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New Results on the Stochastic Gilpin-Ayala Model with Delays

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Abstract. In this paper, a stochastic Gilpin-Ayala model with delays is investigated. Some new sufficient conditions for the existence of positive and global solution of the model are obtained. Moreover, asymptotic behavior of the model is discussed. It is significant that these results improve the previous work (Lian and Hu, Stochastic delay Gilpin-Ayala competition models, Stochastic Dynamics 6 (2006) 561-576).

1. Introduction

After the first ecological population model

$$\frac{dx_i}{dt} = x_i(t)\Big[r_i + \sum_{j=1}^{n} a_{ij}x_j(t)\Big],\ 1 \leqslant i \leqslant n, \tag{1.1}$$

introduced independently by [20, 29], named Lotka-Volterra model, there are a lot of excellent papers on the study of dynamic behavior of model (1.1), such as [2, 3, 7, 8, 10, 11, 21, 25, 26, 32], see the references cited there in for more details. Meanwhile, there are some papers on the study of model (1.1) with stochastic perturbation, see for instance [9, 13, 17–19, 23, 24, 30] and so on. The typical feature of this model is that the rate of change of the size of each species is a linear function of the size of the interacting species, which is simple for study but does not describe the facts precisely. Gilpin and Ayala [5] pointed that Lotka-Volterra system is the linearization of the per capita growth rates about the point of equilibrium. They claimed that a little more complicated model was needed in order to yield more realistic solutions. Hence, they proposed a modified model(known as Gilpin-Ayala model)

$$\frac{dx_i}{dt} = x_i(t)\Big[r_i + \sum_{j=1}^{n} a_{ij}x_j^{\theta_j}(t)\Big],\ 1 \leqslant i \leqslant n, \tag{1.2}$$

where $x_i(1 \leqslant i \leqslant n)$ are the population densities, $r_i(1 \leqslant i \leqslant n)$ are the intrinsic exponential growth rates, and θ_j are the parameters which modify the classical Lotka-Volterra model, $a_{ij}(1 \leqslant i, j \leqslant n)$ represent the effects

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of interspecifi (for $i \neq j$) and intraspecifi (for $i = j$)interaction of populations. There are some papers on the study of dynamic behavior of model (1.2), see [1, 4, 15, 16, 27, 31] for more details.

Kuang [12] pointed out that more realistic models should include some of the past states of the species; that means that a real system should be modeled by differential equations with time delays, and any model of species dynamics without delays is at best an approximate description of the real species system. More detailed arguments on the importance and usefulness of time delays in realistic models may also be found in the classical books of [6, 22]. Motivated by this, Vasilova and Jovanovic [28] considered a stochastic Gilpin-Ayala competition model with infinite delay, and verified that the environmental noise included in the model provide not only an existence of a positive global solution, but also the stochastic ultimate boundedness of the solution. Furthermore, they obtained certain asymptotic results regarding a large time behavior. Lian and Hu [14] replaced the rate r_i by an average value plus a random fluctuation term $r_i + \sum_{j=1}^{n} \sigma_{ij} x_j(t) d\omega(t)$, where σ_{ij} is the intensity of noise, and presented a stochastic Gilpin-Ayala model with constant delays in the following form

$$dx_i = x_i(t)\Big\{\Big[r_i + \sum_{j=1}^{n} a_{ij} x_j^{\theta_j}(t-\tau)\Big]dt + \sum_{j=1}^{n} \sigma_{ij} x_j(t) d\omega(t)\Big\},\ 1 \leqslant i \leqslant n.$$

Under the assumption that $\theta_i < \frac{5}{4}$, they established some sufficient conditions for the existence of positively global solution, and gave the asymptotic moment estimation and pathwise estimation of the solutions of the system.

In this paper we consider the following more general stochastic Gilpin-Ayala model with delays

$$dx_i(t) = x_i(t)\Big\{\Big[r_i - a_{ii} x_i^{\theta_{ii}}(t-\tau_{ii}) + \sum_{j\neq i}^{n} a_{ij} x_j^{\theta_{ij}}(t-\tau_{ij})\Big]dt + \sum_{j=1}^{n} \sigma_{ij} x_j(t) d\omega_j(t)\Big\},\ 1 \leqslant i \leqslant n, \tag{1.3}$$

with the initial conditions

$$x_i(\theta) = \varphi_i(\theta) > 0 \text{ on } [-\tau, 0], \tag{1.4}$$

where $\tau = \max_{i,j}\{\tau_{ij}\}$ and $\tau_{ij} \geqslant 0$, $a_{ii} > 0$, $\theta_{ij} > 0\,(1 \leqslant i, j \leqslant n)$ are real constants.

Remark 1. System (1.3) is a general model, which includes the following three types of n-spaces population models:

- Competition model: $a_{ij} < 0$ for all $i \neq j$.
- Corporation model: $a_{ij} > 0$ for all $i \neq j$.
- Competition-Corporation model: $a_{ij} > 0$ for some $i \neq j$, and $a_{ij} < 0$ for the others.

Remark 2. There are some results on dynamic behaviors of the competition Gilpin-Ayala model. But to the last two types of models, there are few results on them. Thus, in this paper we mainly discuss the stochastic dynamic behavior of the corporation and competition-corporation models, respectively.

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with filtration $\{\mathfrak{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions, i.e., it is increasing and right continuous while $\mathfrak{F}_0$ contains all P-null sets, see [24] for more details. Moreover, let $\omega(t)$ be a one-dimensional Brownian motion defined on the filtered space and $\Re_+^n = \{x \in \Re^n | \ x_i > 0 \text{ for all } 1 \leqslant i \leqslant n\}$. Finally, denote the trace norm of a matrix A by $|A| = \sqrt{\text{trace}(A^\top A)}$(where $A^\top$ denotes the transpose of a vector or matrix A) and its operator norm by $A = \sup\{|Ax| : |x| = 1\}$.

We assume $r_i \in \Re, a_{ij} \in \Re$ and $\varphi \in C([-\tau, 0], \Re_+^n)$, and present the following conditions

$$(H_1) \begin{cases} \sigma_{ii} > 0, \text{ for } 1 \leqslant i \leqslant n, \\ \sigma_{ij} \geqslant 0, \text{ for } i \neq j. \end{cases}$$

$$(H_2)\ \theta_{ij} < 2 \ \text{ for } i \neq j.$$

(H_2') $\theta_{ij} \leqslant 2$ for $i \neq j$.

The aim of the paper is to establish some results on the existence of positively global solutions, asymptotic moment estimation and pathwise estimation to model (1.3)-(1.4), which are weaker than the corresponding ones presented in [14].

2. Positive and Global Solution

In order to a stochastic differential equation have a unique global solution, namely no explosion in a finite time, for any given initial value, the coefficients of Eqs.(1.3)-(1.4) are generally required to satisfy both the linear growth condition and the local Lipschitz condition. However, the coefficients of Eq.(1.3) do not satisfy the linear growth condition but they are locally Lipschitz continuous, thus the solution of Eq.(1.3)-(1.4) may explode in a finite time. Under some simple conditions, the following theorem shows that there exists a positive and global solution of the system.

Theorem 2.1. *Assume that (H_1) and (H_2) hold, then there is a unique solution $x(t)$ of Eqs.(1.3)-(1.4) for $t \geqslant 0$, and the solution remains in $\Re_+^n$ with probability 1.*

Proof. Since the coefficients of the system are locally Lipschitz continuous, for any given initial value $\varphi \in C([-\tau, 0], \Re_+^n)$ there is a unique local positive solution $x(t)$ on interval $[0, \rho)$, where ρ is the explosion time. In order to prove that the solution is global, we only need to show that $\rho = \infty$ a.s.

Let k_0 be sufficiently large for every component of φ lying within the interval $[k_0^{-1}, k_0]$, and for every integer $k > k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \rho) : x_i(t) \notin (k^{-1}, k), \text{ for some } i = 1, 2, ..., n\}$$

and let $\inf \phi = \infty$, where ϕ denotes the empty set as usual. One can easily see that τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$. If we can show $\tau_\infty = \infty$ a.s., then $\rho = \infty$, a.s. and furthermore $x(t) \in \Re_+^n$ a.s. for $t \geqslant -\tau$. In other words, in order to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s., or for all $T > 0$, $P(\tau_k \leqslant T) \to 0$ as $k \to \infty$.

To show above statement, we define a C^2-function $V_1 : \Re_+^n \to \Re_+$ by the form

$$V_1(x) = \sum_{i=1}^{n} \Big(x_i^{\alpha_i} - \ln x_i - \frac{\ln \alpha_i}{\alpha_i} \Big),$$

where $\alpha_i \in (0, 1) (1 \leqslant i \leqslant n)$ are real constants. According to Itô's formula, we have

$$\begin{aligned} dV_1(x) = & \sum_{i=1}^{n} \Big(\alpha_i x_i^{\alpha_i} - 1\Big) \Big[r_i - a_{ii} x_i^{\theta_{ii}}(t - \tau_{ii}) + \sum_{j\neq i}^{n} a_{ij} x_j^{\theta_{ij}}(t - \tau_{ij}) \Big] dt \\ & + \frac{1}{2} \sum_{i=1}^{n} \Big[\alpha_i(\alpha_i - 1) x_i^{\alpha_i} + 1 \Big] \Big(\sum_{j=1}^{n} \sigma_{ij} x_j \Big)^2 dt + \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i x_i^{\alpha_i} - 1) \sigma_{ij} x_j d\omega(t). \end{aligned}$$

Here and in the following, we drop (t) from $x(t)$ in case of no confusion. Moreover, it is easy to see that

$$\sum_{i=1}^{n} \alpha_i(1 - \alpha_i) x_i^{\alpha_i} \Big(\sum_{j=1}^{n} \sigma_{ij} x_j \Big)^2 \geqslant \sum_{i=1}^{n} \alpha_i (1 - \alpha_i) \sigma_{ii}^2 x_i^{2+\alpha_i}$$

and by Young inequality,

$$\sum_{i=1}^{n} \sum_{j\neq i}^{n} \Big(\alpha_i x_i^{\alpha_i} - 1 \Big) a_{ij} x_j^{\theta_{ij}}(t - \tau_{ij}) \leqslant \frac{4}{27} \sum_{i=1}^{n} \sum_{j\neq i}^{n} |a_{ij}(\alpha_i x_i^{\alpha_i} - 1)|^3 + \sum_{i=1}^{n} \sum_{j\neq i}^{n} x_i^{1.5\theta_{ij}}(t - \tau_{ij}).$$

As a result, we obtain

$$dV_1(x) \leqslant \Big\{ \sum_{i=1}^n r_i \left(\alpha_i x_i^{\alpha_i} - 1\right) - \sum_{i=1}^n a_{ii} \left(\alpha_i x_i^{\alpha_i} - 1\right) x_i^{\theta_{ii}}(t-\tau_{ii}) + \frac{4}{27} \sum_{i=1}^n \sum_{j \neq i}^n |a_{ij}(\alpha_i x_i^{\alpha_i} - 1)|^3$$

$$+ \sum_{i=1}^n \sum_{j\neq i}^n x_j^{1.5\theta_{ij}}(t-\tau_{ij}) - \frac{1}{2} \sum_{i=1}^n \alpha_i(1-\alpha_i)\sigma_{ii}^2 x_i^{2+\alpha_i} + \frac{1}{2}|\sigma x|^2 \Big\} dt + \sum_{i=1}^n \sum_{j=1}^n (\alpha_i x_i^{\alpha_i} - 1)\sigma_{ij} x_j d\omega(t),$$

where $\sum_{i=1}^n \left(\sum_{j=1}^n \sigma_{ij} x_j \right)^2 = |\sigma x|^2$. Define further that

$$V_2(t) = \int_{t-\tau_{ij}}^t \sum_{i=1}^n \sum_{j\neq i}^n x_j^{1.5\theta_{ij}}(s) ds.$$

By using Itô's formula again, we have

$$d\left(V_1(x) + V_2\right) \leqslant F(x,t)dt - \frac{1}{4} \sum_{i=1}^n \alpha_i(1-\alpha_i)\sigma_{ii}^2 x_i^{2+\alpha_i} dt + \sum_{i=1}^n \sum_{j=1}^n (\alpha_i x_i^{\alpha_i} - 1)\sigma_{ij} x_j d\omega(t), \tag{2.1}$$

where

$$F(x,t) = \sum_{i=1}^n \sum_{j\neq i}^n x_j^{1.5\theta_{ij}} + \sum_{i=1}^n \left(\alpha_i x_i^{\alpha_i} - 1\right)\left[r_i - a_{ii} x_i^{\theta_{ii}}(t-\tau_{ii})\right]$$

$$+ \frac{4}{27} \sum_{i=1}^n \sum_{j\neq i}^n |a_{ij}(\alpha_i x_i^{\alpha_i} - 1)|^3 - \frac{1}{4} \sum_{i=1}^n \alpha_i(1-\alpha_i)\sigma_{ii}^2 x_i^{2+\alpha_i} + \frac{1}{2}|\sigma x|^2.$$

Under the assumptions of this theorem, we can choose some suitable constants $\alpha_i \in (0,1)$ such that, for $(x,t) \in \Re_+^n \times [-\tau, \infty)$, $F(x,t)$ has an upper positive bound, say K. In fact, if $x \in D := \{x | x_i > M > \alpha_i^{-\alpha_i}\}$, then

$$F(x,t) \leqslant \sum_{i=1}^n \sum_{j\neq i}^n x_j^{1.5\theta_{ij}} + \sum_{i=1}^n r_i \left(\alpha_i x_i^{\alpha_i} - 1\right) + \frac{4}{27} \sum_{i=1}^n \sum_{j\neq i}^n |a_{ij}(\alpha_i x_i^{\alpha_i} - 1)|^3 - \frac{1}{4} \sum_{i=1}^n \alpha_i(1-\alpha_i)\sigma_{ii}^2 x_i^{2+\alpha_i} + \frac{1}{2}|\sigma x|^2,$$

which is bounded on $(x,t) \in D \times [-\tau, \infty)$. Meanwhile, the continuousness of the second variable of function $F(x,t)$ on $[-\tau, \infty)$ yields that $F(x,t)$ is bounded on the domain $(x,t) \in D^c \times [-\tau, \infty)$.

Thus, it follows from inequality (2.1) that

$$V_1(x(t)) + V_2(t) + \frac{1}{4} \int_0^t \sum_{i=1}^n \alpha_i(1-\alpha_i)\sigma_{ii}^2 x_i^{2+\alpha_i} ds \leqslant V_1(x(0)) + V_2(0) + \int_0^t K ds + M(t), \tag{2.2}$$

where

$$M(t) = \int_0^t \sum_{i=1}^n \sum_{j=1}^n (\alpha_i x_i^{\alpha_i} - 1)\sigma_{ij} x_j d\omega(s)$$

is a real-valued continuous local martingale vanishing at $t = 0$. Let $t = \tau_k \wedge T$ and take expectations on both sides of (2.2), we obtain

$$E\Big[V_1(x(\tau_k \wedge T)) + \frac{1}{4} \int_0^{\tau_k \wedge T} \sum_{i=1}^n \alpha_i(1-\alpha_i)\sigma_{ii}^2 x_i^{2+\alpha_i} ds \Big] \leqslant V_1(x(0)) + V_2(0) + KT < \infty. \tag{2.3}$$

Considering the fact that for any $\omega \in \{\tau_k \leqslant T\}$ there exists an i such that $x_i(\tau_k, \omega) = k$ or k^{-1}, then for every i, we have $V(x(\tau_k, \omega)) \notin (k^{-1}, k)$. Thus

$$V_1(x(\tau_k, \omega)) \geqslant (k^{-\alpha_i} + \ln k - \frac{\ln \alpha_i}{\alpha_i}) \wedge (k^{\alpha_i} - \ln k - \frac{\ln \alpha_i}{\alpha_i}),$$

which together with (2.3) yields

$$EV_1(x(\tau_k \wedge T)) \geqslant E\left[I_{\{\tau_k \leqslant T\}}(\omega) V_1(x(\tau_k, \omega))\right] \geqslant P\{\tau_k \leqslant T\}(k^{-\alpha_i} + \ln k - \frac{\ln \alpha_i}{\alpha_i}) \wedge (k^{\alpha_i} - \ln k - \frac{\ln \alpha_i}{\alpha_i}),$$

which implies that

$$\lim_{k \to +\infty} P\{\tau_k \leqslant T\} = 0, \text{ i.e., } P\{\tau_\infty \leqslant T\} = 0.$$

It follows from the arbitrariness of T that $P\{\tau_\infty = \infty\} = 1$. □

3. Asymptotic Moment Estimation

Since Eqs.(1.3)-(1.4) does not have an explicit solution, the study of asymptotic moment behavior is essential if we want to get a deeper understanding of the underlying process. In the following, we will discuss the asymptotic moment estimation of the solutions.

Theorem 3.1. *Assume that hypotheses (H_1) and (H_2) hold, then for any $\alpha_i \in (0, 1)$, there exists a positive constant K_α such that the solutions of Eqs.(1.3)-(1.4) have the following property*

$$\limsup_{t \to \infty} \frac{1}{t} E\Big[\int_0^t \sum_{i=1}^n x_i^{2+\alpha_i} ds\Big] \leqslant K_\alpha. \tag{3.1}$$

Proof. It follows from (2.2) that

$$E\Big[V_1(x(t)) + V_2(t) + \frac{1}{4}\int_0^t \sum_{i=1}^n \alpha_i(1-\alpha_i)\sigma_{ii}^2 x_i^{2+\alpha_i} ds\Big] \leqslant V_1(x(0)) + V_2(0) + Kt,$$

which yields that there exists a positive constant K_α such that

$$E\Big[\int_0^t \sum_{i=1}^n x_i^{2+\alpha_i} ds\Big] \leqslant K_\alpha t + M,$$

where

$$K_\alpha = \frac{4K}{\min\{\alpha_i(1-\alpha_i)\sigma_{ii}^2,\ 1 \leqslant i \leqslant n\}} \text{ and } M = \frac{4(V_1(x(0)) + V_2(0))}{\min\{\alpha_i(1-\alpha_i)\sigma_{ii}^2,\ 1 \leqslant i \leqslant n\}}.$$

The required assertion (3.1) follows immediately. □

4. Pathwise Estimation

In this section, we will investigate the pathwise estimation of the solutions to Eqs.(1.3)-(1.4).

Theorem 4.1. *Assume that hypotheses (H_1) and (H_2) hold, then there exists a positive constant K such that the solutions of Eqs.(1.3)-(1.4) have the property*

$$\limsup_{t \to \infty} \frac{1}{t}\Big[\ln \prod_{i=1}^n x_i(t) + \frac{1}{4}\lambda_{\min}(\sigma^\top \sigma)\int_0^t |x(s)|^2 ds\Big] \leqslant K \text{ a.s.}, \tag{4.1}$$

where $\lambda_{\min}(\sigma^\top\sigma)$ is the smallest eigenvalue of the matrix $\sigma^\top\sigma$.

Proof. For each $1 \leqslant i \leqslant n$, by applying Itô's formula to $\ln x_i(t)$, we obtain

$$d\ln x_i(t) = \Big[r_i - a_{ii}x_i^{\theta_{ii}}(t-\tau_{ii})\Big]dt + \Big[\sum_{j\neq i}^{n} a_{ij}x_j^{\theta_{ij}}(t-\tau_{ij}) - \frac{1}{2}\Big(\sum_{j=1}^{n}\sigma_{ij}x_j\Big)^2\Big]dt + \sum_{j=1}^{n}\sigma_{ij}x_j d\omega(t).$$

Integration of both sides of above equality from 0 to t gives

$$\ln(x_i(t)) = \ln(x_i(0)) + M_i(t) + \int_0^t \Big[r_i - a_{ii}x_i^{\theta_{ii}}(s-\tau_{ii}) + \sum_{j\neq i}^{n} a_{ij}x_j^{\theta_{ij}}(s-\tau_{ij}) - \frac{1}{2}\Big(\sum_{j=1}^{n}\sigma_{ij}x_j\Big)^2\Big]ds, \tag{4.2}$$

where $M_i(t) = \int_0^t \sum_{j=1}^{n}\sigma_{ij}x_j d\omega(s)$ is a real-valued continuous local martingale vanishing at $t = 0$ with quadratic form

$$\langle M_i(t), M_i(t)\rangle = \int_0^t \Big(\sum_{j=1}^{n}\sigma_{ij}x_j\Big)^2 ds.$$

Fix $\varepsilon \in (0, \frac{1}{2})$ arbitrarily, for each integer $k \geqslant 1$, by using the exponential martingale inequality we get

$$P\Big\{\sup_{0\leqslant t\leqslant k}\Big[M_i(t) - \frac{\varepsilon}{2}\langle M_i(t), M_i(t)\rangle\Big] > \frac{2}{\varepsilon}\ln k\Big\} < \frac{1}{k^2},$$

which together with Borel-Cantelli lemma yields, with probability one,

$$\sup_{0\leqslant t\leqslant k}\Big[M_i(t) - \frac{\varepsilon}{2}\langle M_i(t), M_i(t)\rangle\Big] \leqslant \frac{2}{\varepsilon}\ln k$$

holds for all but finitely many k, i.e., there exist $\Omega_i \subset \Omega$ with $P(\Omega_i) = 1$ such that, for any $\omega \in \Omega_i$, there is an integer $k_i = k_i(\omega)$ such that

$$M_i(t) \leqslant \frac{\varepsilon}{2}\langle M_i(t), M_i(t)\rangle + \frac{2}{\varepsilon}\ln k,\ 0 \leqslant t \leqslant k$$

for any $k \geqslant k_i(\omega)$. Thus, Eq.(4.2) results in

$$\ln(x_i(t)) \leqslant \ln(x_i(0)) + \int_0^t \Big[r_i - a_{ii}x_i^{\theta_{ii}}(s-\tau_{ii}) + \sum_{j\neq i}^{n} a_{ij}x_j^{\theta_{ij}}(s-\tau_{ij}) - \frac{1-\varepsilon}{2}\Big(\sum_{j=1}^{n}\sigma_{ij}x_j\Big)^2\Big]ds + \frac{2}{\varepsilon}\ln k \tag{4.3}$$

for $0 \leqslant t \leqslant k$, and $k \geqslant k_i(\omega)$ where $\omega \in \Omega_i$. Let $\Omega_0 = \bigcap_{i=1}^{n}\Omega_i$, it is obvious that $P(\Omega_0) = 1$. Furthermore, for any $\omega \in \Omega_0$, let $k_0(\omega) = \max\{k_i(\omega) : 1 \leqslant i \leqslant n\}$, then for any $\omega \in \Omega_0$, it follows from (4.3) that

$$\begin{aligned}\sum_{i=1}^{n}\ln(x_i(t)) \leqslant& \sum_{i=1}^{n}\ln(x_i(0)) + \frac{2n}{\varepsilon}\ln k + \int_0^t \sum_{i=1}^{n}\Big[r_i - a_{ii}x_i^{\theta_{ii}}(s-\tau_{ii}) \\ &+ \sum_{j\neq i}^{n} a_{ij}x_j^{\theta_{ij}}(s-\tau_{ij}) - \frac{1-\varepsilon}{2}\Big(\sum_{j=1}^{n}\sigma_{ij}x_j\Big)^2\Big]ds \\ \leqslant& C + \frac{2n}{\varepsilon}\ln k + \int_0^t \sum_{i=1}^{n}\Big[r_i + \sum_{j\neq i}^{n} a_{ij}x_j^{\theta_{ij}}(s) - \frac{1-\varepsilon}{2}\Big(\sum_{j=1}^{n}\sigma_{ij}x_j\Big)^2\Big]ds\end{aligned} \tag{4.4}$$

for $0 \leqslant t \leqslant k$, and $k \geqslant k_0(\omega)$, where

$$C = \sum_{i=1}^{n}\ln(x_i(0)) + \int_{-\tau}^{0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}|a_{ij}|x_j^{\theta_{ij}}(s)ds.$$

Here, in the last inequality, we insert the following inequality

$$\int_0^t \sum_{i=1}^n \sum_{j\neq i}^n a_{ij} x_j^{\theta_{ij}}(s-\tau_{ij})ds = \int_{-\tau_{ij}}^{t-\tau_{ij}} \sum_{i=1}^n \sum_{j\neq i}^n a_{ij} x_j^{\theta_{ij}}(s)ds \leqslant \Big(\int_{-\tau}^0 + \int_0^t\Big) \sum_{i=1}^n \sum_{j\neq i}^n |a_{ij}| x_j^{\theta_{ij}}(s)ds.$$

In view of assumption (H_2), one know that there exists a positive constant K such that, for any $(t,x) \in [-\tau,\infty)\times\Re^n_+$,

$$\sum_{i=1}^n \Big[r_i + \sum_{j\neq i}^n |a_{ij}| x_j^{\theta_{ij}} - \frac{1}{4}\Big(\sum_{j=1}^n \sigma_{ij}x_j\Big)^2 \Big] \leqslant K, \tag{4.5}$$

which together with (4.4) implies

$$\ln\Big(\prod_{i=1}^n x_i(t)\Big) + \Big(\frac{1}{4}-\frac{\varepsilon}{2}\Big)\int_0^t |\sigma x|^2 ds \leqslant C + \frac{2n}{\varepsilon}\ln k + Kt$$

for $0 \leqslant t \leqslant k$, and $k \geqslant k_i(\omega)$. Thus, for any $\omega \in \Omega_0$, if $k-1 \leqslant t \leqslant k$ and $k \geqslant k(\omega)$, then

$$\frac{1}{t}\Big[\ln\Big(\prod_{i=1}^n x_i(t)\Big) + \Big(\frac{1}{4}-\frac{\varepsilon}{2}\Big)\int_0^t |\sigma x(s)|^2 ds\Big] \leqslant \frac{C + \frac{2n}{\varepsilon}\ln k}{t} + K$$

or

$$\limsup_{t\to\infty} \frac{1}{t}\Big[\ln\Big(\prod_{i=1}^n x_i(t)\Big) + \Big(\frac{1}{4}-\frac{\varepsilon}{2}\Big)\int_0^t |\sigma x(s)|^2 ds\Big] \leqslant K,$$

where we use the fact that $|\sigma x|^2 \geqslant \lambda_{\min}(\sigma^\top\sigma)|x|^2$. In view of the arbitrary choose of ε, we can let $\varepsilon \to 0$, then the required assertion (4.1) is obtained. □

Corollary 4.1. *Assume that hypotheses (H_1) and (H_2') hold, and $\sum_{j=1}^n \sigma_{ij}^2 \geqslant 4\sum_{j\neq i}^n a_{ij}$, then there exists a positive constant K such that the solutions of Eqs.(1.3)-(1.4) have the property*

$$\limsup_{t\to\infty} \frac{1}{t}\Big[\ln \prod_{i=1}^n x_i(t) + \frac{1}{4}\lambda_{\min}(\sigma^\top\sigma)\int_0^t |x(s)|^2 ds\Big] \leqslant K \ a.s.$$

It follows from (4.5), in the proof of Theorem 4.1, that the assertion is easily obtained, thus we omit it here.

Theorem 4.2. *Assume that hypotheses (H_1) and (H_2) hold, and there exist positive constants λ and ρ, with $2\rho > \lambda$, such that*

$$(H_3)\ \Big(\sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}x_i x_j\Big)^2 \leqslant \lambda |x|^4 \ \ \text{and}\ \Big(\sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}x_i^2 x_j\Big)^2 \geqslant \rho|x|^6,$$

then there exists a positive constant K such that the solutions of Eqs.(1.3)-(1.4) have the property

$$\limsup_{t\to\infty} \frac{1}{t}\Big[\ln|x(t)|^2 + \frac{2\rho-\lambda}{2}\int_0^t |x(s)|^2 ds\Big] \leqslant K \ a.s. \tag{4.6}$$

Proof. Define a C^2- function $V : \Re^n_+ \to \Re_+$ by the form

$$V(x) = \ln|x|^2.$$

By applying Itô's formula, we obtain

$$\begin{aligned}dV(x) \leqslant & \Big\{ \frac{2}{|x|^2}\Big[\sum_{i=1}^{n} r_i x_i^2 + \sum_{i=1}^{n}\sum_{j\neq i}^{n} a_{ij} x_i^2 x_j^{\theta_{ij}}(t-\tau_{ij})\Big] \\ & + \frac{1}{|x|^2}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j\Big)^2 - \frac{2}{|x|^4}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i^2 x_j\Big)^2\Big\}dt + \frac{2}{|x|^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i^2 x_j d\omega(t).\end{aligned} \tag{4.7}$$

Integration of both sides of (4.7) from 0 to t gives

$$\begin{aligned}V(x) \leqslant & V(x(0)) + \int_0^t \Big\{ \frac{2}{|x|^2}\Big[\sum_{i=1}^{n} r_i x_i^2 + \sum_{i=1}^{n}\sum_{j\neq i}^{n} a_{ij} x_i^2 x_j^{\theta_{ij}}(s-\tau_{ij})\Big] \\ & + \frac{1}{|x|^2}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j\Big)^2 - \frac{2}{|x|^4}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i^2 x_j\Big)^2\Big\}dt + \int_0^t \frac{2}{|x|^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i^2 x_j d\omega(t) \\ \leqslant & V(x(0)) + A\int_{-\tau}^0 \sum_{i=1}^{n}\sum_{j\neq i}^{n} x_j^{\theta_{ij}} ds + \int_0^t \Big\{ 2\Big(R + A\sum_{i=1}^{n}\sum_{j\neq i}^{n} x_j^{\theta_{ij}}\Big) + (\lambda - 2\rho)|x|^2\Big\} ds + M(t),\end{aligned} \tag{4.8}$$

where $A = \max_{0\leqslant i,j\leqslant n}|a_{ij}|$, $R = \max_{0\leqslant i\leqslant n}|r_i|$ and in the last inequality we insert the following formula

$$\int_0^t \sum_{i=1}^{n}\sum_{j\neq i}^{n} a_{ij}x_j^{\theta_{ij}}(s-\tau_{ij})ds = \int_{-\tau_{ij}}^{t-\tau_{ij}} \sum_{i=1}^{n}\sum_{j\neq i}^{n} a_{ij}x_j^{\theta_{ij}}(s)ds \leqslant A\Big(\int_{-\tau}^0 + \int_0^t\Big)\sum_{i=1}^{n}\sum_{j\neq i}^{n} x_j^{\theta_{ij}}(s)ds,$$

where

$$M(t) = \int_0^t \frac{2}{|x|^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i^2 x_j d\omega(t)$$

is a real-valued continuous local martingale vanishing at $t = 0$ with quadratic form

$$\langle M(t), M(t)\rangle = \int_0^t \frac{4}{|x|^4}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i^2 x_j\Big)^2 ds.$$

Fix the arbitrary small positive constant ε, then it follows from the exponential martingale inequality that, for every integer $k \geqslant 1$,

$$P\Big\{\sup_{0\leqslant t\leqslant k}\Big[M(t) - \frac{\varepsilon}{4}\langle M(t), M(t)\rangle\Big] > \frac{4}{\varepsilon}\ln k\Big\} \leqslant k^{-2},$$

which together with the convergence of $\sum_{k=1}^{\infty} k^{-2}$ and Borel-Cantelli lemma implies that, for almost all $\omega \in \Omega$, there must exist a random integer $k_0(\omega)$ such that

$$\sup_{0\leqslant t\leqslant k}\Big[M(t) - \frac{\varepsilon}{4}\langle M(t), M(t)\rangle\Big] \leqslant \frac{4}{\varepsilon}\ln k$$

for all $k \geqslant k_0(\omega)$. Thus, for $t \in [0, k]$,

$$M(t) \leqslant \frac{\varepsilon}{4}\langle M(t), M(t)\rangle + \frac{4}{\varepsilon}\ln k,$$

which together with (4.8) gives

$$V(x) \leqslant V(x(0)) + A\int_{-\tau}^0 \sum_{i=1}^{n}\sum_{j\neq i}^{n} x_j^{\theta_{ij}} ds + \int_0^t \Big\{ 2\Big(R + A\sum_{i=1}^{n}\sum_{j\neq i}^{n} x_j^{\theta_{ij}}\Big) + (\lambda - 2\rho + \varepsilon\rho)|x|^2\Big\} ds + \frac{4}{\varepsilon}\ln k. \tag{4.9}$$

Assumptions $\theta_{ij} < 2$ for all $i \neq j$ and $2\rho > \lambda$ yield that one can choose ε small enough to guarantee that, for $(x,t) \in \Re^n_+ \times [-\tau, \infty)$, there is a positive constant K such that

$$2\Big(R + A\sum_{i=1}^{n}\sum_{j\neq i}^{n} x_j^{\theta_{ij}}\Big) + \frac{1}{2}(\lambda - 2\rho + \varepsilon\rho)|x|^2 \leqslant K.$$

It follows from (4.9) that, for $t \in [0,k]$,

$$V(x(t)) + \frac{2\rho - \lambda - \varepsilon\rho}{2}\int_0^t |x|^2 ds \leqslant C + Kt + \frac{4}{\varepsilon}\ln k,$$

where $C = V(x(0)) + A\int_{-\tau}^{0}\sum_{i=1}^{n}\sum_{j\neq i}^{n} x_j^{\theta_{ij}} ds$, which yields that, for almost $\omega \in \Omega$, if $t \in [k-1,k]$ and $k \geqslant k_0(\omega)$, then

$$\limsup_{t\to\infty}\frac{1}{t}\Big[V(x(t)) + \frac{2\rho - \lambda - \varepsilon\rho}{2}\int_0^t |x(s)|^2 ds\Big] \leqslant K.$$

Thus the assertion (4.6) is obtained by let $\varepsilon \to 0$. $\square$

From the proof of Theorem 4.2 one can easily obtain the following result.

Corollary 4.2. *Assume that hypotheses (H_1) and (H_2') hold, and there exist positive constants ρ and λ, with $2\rho > \lambda + 2A$ such that (H_3) holds, then there exists a positive constant K such that the solutions of Eqs.(1.3)-(1.4) have the property*

$$\limsup_{t\to\infty}\frac{1}{t}\Big[\ln|x(t)|^2 + \frac{2\rho - \lambda - 2A}{2}\int_0^t |x(s)|^2 ds\Big] \leqslant K \ a.s.,$$

where A is defined in (4.8).

5. Conclusions

In this paper, we investigate the existence of positive and global solution, asymptotic moment estimation and pathwise estimation of the solution to a stochastic n-species Gilpin-Ayala model with delays in the form (1.3). Some interesting results are obtained under natural and simple conditions.

Conclusion 1. Consider the special case of the Eq.(1.3) with $\theta_{ij} = \theta_i$ for all $j = 1, 2, ..., n$, in [14] all results obtained are under the condition that $\theta_i < \frac{5}{4}$ for all $i = 1, 2, ..., n$. In this paper, we present a weak condition (H_2) for guarantee the existence of positive and global solution. Moreover, under the condition (H_2'), we give the same asymptotic moment estimation of the solution in large time.

Conclusion 2. If $a_{ij} < 0$, for $i \neq j$, that is Eq.(1.1) describing a n-species Gilpin-Ayala competition model, then the assumptions (H_2) and (H_2') can be removed, i.e., all results presented here still hold without any conditions imposed on θ_{ij}. This shows that some stochastic dynamics of the n-species Gilpin-Ayala competition model are the same as these of the classical Lotka-Volterra model, or the constants θ_{ij} have no substantially influence on stochastic dynamic behavior of the Gilpin-Ayala competition model, such as the global existence of positive solution, asymptotic behavior and so on.

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