$$
\begin{aligned}
\mathfrak{g}_0 = \Big\langle & - w_8 \partial_{w_7} + \frac{1}{3} w_2 \partial_{w_3} + w_7 \partial_{w_8} - \frac{1}{3} w_3 \partial_{w_2} + \frac{2}{3} w_4 \partial_{w_5} - \frac{2}{3} w_5 \partial_{w_4} + \frac{1}{3} iz\partial_z, \\
& \frac{1}{5} z\partial_z + \frac{3}{5} w_3 \partial_{w_3} + \frac{3}{5} w_2 \partial_{w_2} + \frac{4}{5} w_5 \partial_{w_5} + \frac{4}{5} w_6 \partial_{w_6} + \frac{4}{5} w_4 \partial_{w_4} + w_7 \partial_{w_7} + w_8 \partial_{w_8} + \frac{2}{5} w_1 \partial_{w_1} \Big\rangle.
\end{aligned}
$$

One easily concludes from the results of the above table, all the above twelve models have rigidity, as is the main result of Mamai in [20]. Together with the library CRAut*, we also have put a sample file concerning the computations of this example.*

Remark 5.4. *A glance on the above timings shows how it increases mostly the complexity of computations as one passes from each model to the next by adding just one variable and one defining equation to the previous ones. The* *following diagram may be helpful to compare the appearing timings. Moreover, actually the computations in the case of parametric defining equations are more complicated in comparison to those of the non-parametric case. For example, compare the timings corresponding to the models M_{11} and M_{12} — notice that the defining equations of M_{12} are non-parametric.*



Example 5.5. *Add the following rigid defining equations to the list* (24)*:*

$$
\begin{aligned}
&w_{13} - \overline{w}_{13} = 2i\left(z^5\overline{z} + \overline{z}^5 z\right), \quad w_{14} - \overline{w}_{14} = 2\left(z^5\overline{z} - \overline{z}^4 z\right), \quad w_{15} - \overline{w}_{15} = 2i\left(z^4\overline{z}^2 + \overline{z}^4 z^2\right),\\
&w_{16} - \overline{w}_{16} = 2\left(z^4\overline{z}^2 - \overline{z}^4 z^2\right), \quad w_{17} - \overline{w}_{17} = 2i\, z^3\overline{z}^3,
\end{aligned}
\tag{26}
$$

and for $\mathsf{k} = 13, \ldots, 17$ *let* M_{k} *to be the CR manifold of CR-dimension one and codimension* k *represented as the graph of the equations of* (24) *together the first* $\mathsf{k} - 12$ *equations of the above list. These are the next five rigid Beloshapka's models which are the very first models of the length six. Here, we also compute — for the first time — the desired Lie algebras associated to these models and the results are displayed in the following table:*

Model	M_{13}	M_{14}	M_{15}	M_{16}	M_{17}
Time (sec.)	*83.5*	*152*	*286*	*545*	*1157*
ρ	*6*	*6*	*6*	*6*	*6*
ϱ	*-1*	*-1*	*-1*	*-1*	*-1*
dim	*15*	*16*	*17*	*18*	*19*

Acknowledgment

The authors gratefully acknowledge the helpful discussions of Valerii Beloshapka during the preparation of this paper.

References

[1] M. Aghasi, J. Merker, M. Sabzevari, Connections the Cartan-Tanaka effectives pour les hypersurfaces strictement pseudoconvexes $M^3 \subset \mathbb{C}^2$ de classe C^6, C. R. Acad. Sci. Paris, Sr. I 349 (2011) 845–848.
[2] D. V. Alekseevsky, A. F. Spiro, Prolongations of Tanaka structures and regular CR structures, Selected Topics in Cauchy-Riemann Geometry, Dept. Math., Seconda Univ. Napoli, Caserta, (2001) 1–37.
[3] S. Baouendi, P. Ebenfelt, L. P. Rothschild, Real Submanifolds in Comlex Space and their Mappings, Princeton Math. Series, 1999.
[4] L. Baracco, G. Zampieri, CR extension from manifolds of higher type, Canadian Journal of Mathematics 60 (2008) 1219–1239.
[5] V. K. Beloshapka, Universal models for real submanifolds, Mathematical Notes 75(4) (2004) 475–488.
[6] V. K. Beloshapka, Real submanifolds in complex space: polynomial models, automorphisms, and classification problems, Russian Mathematical Surveys 57(1) (2002) 1–41.
[7] V. K. Beloshapka, Polynomial models of real manifolds, Izvestiya: Mathematics 65(4) (2001) 641–657.
[8] V. K. Beloshapka, CR-Varieties of the type (1, 2) as varieties of super-high codimension, Russian Journal of Mathematical Physics 5(2) (1997) 399–404.
[9] T. Bloom, I. Graham, On type conditions for generic real submanifolds in $\mathbb{C}^n$, Inventions Mathematicae 40 (1977) 217–243.
[10] A. Boggess, CR Manifolds and the Tangential Cauchy-Riemann Complex, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1991.
[11] S. S. Chern, Y. Moser, Real hypersurfaces in complex manifold, Acta Mathematica 133 (1975) 219–271.
[12] S. K. Donaldson, Lectures on Lie Groups and Geometry, 2011, available online.

[13] G. Fels, W. Kaup, Classification of Levi degenerate homogeneous CR manifolds in dimension 5, Acta Mathematica 201 (2008) 1–82.
[14] R. V. Gammel, I. Kossovskiy, The envelope of holomorphy of a model surface of the third degree and the 'rigidity' phenomenon, Proceedings of the Steklov Institute of Mathematics 253 (2006) 22–37.
[15] H. Gaussier, J. Merker, Nonalgebraizable real analytic tubes in $\mathbb{C}^n$, Mathematische Zeitschrift 247(2) (2004) 337–383.
[16] R. Juhlin, B. Lamel, Automorphism groups of minimal real-analytic CR manifolds, Journal of European Mathematical Society 15(2) (2013) 509–537.
[17] D. Kapur, Y. Sun, D. Wang, An efficient algorithm for computing a comprehensive Gröbner system of a parametric polynomial system, Journal of Symbolic Computations 49 (2013) 27–44.
[18] D. Kapur, Y. Sun, D. Wang, A new algorithm for computing comprehensive Gröbner systems, Proceedings of ISSAC'10, ACM Press, New York, 29–36, 2010.
[19] S. G. Krantz, Function Theory of Several Complex Variables (2nd ed.), AMS Chelsea Publications, 2001.
[20] I. B. Mamai, Model CR manifolds with one-dimensional complex tangent, Russian Journal of Mathematical Physics 16(1) (2009) 97–102.
[21] C. Medori, M. Nacinovich, Levi-Tanaka algebras and homogeneous CR manifolds, Compositio Mathematica 109 (1997) 195–250.
[22] J. Merker, Sophus Lie's Theory of Transformation Groups I, general properties of continuous transformation groups: contemporary approach and translation, Springer-Verlag, Heidelberg, 2015.
[23] J. Merker, S. Pocchiola, M. Sabzevari, Equivalence of 5-dimensional CR manifolds **II**: General classes I, II, $\mathsf{III}_1, \mathsf{III}_2, \mathsf{IV}_1, \mathsf{IV}_2$, arxiv:1311.5669v1.
[24] J. Merker, E. Porten, Holomorphic extension of CR functions, envelopes of holomorphy and removable singularities, International Mathematics Research Surveys, 2006.
[25] J. Merker, M. Sabzevari, Cartan equivalence problem for 5-dimensional bracket-generating CR-manifolds in $\mathbb{C}^4$, J. Geom. Anal., 58 pp., expanded form arxiv:1401.4297, 172 pp. DOI 10.1007/s12220-015-9667-6
[26] J. Merker, M. Sabzevari, Explicit expression of Cartan's connections for Levi-nondegenerate 3-manifolds in complex surfaces, and identification of the Heisenberg sphere, Central European Journal of Mathematics 10(5) (2012) 1801–1835.
[27] A. Montes, M. Wibmer, Gröbner bases for polynomial systems with parameters, Journal of Symbolic Computations 45(12) (2010) 1391–1425.
[28] A. Montes, A new algorithm for discussing Gröbner bases with parameters, Journal of Symbolic Computations 33(1-2) (2002) 183–208.
[29] S. Pocchiola, Le Problème d'èquivalence Pour les Variétés de Cauchy-Riemann en Dimension 5, Ph. D. Thesis, Paris-Sud 11, 2014.
[30] H. Poincaré, Les fonction analytiques de deux variables et la représentation conforme, Rendiconti del Circolo Matematico di Palermo 23 (1907) 185–220.
[31] M. Sabzevari, The Beloshapka's maximum conjecture is correct, 31 pp., arxiv:1601.01164.
[32] M. Sabzevari, Moduli spaces of model real submanifolds: two alternative approaches, Sci. China Math. 58(11) (2015) 2261–2278.
[33] M. Sabzevari, A. Hashemi, B. M.-Alizadeh, J. Merker, Applications of differential algebra for computing Lie algebras of infinitesimal CR automorphisms, Science China Mathematics 57(9) (2014) 1811-1834.
[34] M. Sabzevari, A. Hashemi, B. M.-Alizadeh, J. Merker, Implementation of the algorithm in Maple, Available online at `http://amirhashemi.iut.ac.ir/softwares.html`.
[35] M. Sabzevari, J. Merker, S. Pocchiola, Canonical Cartan connections on maximally minimal generic submanifolds $M^5 \subset \mathbb{C}^4$, Elect. Res. Ann. Math. Sci. (ERA-MS), **21** (2014) 153–166.
[36] Y. Sato, A. Suzuki, A simple algorithm to compute comprehensive Gröbner bases using Gröbner bases, Proceedings of ISSAC'2006, ACM Press, New York, 326–331, 2006.
[37] E. N. Shananina, Models for CR manifolds of type $(1, K)$ for $3 \leqslant K \leqslant 7$ and their automorphisms, Mathematical Notes 67(3) (2000) 382–388.
[38] N. K. Stanton, Infinitesimal CR automorphisms of real hypersurfaces, American Journal of Mathematics 118(1) (1996) 209–233.
[39] N. Tanaka, On differential systems, graded Lie algebras and pseudo-groups, Journal of Mathematics of Kyoto University 10 (1970) 1–82.
[40] A. E. Tumanov, Finite-dimensionality of the group of CR automorphisms of a standard CR manifold, and proper holomorphic mappings of Siegel domains, Mathematics of the USSR Izvestiya 32(3) (1989) 655–662.
[41] V. Weispfenning, Comprehensive Gröbner bases, Journal of Symbolic Computations 14(1) (1992) 1–29.