Finite Difference Approximation of an Elliptic Problem
with Nonlocal Boundary Condition

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Abstract.

We consider Poisson's equation on the unit square with a nonlocal boundary condition. The existence and uniqueness of its weak solution in Sobolev space H^1 is proved. A finite difference scheme approximating this problem is proposed. An error estimate compatible with the smoothness of input data in discrete H^1 Sobolev norm is obtained.

1. Introduction

Differential equations with nonlocal boundary conditions have received much attention in the last decades (see, e.g. [3, 4, 6, 7, 13] and references therein). As a rule, nonlocal boundary condition contains an integral term over the spatial domain (or its boundary) of some function of the problem solution. When integral term is involved in the governing partial differential equation, it is referred as partial integro-differential equation.

Nonlocal boundary value problems have a great theoretical and practical significance. On the one hand, they represent interesting generalization of classical boundary value problems. On the other hand, they can serve as mathematical models of some physical phenomena related to heat propagation, moisture transfer in porous media, chemical diffusion, population dynamics, thermoelasticity, thermodynamics, plasma physics, medical science, some biological and technological processes, etc. Nonlocal boundary conditions arise mainly in the case when the data on the boundary can not be measured directly. Therefore, these conditions are often encountered in inverse problems.

The layout of the paper is as follows. In Section 2 we introduce a nonlocal boundary-value problem (BVP) for Poisson equation, briefly expose its properties and prove the existence and uniqueness of its weak solution. In Section 3 we introduce meshes, finite-difference operators and discrete Sobolev-like norms and define a finite difference scheme (FDS) approximating BVP (1)-(2). Further, we investigate the properties of FDS (10). Section 4 is devoted to the error analysis of FDS (10). A convergence rate estimate, compatible with the smoothness of the input data (up to a logarithmic factor of mesh-size), is obtained. In Section 5 we consider the case when the coerciveness assumption is not met.

2. Formulation of the Problem

As a model example, we consider Poisson's equation in the unit square $\Omega = (0,1)^2$

$$-\Delta u = f, \qquad x = (x_1, x_2) \in \Omega \tag{1}$$

subject to nonlocal boundary condition

$$\frac{\partial u}{\partial \nu} \cos(\nu, x_i) + \alpha_{ij} u = \int_{\Gamma_{ij}} \beta_{ij}(x, x') u(x') \, d\Gamma_{x'}, \qquad x \in \Gamma_{ij}, \qquad i, j = 1, 2 \tag{2}$$

where $\Gamma = \partial\Omega = \cup_{i,j=1}^{2} \Gamma_{ij}$, $\Gamma_{ij} = \{x = (x_1, x_2) \in \Gamma \,|\, x_i = j - 1\}$, and ν is the unit outward normal to Γ.

Boundary-value problem (1)-(2) represent linearized symmetric transmission problem of heat radiation (see [2, 11]).

We assume that

$$\alpha_{ij} \in L^\infty(\Gamma_{ij}), \qquad \beta_{ij} \in L^\infty(\Gamma_{ij}^2). \tag{3}$$

By C and c_i we denote positive constants, independent of the solution of the boundary-value problem and the mesh-size. In particular, C may take different values in the different formulas.

Let $H^s(\Omega)$ be the standard Sobolev space and $H^0(\Omega) = L^2(\Omega)$ [1]. In the standard manner we introduce the weak form of boundary-value problem (1)-(2): *Find $u \in H^1(\Omega)$ such that*

$$a(u, v) = l(v), \qquad \forall\, v \in H^1(\Omega), \tag{4}$$

where

$$a(u, v) = \sum_{i=1}^{2} \iint_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx_1 \, dx_2 + \sum_{i,j=1}^{2} \left(\int_{\Gamma_{ij}} \alpha_{ij} u v \, d\Gamma - \iint_{\Gamma_{ij}} \int_{\Gamma_{ij}} \beta_{ij}(x, x') u(x') v(x) \, d\Gamma_{x'} \, d\Gamma_x \right) \tag{5}$$

is bilinear form associated with the boundary-value problem (1)-(2) and

$$l(v) = \iint_\Omega f v \, dx_1 \, dx_2. \tag{6}$$

Analogously we define the corresponding weak eigenvalue problem: *Find the pair $(\lambda, u) \in \mathbb{C} \times H^1(\Omega)$, $u \not\equiv 0$, such that*

$$a(u, v) = \lambda\, (u, v)_{L^2(\Omega)}, \qquad \forall\, v \in H^1(\Omega). \tag{7}$$

Lemma 2.1. *Under the conditions (3) the bilinear form a, defined by (5), is bounded on $H^1(\Omega) \times H^1(\Omega)$. This form also satisfies the Gårding's inequality on $H^1(\Omega)$, i.e. there exist positive constants m and κ such that*

$$a(u, u) + \kappa\|u\|_{L^2(\Omega)}^2 \geq m\|u\|_{H^1(\Omega)}^2, \qquad \forall\, u \in H^1(\Omega). \tag{8}$$

Proof. Boundedness of a follows from (3) and the trace theorem for the Sobolev spaces. Gårding's inequality (8) follows from the multiplicative trace inequality (see, e.g., Proposition 1.6.3 in [5])

$$\|u\|_{L^2(\partial\Omega)}^2 \leq C\, \|u\|_{L^2(\Omega)}\, \|u\|_{H^1(\Omega)},$$

Cauchy-Schwarz and ε-inequalities, for sufficiently small $\varepsilon > 0$. $\square$

If $\alpha_{ij} > 0$ and β_{ij} $(i, j = 1, 2)$ are sufficiently small, then the bilinear form a is coercive (i.e. $\kappa = 0$). Sufficient conditions are

$$\alpha_{ij}(x) \geq \varepsilon > 0, \qquad |\beta_{ij}(x, x') + \beta_{ij}(x', x)| \leq 2 \sqrt{\left[\alpha_{ij}(x) - \varepsilon\right] \left[\alpha_{ij}(x') - \varepsilon\right]}. \tag{9}$$

From Lemma 2.1 and Lax-Milgram lemma (see Theorem 1.13 in [10]) one immediately obtains the following result.

Theorem 2.2. *Under the conditions (3) and (9) the problem (4)-(6) has a unique solution $u \in H^1(\Omega)$, and it depends continuously on $f \in L^2(\Omega)$.*

Analogous result holds in general case (without assumption (9)) if 0 is not eigenvalue of the boundary-value problem (7) (as consequence of Theorem 17.11 in [15]).

3. Finite Difference Approximation

Let $\bar{\omega}$ be a uniform mesh in $\bar{\Omega}$, with the step size $h = 1/n$, $n \in \mathbb{N}$. We denote $\omega = \bar{\omega} \cap \Omega$, $\gamma = \bar{\omega} \cap \Gamma$, $\bar{\gamma}_{ij} = \bar{\omega} \cap \Gamma_{ij}$, $\gamma_{ij} = \{x \in \bar{\gamma}_{ij} \,|\, 0 < x_{3-i} < 1\}$, $\gamma_{ij}^- = \{x \in \bar{\gamma}_{ij} \,|\, 0 \leq x_{3-i} < 1\}$, $\gamma_{ij}^\star = \bar{\gamma}_{ij} \setminus \gamma_{ij}$ and $\gamma^\star = \cup_{i,j}\gamma_{ij}^\star = \gamma \setminus \cup_{i,j}\gamma_{ij}$. We will consider mesh functions $v, w, \ldots$ defined on $\bar{\omega}$ or its submeshes.

The finite difference operators are defined in the usual manner [14]:

$$v_{x_i} = \frac{v^{+i} - v}{h}, \qquad v_{\bar{x}_i} = \frac{v - v^{-i}}{h},$$

where $v^{\pm i}(x) = v(x \pm he_i)$ and e_i is the unit vector of the axis x_i, $i = 1, 2$.

We define the following discrete inner products and norms:

$$[v, w] = h^2 \sum_{x \in \omega} v(x)w(x) + \frac{h^2}{2} \sum_{x \in \gamma \setminus \gamma^\star} v(x)w(x) + \frac{h^2}{4} \sum_{x \in \gamma^\star} v(x)w(x),$$

$$[v, w]_i = h^2 \sum_{x \in \omega \cup \gamma_{i1}} v(x)w(x) + \frac{h^2}{2} \sum_{x \in \gamma_{3-i,1}^- \cup \gamma_{3-i,2}^-} v(x)w(x),$$

$$\|[v]\|^2 = [v, v], \qquad \|[v]\|_i^2 = [v, v]_i, \qquad \|[v]\|_{C(\bar{\omega})} = \max_{x \in \bar{\omega}} |v(x)|,$$

$$\|[v]\|_{H^1(\bar{\omega})}^2 = \|[v]\|^2 + \|[v_{x_1}]\|_1^2 + \|[v_{x_2}]\|_2^2,$$

$$[v, w]_{\bar{\gamma}_{ij}} = h \sum_{x \in \bar{\gamma}_{ij}}' v(x)w(x) = h \sum_{x \in \gamma_{ij}} v(x)w(x) + \frac{h}{2} \sum_{x \in \gamma_{ij}^\star} v(x)w(x),$$

$$[v, w]_{\gamma_{ij}^-} = h \sum_{x \in \gamma_{ij}^-} v(x)w(x), \qquad (v, w)_{\gamma_{ij}} = h \sum_{x \in \gamma_{ij}} v(x)w(x),$$

$$\|v\|_{\gamma_{ij}}^2 = (v, v)_{\gamma_{ij}}, \qquad \|[v]\|_{\gamma_{ij}^-}^2 = [v, v]_{\gamma_{ij}^-}, \qquad \|[v]\|_{\bar{\gamma}_{ij}}^2 = [v, v]_{\bar{\gamma}_{ij}},$$

$$|v|_{H^{1/2}(\gamma_{ij}^-)}^2 = h^2 \sum_{x, x' \in \gamma_{ij}^-, \, x' \neq x} \left[\frac{v(x) - v(x')}{x_{3-i} - x'_{3-i}}\right]^2,$$

$$\|[v]\|_{\dot{H}^{1/2}(\gamma_{ij}^-)}^2 = |v|_{H^{1/2}(\gamma_{ij}^-)}^2 + h \sum_{x \in \gamma_{ij}^-} \left(\frac{1}{x_{3-i} + h/2} + \frac{1}{1 - x_{3-i} - h/2}\right) |v(x)|^2.$$

We also define the Steklov smoothing operators (see [10]):

$$T_i^+ f(x) = \int_0^1 f(x + hx_i' e_i)\, dx_i' = T_i^- f(x + he_i) = T_i f(x + 0.5he_i),$$

$$T_i^{2\pm} f(x) = 2 \int_0^1 (1 - x_i') f(x \pm hx_i' e_i)\, dx_i', \qquad i = 1, 2.$$

These operators commute and transform derivatives into differences, for example:

$$T_i^+ \left(\frac{\partial u}{\partial x_i} \right) = u_{x_i}, \qquad T_i^- \left(\frac{\partial u}{\partial x_i} \right) = u_{\bar{x}_i}, \qquad T_i^2 \left(\frac{\partial^2 u}{\partial x_i^2} \right) = u_{\bar{x}_i x_i}.$$

We approximate the boundary-value problem (1)-(2) with the following finite difference scheme:

$$-\bar{\Delta}_h v = \bar{f}, \qquad x \in \bar{\omega}, \tag{10}$$

where

$$\bar{\Delta}_h v = \begin{cases} v_{\bar{x}_1 x_1} + v_{\bar{x}_2 x_2}, & x \in \omega \\[2mm] \frac{2}{h}\left(v_{x_1} - \bar{\alpha}_{11} v + [\bar{\beta}_{11}(x, \cdot), v(\cdot)]_{\bar{\gamma}_{11}} \right) + v_{\bar{x}_2 x_2}, & x \in \gamma_{11} \\[2mm] \frac{2}{h}\left(v_{x_1} - \bar{\alpha}_{11} v + [\bar{\beta}_{11}(x, \cdot), v(\cdot)]_{\bar{\gamma}_{11}} \right) \\[1mm] \quad + \frac{2}{h}\left(v_{x_2} - \bar{\alpha}_{21} v + [\bar{\beta}_{21}(x, \cdot), v(\cdot)]_{\bar{\gamma}_{21}} \right), & x = (0, 0) \\[2mm] \text{and analogously at the other boundary nodes,} \end{cases}$$

$$\bar{f} = \begin{cases} T_1^2 T_2^2 f, & x \in \omega \\ T_i^{2\pm} T_{3-i}^2 f, & x \in \gamma_{ij} \\ T_1^{2\pm} T_2^{2\pm} f, & x \in \gamma^\star \end{cases}, \qquad \bar{\alpha}_{ij} = \begin{cases} T_{3-i}^2 \alpha_{ij}, & x \in \gamma_{ij} \\ T_{3-i}^{2\pm} \alpha_{ij}, & x \in \gamma_{ij}^\star \end{cases}$$

and

$$\bar{\beta}_{ij} = \begin{cases} T_{3-i}^2 T_{3-i}'^2 \beta_{ij}, & x \in \gamma_{ij}, & x' \in \gamma_{ij} \\ T_{3-i}^{2\pm} T_{3-i}'^2 \beta_{ij}, & x \in \gamma_{ij}^\star, & x' \in \gamma_{ij} \\ T_{3-i}^2 T_{3-i}'^{2\pm} \beta_{ij}, & x \in \gamma_{ij}, & x' \in \gamma_{ij}^\star \\ T_{3-i}^{2\pm} T_{3-i}'^{2\pm} \beta_{ij}, & x \in \gamma_{ij}^\star, & x' \in \gamma_{ij}^\star \end{cases}$$

(T'_{3-i} denotes Steklov averaging operator on the variable x'_{3-i}).

In the sequel we will assume that the generalized solution of the problem (1)-(2) belongs to the Sobolev space $H^s(\Omega)$, $2 < s \le 3$, while the data satisfy the following smoothness conditions:

$$\alpha_{ij} \in H^{s-3/2}(\Gamma_{ij}), \qquad \beta_{ij} \in H^{s-1}(\Gamma_{ij}^2), \qquad i, j = 1, 2; \qquad f \in H^{s-2}(\Omega). \tag{11}$$

We introduce the bilinear form $a_h(v, w)$ associated with difference operator $-\bar{\Delta}_h$:

$$a_h(v, w) = [-\bar{\Delta}_h v, w] = \sum_{i=1}^2 [v_{x_i}, w_{x_i})_i + \sum_{i,j=1}^2 [\bar{\alpha}_{ij} v, w]_{\bar{\gamma}_{ij}} - h^2 \sum_{i,j=1}^2 \sideset{}{'}\sum_{x \in \bar{\gamma}_{ij}} \sideset{}{'}\sum_{x' \in \bar{\gamma}_{ij}} \bar{\beta}_{ij}(x, x') v(x') w(x). \tag{12}$$

The following counterpart of Lemma 2.1 holds.

Lemma 3.1. *Under the conditions (11) the bilinear form a_h, defined by (12), is bounded on $H^1(\bar{\omega}) \times H^1(\bar{\omega})$. This form also satisfies the discrete Gårding's inequality on $H^1(\bar{\omega})$, i.e. there exist positive constants $\bar{m}$ and $\bar{\kappa}$ such that*

$$a_h(v,v) + \bar{\kappa}\|[v]\|^2 \geq \bar{m}\|[v]\|^2_{H^1(\bar{\omega})}. \tag{13}$$

If $\bar{\alpha}_{ij} > 0$ and $\bar{\beta}_{ij}$ $(i,j = 1,2)$ are sufficiently small then, as in the continuous case, the bilinear form a_h is coercive. When α_{ij} and β_{ij} satisfy the assumptions (11) and the step-size h is sufficiently small, the conditions (9) are sufficient for this. Then there exist positive constants c_1 and c_2 such that

$$c_1\|[v]\|^2_{H^1(\bar{\omega})} \leq a_h(v,v) = [-\bar{\Delta}_h v,\, v] \leq c_2\|[v]\|^2_{H^1(\bar{\omega})}. \tag{14}$$

From Lemma 3.1 and Lax-Milgram lemma one immediately obtains the following result.

Theorem 3.2. *Under the conditions (11) and (9), for sufficiently small step-size h, the finite difference scheme (10) has a unique solution.*

4. Convergence of the Finite Difference Scheme

Let u be the solution of the BVP (1)-(2), and let v denote the solution of the FDS (10). The error $z = u - v$ satisfies the following conditions

$$-\bar{\Delta}_h z = \psi, \qquad x \in \bar{\omega}, \tag{15}$$

where

$$\psi = \begin{cases} \displaystyle\sum_{i=1} \eta_{i,\bar{x}_i}, & x \in \omega, \\[2mm] \displaystyle\frac{2}{h}\left(\eta_1 + \zeta_{11} + \chi_{11}\right) + \tilde{\eta}_{2,\bar{x}_2}, & x \in \gamma_{11}, \\[2mm] \displaystyle\frac{2}{h}\left(\tilde{\eta}_1 + \tilde{\zeta}_{11} + \chi_{11}\right) + \frac{2}{h}\left(\tilde{\eta}_2 + \tilde{\zeta}_{21} + \chi_{21}\right), & x = (0,0), \\[2mm] \text{and analogously at the other boundary nodes,} \end{cases}$$

$$\eta_i = T_i^+ T_{3-i}^2\left(\frac{\partial u}{\partial x_i}\right) - u_{x_i}, \qquad x \in \omega,$$

$$\tilde{\eta}_i = T_i^+ T_{3-i}^{2\pm}\left(\frac{\partial u}{\partial x_i}\right) - u_{x_i}, \qquad x \in \gamma_{3-i,1}^- \,/\, x \in \gamma_{3-i,2}^-,$$

$$\zeta_{ij} = (T_{3-i}^2 \alpha_{ij})u - T_{3-i}^2(\alpha_{ij}u), \qquad x \in \gamma_{ij},$$

$$\tilde{\zeta}_{ij} = (T_{3-i}^{2\pm} \alpha_{ij})u - T_{3-i}^{2\pm}(\alpha_{ij}u), \qquad x \in \gamma_{ij}^\star,$$

$$\chi_{ij} = \int_{\Gamma_{ij}} T_{3-i}^2 \beta_{ij}(x,x')u(x')\,d\Gamma - h\sum_{x'\in\gamma_{ij}} T_{3-i}^2 T_{3-i}'^2 \beta_{ij}(x,x')u(x') - \frac{h}{2}\sum_{x'\in\gamma_{ij}^\star} T_{3-i}^2 T_{3-i}'^{2\pm}\beta_{ij}(x,x')u(x'), \qquad x \in \gamma_{ij},$$

$$\chi_{ij} = \int_{\Gamma_{ij}} T_{3-i}^{2\pm} \beta_{ij}(x,x')u(x')\,d\Gamma - h\sum_{x'\in\gamma_{ij}} T_{3-i}^{2\pm} T_{3-i}'^2 \beta_{ij}(x,x')u(x') - \frac{h}{2}\sum_{x'\in\gamma_{ij}^\star} T_{3-i}^{2\pm} T_{3-i}'^{2\pm}\beta_{ij}(x,x')u(x'), \qquad x \in \gamma_{ij}^\star.$$

Let us further denote $\tilde{\eta}_i = \eta_i + \bar{\eta}_i$, where

$$\bar{\eta}_i = \pm\frac{h}{3}\, T_i^+\left(\frac{\partial^2 u}{\partial x_{3-i}\partial x_i}\right), \qquad x \in \gamma_{3-i,1} \,/\, x \in \gamma_{3-i,2},$$

We shall prove a suitable a priori estimate for the FDS (15). For this purpose we need the following auxiliary lemmas:

Lemma 4.1. (see [9]) *The following inequality holds true:*

$$\left| [v, w_{x_{3-i}})_{\gamma_{ij}^-} \right| \le C |[v]|_{\dot{H}^{1/2}(\gamma_{ij}^-)} |[w]|_{H^1(\bar{\omega})}.$$

Lemma 4.2. (see [9]) *Let v be a mesh function on $\bar{\omega}$, then*

$$|[v]|_{C(\bar{\omega})} \le C \sqrt{\log \tfrac{1}{h}} \, |[v]|_{H^1(\bar{\omega})}.$$

Theorem 4.3. *Let the conditions (9) and (11) hold. Then, for sufficiently small step-size h, the FDS (15) is stable in the sense of a priori estimate*

$$|[z]|_{H^1(\bar{\omega})} \le C\left[\sum_{i=1}^2 |[\eta_i]|_i + \sum_{i,j=1}^2 \left(\|\zeta_{ij}\|_{\gamma_{ij}} + |[\chi_{ij}]|_{\bar{\gamma}_{ij}} + h\,|[\bar{\eta}_i]|_{\dot{H}^{1/2}(\gamma_{3-i,j}^-)} \right) + h \sqrt{\log \tfrac{1}{h}} \sum_{i,j=1}^2 \sum_{x\in\gamma_{ij}^\star} |\tilde{\zeta}_{ij}(x)| \right]. \tag{16}$$

Proof. Taking inner product of (15) with z and performing partial summation one obtains:

$$[-\Delta_h z, z] = [\psi, z] = -\sum_{i=1}^2 [\eta_i, z_{x_i})_i + \sum_{i,j=1}^2 \left\{ -\frac{h}{2}[\bar{\eta}_i, z_{x_i})_{\gamma_{3-i,j}} + (\zeta_{ij}, z)_{\gamma_{ij}} + [\chi_{ij}, z]_{\bar{\gamma}_{ij}} + \frac{h}{2} \sum_{x\in\gamma_{ij}^\star} \tilde{\zeta}_{ij}(x)\, z(x) \right\}.$$

Result follows applying Lemmas 4.1 and 4.2, inequality Cauchy-Schwarz and inequality (14). □

Theorem 4.4. *Let the assumptions of Theorem 4.3 hold. Then the solution of FDS (10) converges to the solution of BVP (1)-(2) and the convergence rate estimates*

$$|[u-v]|_{H^1(\bar{\omega})} \le Ch^{s-1} \sqrt{\log \tfrac{1}{h}} \left(1 + \max_{i,j} \|\alpha_{ij}\|_{H^{s-3/2}(\Gamma_{ij})} + \max_{i,j} \|\beta_{ij}\|_{H^{s-1}(\Gamma_{ij}^2)} \right) \|u\|_{H^s(\Omega)}, \qquad 2.5 < s < 3 \tag{17}$$

and

$$|[u-v]|_{H^1(\bar{\omega})} \le Ch \left(\log \tfrac{1}{h} \right)^{3/2} \left(1 + \max_{i,j} \|\alpha_{ij}\|_{H^{3/2}(\Gamma_{ij})} + \max_{i,j} \|\beta_{ij}\|_{H^2(\Gamma_{ij}^2)} \right) \|u\|_{H^3(\Omega)}, \qquad s = 3 \tag{18}$$

hold.

Proof. To prove the theorem it is sufficient to estimate the right-hand side terms in (16). The term η_i at the internal nodes of the mesh $\bar{\omega}$ can be estimated using Bramble-Hilbert lemma (see Theorem 2.27 in [10]), in the same manner as in the case of the Dirichlet BVP (see Sections 2.3 and 2.6 in [10]):

$$h^2 \sum_{x\in\omega\cup\gamma_{i1}} \eta_i^2 \le Ch^{2s-2} \|u\|_{H^s(\Omega)}^2, \qquad 1 < s \le 3.$$

At the boundary nodes η_i satisfy the same assumptions as at the internal nodes, but for $2 < s \le 3$. Hence

$$h^2 \sum_{x\in\gamma_{3-i,1}^-\cup\gamma_{3-i,2}^-} \eta_i^2 \le Ch^{2s-2} \|u\|_{H^s(\Omega)}^2, \qquad 2 < s \le 3.$$

From these inequalities follows

$$|[\eta_i]|_i \le Ch^{s-1} \|u\|_{H^s(\Omega)}, \qquad 2 < s \le 3. \tag{19}$$

Let us set $\chi_{ij} = \bar{\chi}_{ij} + \hat{\chi}_{ij}$, where

$$\bar{\chi}_{ij} = \int_{\Gamma_{ij}} T_{3-i}^2 \beta_{ij}(x,x')u(x')\,\mathrm{d}\Gamma - h \sum_{x'\in\bar{\gamma}_{ij}}{}' T_{3-i}^2 \beta_{ij}(x,x')u(x'), \qquad x\in\gamma_{ij},$$

$$\bar{\chi}_{ij} = \int_{\Gamma_{ij}} T_{3-i}^{2\pm} \beta_{ij}(x,x')u(x')\,\mathrm{d}\Gamma - h \sum_{x'\in\bar{\gamma}_{ij}}{}' T_{3-i}^{2\pm} \beta_{ij}(x,x')u(x'), \qquad x\in\gamma_{ij}^\star.$$

$$\hat{\chi}_{ij} = h \sum_{x' \in \gamma_{ij}} \left(T_{3-i}^2 \beta_{ij}(x,x') - T_{3-i}^2 T_{3-i}'^2 \beta_{ij}(x,x') \right) u(x')$$
$$+ \frac{h}{2} \sum_{x' \in \gamma_{ij}^\star} \left(T_{3-i}^2 \beta_{ij}(x,x') - T_{3-i}^2 T_{3-i}'^{2\pm} \beta_{ij}(x,x') \right) u(x'), \qquad x \in \gamma_{ij},$$

$$\hat{\chi}_{ij} = h \sum_{x' \in \gamma_{ij}} \left(T_{3-i}^{2\pm} \beta_{ij}(x,x') - T_{3-i}^{2\pm} T_{3-i}'^2 \beta_{ij}(x,x') \right) u(x')$$
$$+ \frac{h}{2} \sum_{x' \in \gamma_{ij}^\star} \left(T_{3-i}^{2\pm} \beta_{ij}(x,x') - T_{3-i}^{2\pm} T_{3-i}'^{2\pm} \beta_{ij}(x,x') \right) u(x'), \qquad x \in \gamma_{ij}^\star.$$

Terms analogous to ζ_{ij}, $\tilde{\zeta}_{ij}$, $\bar{\chi}_{ij}$ and $\bar{\eta}_i$ have been estimated in [8, 11] whereby it follows that:

$$\|\zeta_{ij}\|_{\gamma_{ij}} \le Ch^{s-1} \|\alpha_{ij}\|_{H^{s-3/2}(\Gamma_{ij})} \|u\|_{H^s(\Omega)}, \qquad 2 < s \le 3, \tag{20}$$

$$\sum_{i,j=1}^2 \sum_{x \in \gamma_{ij}^\star} |\tilde{\zeta}_{ij}(x)| \le Ch \max_{i,j} \|\alpha_{ij}\|_{H^{s-3/2}(\Gamma_{ij})} \|u\|_{H^s(\Omega)}, \qquad s > 2, \tag{21}$$

$$\|[\bar{\chi}]\|_{\bar{\gamma}_{ij}} \le Ch^{s-1} \|\beta_{ij}\|_{H^{s-1}(\Gamma_{ij}^2)} \|u\|_{H^s(\Omega)}, \qquad 2 < s \le 3, \tag{22}$$

$$\|\bar{\eta}_i\|_{\dot{H}^{1/2}(\gamma_{3-i,j}^-)} \le Ch^{s-2} \sqrt{\log \frac{1}{h}} \|u\|_{H^s(\Omega)}, \qquad 2.5 < s < 3 \tag{23}$$

and

$$\|\bar{\eta}_i\|_{\dot{H}^{1/2}(\gamma_{3-i,j}^-)} \le Ch \left(\log \frac{1}{h} \right)^{3/2} \|u\|_{H^s(\Omega)}, \qquad s = 3. \tag{24}$$

To estimate $\hat{\chi}$ let us consider a function $U(x_k)$ of one variable $x_k \in [0,1]$, $k = 1,2$. Then the expression $U(x_k) - T_k^2 U(x_k)$ is a bounded linear functional of $U \in H^{s-1}(0,1)$, $s > 1.5$, which vanishes for $U = 1$ and $U = x_k$. Using the Bramble-Hilbert lemma one obtains

$$|U(x_k) - T_k^2 U(x_k)| \le Ch^{s-1.5} \|U\|_{H^{s-1}(x_k-h,x_k+h)}, \qquad 1.5 < s \le 3.$$

Analogously, using inequality [12]

$$\|U\|_{L^2(0,h)} \le Ch^{1/2} \|U\|_{H^r(0,1)}, \qquad r > 0.5$$

one obtains

$$|U(0) - T_k^{2+} U(0)| \le Ch^{1/2} \|U\|_{H^1(0,h)} \le Ch \|U\|_{H^{s-1}(0,1)}, \qquad s > 2.5.$$

Analogous bound holds for $|U(1) - T_k^{2-} U(1)|$. From these inequalities we immediately obtain

$$|\hat{\chi}(x)| \le Ch^{s-1} \|T_{3-i}^2 \beta(x,\cdot)\|_{H^{s-1}(\Gamma_{ij})} \|u\|_{C(\bar{\Omega})}, \qquad x \in \gamma_{ij}, \quad 2.5 < s \le 3$$

and

$$|\hat{\chi}(x)| \le Ch^{s-1} \|T_{3-i}^{2\pm} \beta(x,\cdot)\|_{H^{s-1}(\Gamma_{ij})} \|u\|_{C(\bar{\Omega})}, \qquad x \in \gamma_{ij}^\star, \quad 2.5 < s \le 3.$$

Summing over the nodes $x \in \bar{\gamma}_{ij}$, after obvious majoration, one obtains:

$$\|[\hat{\chi}]\|_{\bar{\gamma}_{ij}} \le Ch^{s-1} \|\beta_{ij}\|_{H^{s-1}(\Gamma_{ij}^2)} \|u\|_{H^s(\Omega)}, \qquad 2.5 < s \le 3, \tag{25}$$

The assertion follows from (16)-(25). $\square$

5. The Case of Non-coercive Operator

Let us consider now the case when the coerciveness condition (14) is not satisfied. For the sake of simplicity we assume that

$$\beta_{ij}(x, x') = \beta_{ij}(x', x), \qquad i, j = 1, 2. \tag{26}$$

Hence, the operator $A_h = -\bar{\Delta}_h$ is selfadjoint, its eigenvalues λ_i^h are real and the eigenfunctions $v_i(x)$ can be orthonormed with regard to the inner product $[\cdot, \cdot]$. From (12) and (13) it follows that

$$\lambda_i^h + \bar{\kappa} > 0, \qquad i = 1, 2, \dots, N, \qquad N = (n + 1)^2,$$

therefore there exists the index k, $1 \le k \ll N$, such that

$$-\bar{\kappa} < \lambda_1^h \le \lambda_2^h \le \cdots \le \lambda_k^h \le 0 < \lambda_{k+1}^h \le \cdots \le \lambda_N^h \asymp n^2 = h^{-2}.$$

Let us introduce linear operator $\tilde{A}_h = A_h + \bar{\kappa} I_h$, where I_h is identity operator, and the corresponding bilinear form $\tilde{a}_h(v, w) = [\tilde{A}_h v, w] = a_h(v, w) + \tilde{\kappa}[v, w]$. Operator $\tilde{A}_h$ is selfadjoint and positive definite, so we can define the energy norms

$$\|[v]\|_{\tilde{A}_h}^2 = [\tilde{A}_h v, v] \qquad \text{and} \qquad \|[v]\|_{\tilde{A}_h^{-1}}^2 = [\tilde{A}_h^{-1} v, v].$$

From (12) and (13) follows that

$$c_3 \|[v]\|_{H^1(\bar{\omega})}^2 \le \|[v]\|_{\tilde{A}_h}^2 = \tilde{a}_h(v, v) \le c_4 \|[v]\|_{H^1(\bar{\omega})}^2, \tag{27}$$

where $c_3 = \bar{m}$ and $c_4 = c_2 + \bar{\kappa}$.

Let us assume that 0 is not the eigenvalue of A_h. Then there exists the inverse operator A_h^{-1} and from (10) follows $v = A_h^{-1} \bar{f}$. Let $f_i = [\bar{f}, v_i]$, $i = 1, 2, \dots, N$, be Fourier's coefficients of $\bar{f}$. Using Parseval's equality we immediately obtain

$$\|[v]\|_{\tilde{A}_h} = \left\{ \sum_{i=1}^N (\lambda_i^h + \bar{\kappa}) \left(\frac{f_i}{\lambda_i^h} \right)^2 \right\}^{1/2} = \left\{ \sum_{i=1}^N \left(\frac{\lambda_i^h + \bar{\kappa}}{\lambda_i^h} \right)^2 \frac{f_i^2}{\lambda_i^h + \bar{\kappa}} \right\}^{1/2} \le \max_i \left| \frac{\lambda_i^h + \bar{\kappa}}{\lambda_i^h} \right| \left(\sum_{i=1}^N \frac{f_i^2}{\lambda_i^h + \bar{\kappa}} \right)^{1/2}$$

$$= \max_i \left| \frac{\lambda_i^h + \bar{\kappa}}{\lambda_i^h} \right| \|[\bar{f}]\|_{\tilde{A}_h^{-1}}.$$

For $1 \le i \le k$ we have

$$\left| \frac{\lambda_i^h + \bar{\kappa}}{\lambda_i^h} \right| = \frac{\lambda_i^h + \bar{\kappa}}{|\lambda_i^h|} = \frac{\bar{\kappa}}{|\lambda_i^h|} - 1 \le \frac{\bar{\kappa}}{|\lambda_k^h|} - 1,$$

while for $i \ge k + 1$ holds

$$\left| \frac{\lambda_i^h + \bar{\kappa}}{\lambda_i^h} \right| = \frac{\lambda_i^h + \bar{\kappa}}{\lambda_i^h} = \frac{\bar{\kappa}}{\lambda_i^h} + 1 \le \frac{\bar{\kappa}}{\lambda_{k+1}^h} + 1.$$

In such a way, for the solution of (10) we obtained a priori estimate

$$\|[v]\|_{\tilde{A}_h} \le c_5 \|[\bar{f}]\|_{\tilde{A}_h^{-1}}, \qquad \text{where} \qquad c_5 = \max \left\{ \frac{\bar{\kappa}}{|\lambda_k^h|} - 1, \frac{\bar{\kappa}}{\lambda_{k+1}^h} + 1 \right\}. \tag{28}$$

Applying (28) to (15) and using (27) we obtain

$$\|[z]\|_{H^1(\bar{\omega})} \le \frac{c_5}{\sqrt{c_3}} \|[\psi]\|_{\tilde{A}_h^{-1}} = \frac{c_5}{\sqrt{c_3}} \sup_{w \ne 0} \frac{[\psi, w]}{\|[w]\|_{\tilde{A}_h}},$$

whereby, in the same manner as in the proof of Theorem 4.3, one obtains a priori estimate of the form (16). Notice that the constant C in this a priori estimate now depends on c_5 (i.e. on $1/\min_i |\lambda_i^h|$).

In such a manner, we proved the following assertion.

Theorem 5.1. *Let the conditions (11) and (26) hold and let 0 is not eigenvalue of the problem (7) nor of the difference operator $A_h = -\bar{\Delta}_h$. Then the solution of FDS (10) converges to the solution of BVP (1)-(2) and the convergence rate estimates (17) and (18) hold.*

Remark 5.2. *Since eigenvalues of the difference problem converge to the corresponding eigenvalues of the BVP when h tends to 0 (see e.g. [14]) it is enough to assume that 0 is not eigenvalue of the problem (7) and h is sufficiently small.*

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