Filomat 30:7 (2016), 1747–1755
DOI 10.2298/FIL1607747L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Regular Functions with Values in a Noncommutative Algebra using Clifford Analysis

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Abstract. We construct a noncommutative algebra $\mathbf{C}(2)$ that is a subalgebra of the Pauli matrices of $M(2;\mathbf{C})$, and investigate the properties of solutions with values in $\mathbf{C}(2)$ of the inhomogeneous Cauchy-Riemann system of partial differential equations with coefficients in the associated Pauli matrices. In addition, we construct a commutative subalgebra $\mathbf{C}(4)$ of $M(4;\mathbf{C})$, obtain some properties of biregular functions with values in $\mathbf{C}(2)$ on Ω in $\mathbf{C}^2 \times \mathbf{C}^2$, define a J-regular function of four complex variables with values in $\mathbf{C}(4)$, and examine some properties of J-regular functions of partial differential equations.

1. Introduction

Let $\mathcal{A}_n$ be the universal Clifford algebra constructed over a real anti-Euclidean quadratic n-dimensional vector space. Then $\mathcal{A}_n$ is generalized by the field $\mathbf{C}$ of complex numbers and bases $\{e_j : j = 0, 1, \cdots, n-1\}$, where $e_0 = id$, $e_j^2 = -1$ and $e_j e_k + e_k e_j = 0$ ($j \neq k$, $j, k = 1, 2, \cdots, n-1$). Sudbery [14] and Naser [9] developed quaternionic function theory, and Nôno [10, 11] gave some properties of hyperholomorphic functions in quaternion and octonion analysis. Gotô and Nôno [1] constructed a commutative algebra as a commutative subalgebra of the four-dimensional real matrix algebra and gave several properties of regular functions. Stern [13] researched the boundary value problems for generalized Cauchy-Riemann systems in the space. Kajiwara et al. [3, 4] regenerated hyperholomorphic functions in quaternion and Clifford analysis, and examined the properties of solutions of the inhomogeneous Cauchy-Riemann system in Clifford analysis. Koriyama et al. [5] obtained some properties of hyperholomorphic and holomorphic functions in quaternion analysis. Song et al. [12] gave some concepts on quaternion matrix ans researched the iterative solution to the coupled quaternion matrix equations. Lian and Chiang [6] formulated the maximal rank of a kind of 3×3 partial banded block matrix, by using properties of the set of all $m \times m$ matrices over the quaternion algebra. In addition, we [7, 8] obtained some properties of hyperholomorphic functions and solved the Cauchy theorem for hyperholomorphic functions of the quaternion product spaces and octonion variables.

In this paper, we define a noncommutative subalgebra $\mathbf{C}(2)$ of $M(2;\mathbf{C})$ according to associated Pauli matrices and examine the properties of solutions of the inhomogeneous generalized Cauchy-Riemann

2010 *Mathematics Subject Classification*. 30G35; 32W50; 32A99; 11E88; 15A66
Keywords. regular function, generalized Cauchy-Riemann system, Pauli matrix, quaternion, octonion, elliptic, Clifford analysis
Received: 11 September 2014; Accepted: 25 January 2015
Communicated by Hari M. Srivastava
Kwang Ho Shon was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2013R1A1A2008978).
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system for biregular functions on $\Omega \subset \mathbf{C}^2 \times \mathbf{C}^2$. In addition, we construct a commutative subalgebra $\mathbf{C}(4)$ of the four dimensional matrix algebra $M(4;\mathbf{C})$ on the field $\mathbf{C}$ of complex numbers generated by eight bases $\{\varepsilon_j : j = 0, 1, \cdots, 7\}$, give a J-regular function with values in $\mathbf{C}(4)$ and investigate the properties of J-regular functions on $\Omega \subset \mathbf{C}^4$.

2. Biregular Functions with Values in C(2)

Let $a_0 = id$, $a_1 = i$, $a_2 = j$, and $a_3 = k$ be hypercomplex numbers. The field $\mathcal{T} \cong \mathbf{C}^2$ of quaternions $z = \sum_{j=0}^{3} a_j x_j = z_1 + z_2 a_2$ $(x_0, x_1, x_2, x_3 \in \mathbf{R})$ is a four-dimensional noncommutative real field generated by four bases a_0, a_1, a_2, and a_3 with the following rule:

$$a_1^2 = a_2^2 = a_3^2 = -1,\ \ a_1a_2 = -a_2a_1 = a_3,\ \ a_2a_3 = -a_3a_2 = a_1,\ \ a_3a_1 = -a_1a_3 = a_2.$$

Here this is said to satisfy the triple rule in Clifford analysis.

Consider the Pauli matrices of $M(2;\mathbf{C})$ by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \ \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \ \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$. The multiplications of σ_1, σ_2 and σ_3 produces

$$\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I, \quad \sigma_l\sigma_m = -i\sigma_n,$$

where $\{l, m, n\}$ is a cyclic permutation of $\{1, 2, 3\}$. The following matrices occur in the theory of Clifford algebras:

$$\begin{aligned}
\mathcal{U}_{1,0} &= \left\{\begin{pmatrix} x & y \\ y & x \end{pmatrix} \mid x, y \in \mathbf{R}\right\} \cong \mathbf{R} \oplus \mathbf{R}, \\
\mathcal{U}_{0,1} &= \left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x, y \in \mathbf{R}\right\} \cong \mathbf{C}, \\
\mathcal{U}_{0,2} &= \left\{\begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \mid x_n \in \mathbf{R}\ (n = 0, 1, 2, 3)\right\} \\
&= \left\{\begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \mid z_n \in \mathbf{C}\ (n = 1, 2)\right\} \cong \mathbf{H},
\end{aligned}$$

where $\mathbf{H}$ is the Hamilton's algebra of quaternions.

Consider the associated Pauli matrices

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \ e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ \ e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \ e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then these associated Pauli matrices satisfy the triple rule.

The algebra

$$\mathbf{C}(2) := \{z = \sum_{j=0}^{3} e_j x_j \mid x_j \in \mathbf{R}\ (j = 0, 1, 2, 3)\} \cong \mathcal{T}$$

is a noncommutative subalgebra of $M(2;\mathbf{C})$. Here $\mathbf{C}(2)$ can be identified with $\mathbf{C}^2$. The numbers of the field $\mathbf{C}(2)$ are

$$z = \sum_{j=0}^{3} e_j x_j = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \in \mathcal{U}_{0,2}$$

and

$$w=\sum_{j=0}^{3}e_jy_j=\begin{pmatrix} y_0+iy_1 & y_2+iy_3 \\ -y_2+iy_3 & y_0-iy_1 \end{pmatrix}\in\mathcal{U}_{0,2}.$$

The quaternionic conjugate z^* of z is $z^*=\sum_{j=0}^{3}\overline{e_j}x_j$, where $\overline{e_0}=e_0$ and $\overline{e_k}=-e_k$ $(k=1,2,3)$. Let Ω be an open subset of $\mathbf{C}^2\times\mathbf{C}^2$. Consider a function:

$$f:\Omega\longrightarrow\mathbf{C}(2)$$

satisfies

$$(z,w)\in\Omega\longmapsto f(z,w)=\sum_{j=0}^{3}e_ju_j\in\mathbf{C}(2).$$

We use quaternion differential operators:

$$D_x := \begin{pmatrix} \frac{\partial}{\partial x_0}-i\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2}-i\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2}-i\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0}+i\frac{\partial}{\partial x_1} \end{pmatrix},$$

$$D_y := \begin{pmatrix} \frac{\partial}{\partial y_0}-i\frac{\partial}{\partial y_1} & -\frac{\partial}{\partial y_2}-i\frac{\partial}{\partial y_3} \\ \frac{\partial}{\partial y_2}-i\frac{\partial}{\partial y_3} & \frac{\partial}{\partial y_0}+i\frac{\partial}{\partial y_1} \end{pmatrix},$$

and conjugate differential operators

$$D_x^* = \begin{pmatrix} \frac{\partial}{\partial x_0}+i\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}+i\frac{\partial}{\partial x_3} \\ -\frac{\partial}{\partial x_2}+i\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0}-i\frac{\partial}{\partial x_1} \end{pmatrix},$$

$$D_y^* = \begin{pmatrix} \frac{\partial}{\partial y_0}+i\frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2}+i\frac{\partial}{\partial y_3} \\ -\frac{\partial}{\partial y_2}+i\frac{\partial}{\partial y_3} & \frac{\partial}{\partial y_0}-i\frac{\partial}{\partial y_1} \end{pmatrix}.$$

The operators act for a function $f(z,w)$ in $\mathbf{C}(2)$:

$$\begin{aligned} D_x^*f(z,w) &= \Big(\sum_{j=0}^{3}e_j\frac{\partial}{\partial x_j}\Big)\Big(\sum_{k=0}^{3}e_ku_k\Big) \\ &= \Big(\frac{\partial u_0}{\partial x_0}-\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}-\frac{\partial u_3}{\partial x_3}\Big)+\Big(\frac{\partial u_1}{\partial x_0}+\frac{\partial u_0}{\partial x_1}+\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\Big)e_1 \\ &\quad+\Big(\frac{\partial u_2}{\partial x_0}-\frac{\partial u_3}{\partial x_1}+\frac{\partial u_0}{\partial x_2}+\frac{\partial u_1}{\partial x_3}\Big)e_2+\Big(\frac{\partial u_3}{\partial x_0}+\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}+\frac{\partial u_0}{\partial x_3}\Big)e_3, \end{aligned}$$

$$\begin{aligned} f(z,w)D_y^* &= \Big(\sum_{k=0}^{3}e_ku_k\Big)\Big(\sum_{j=0}^{3}e_j\frac{\partial}{\partial y_j}\Big) \\ &= \Big(\frac{\partial u_0}{\partial y_0}-\frac{\partial u_1}{\partial y_1}-\frac{\partial u_2}{\partial y_2}-\frac{\partial u_3}{\partial y_3}\Big)+\Big(\frac{\partial u_1}{\partial y_0}+\frac{\partial u_0}{\partial y_1}-\frac{\partial u_3}{\partial y_2}+\frac{\partial u_2}{\partial y_3}\Big)e_1 \\ &\quad+\Big(\frac{\partial u_2}{\partial y_0}+\frac{\partial u_3}{\partial y_1}+\frac{\partial u_0}{\partial y_2}-\frac{\partial u_1}{\partial y_3}\Big)e_2+\Big(\frac{\partial u_3}{\partial y_0}-\frac{\partial u_2}{\partial y_1}+\frac{\partial u_1}{\partial y_2}+\frac{\partial u_0}{\partial y_3}\Big)e_3. \end{aligned}$$

The point norm at $p=(z,w)\in\Omega$ is defined by

$$\|p\|=\frac{1}{\sqrt{2}}\sqrt{\mathrm{tr}(z^*z)+\mathrm{tr}(w^*w)}=\sqrt{\sum_{j=0}^{3}(x_j^2+y_j^2)}.$$

Definition 1. *Let Ω be an open set of $\mathbf{C}^2 \times \mathbf{C}^2$. A function $f : \Omega \longrightarrow \mathbf{C}(2)$ is said to be biregular on Ω if the following two conditions are satisfied:*
(a) For each $w \in \mathbf{C}(2)$ fixed, f is of class C^1 in $z \in \Omega_w = \{z \in \mathbf{C}(2) \mid (z, w) \in \Omega\}$ and satisfies $D_x^* f(z, w) = 0$ on Ω.
(b) For each $z \in \mathbf{C}(2)$ fixed, f is of class C^1 in $w \in \Omega_z = \{w \in \mathbf{C}(2) \mid (z, w) \in \Omega\}$ and satisfies $f(z, w)D_y^* = 0$ on Ω.

Let (g, h) be a pair of $\mathbf{C}(2)$-valued functions g and h of class C^∞ on Ω in $\mathbf{C}^2 \times \mathbf{C}^2$. Consider the inhomogeneous Cauchy-Riemann system of partial differential equations

$$D_x^* f = g, \quad f D_y^* = h. \tag{1}$$

If the system (1) has a solution f with values in $\mathbf{C}(2)$, then

$$g D_y^* = D_x^* f D_y^* = D_x^* h.$$

For the system (1), the solvability condition

$$g D_y^* = D_x^* h \tag{2}$$

is necessary for the existence of a solution f with values in $\mathbf{C}(2)$ to the system (1). Let Ω be a product domain $\Omega_1 \times \Omega_2$ of simply connected domains in $\mathbf{C}^2$ and $\mathbf{C}^2$, respectively, and let $Tf := (D_x^* f,\ f D_y^*)$. Then $Tf = (g, h) = 0$ satisfies $g D_y^* = D_x^* h$. For $Tf : \Omega \longrightarrow \mathbf{C}(2)$ of class C^2, consider $S(g, h) = g D_y^* - D_x^* h$. Then the Cousin 1 problem according to $\mathbf{C}(2)$-valued functions on Ω has a solution.

Theorem 2. *If a function $f(z, w)$ is biregular on $\Omega \subset \mathbf{C}^2 \times \mathbf{C}^2$, then each component of $f(z, w)$ is harmonic on Ω.*

Proof. Consider

$$\begin{aligned} D_x D_x^* &= \begin{pmatrix} \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3} \\ -\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=0}^3 \frac{\partial^2}{\partial x_j^2} & 0 \\ 0 & \sum_{j=0}^3 \frac{\partial^2}{\partial x_j^2} \end{pmatrix} = \Delta_x \end{aligned}$$

and

$$\begin{aligned} D_y^* D_y &= \begin{pmatrix} \frac{\partial}{\partial y_0} + i\frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} + i\frac{\partial}{\partial y_3} \\ -\frac{\partial}{\partial y_2} + i\frac{\partial}{\partial y_3} & \frac{\partial}{\partial y_0} - i\frac{\partial}{\partial y_1} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y_0} - i\frac{\partial}{\partial y_1} & -\frac{\partial}{\partial y_2} - i\frac{\partial}{\partial y_3} \\ \frac{\partial}{\partial y_2} - i\frac{\partial}{\partial y_3} & \frac{\partial}{\partial y_0} + i\frac{\partial}{\partial y_1} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=0}^3 \frac{\partial^2}{\partial y_j^2} & 0 \\ 0 & \sum_{j=0}^3 \frac{\partial^2}{\partial y_j^2} \end{pmatrix} = \Delta_y. \end{aligned}$$

Since f is biregular on Ω, $\Delta_x u_0 = 0$ and $\Delta_y u_0 = 0$. Therefore, the function $u_0(z, w)$ is harmonic on Ω. Similarly, the functions $u_k(z, w)$ $(k = 1, 2, 3)$ are also harmonic on Ω. □

Theorem 3. *Let Ω be a domain in $\mathbf{C}^2 \times \mathbf{C}^2$ of $\mathbf{C}(2)$ variables z and w, and let (g, h) be $\mathbf{C}(2)$-valued functions on Ω of class C^∞. If $f(z, w) = \sum_{k=0}^3 e_k u_k$ is a locally integrable function with $\mathbf{C}(2)$ values satisfying the system (1) in the sense of distribution, that is, $f(z, w)$ as a weak solution of (1), then $f(z, w)$ is of class C^∞ and a strong solution of (1) on Ω.*

Proof. Because $D_x D_x^* = \Delta_x$ and $D_y^* D_y = \Delta_y$,

$$(\Delta_x + \Delta_y)u_0 = \sum_{j=0}^{3}\Big(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\Big)u_0.$$

Because u_0, u_1, u_2 and u_3 are solutions of the elliptic equation

$$(\Delta_x + \Delta_y)u_j = D_x D_x^* u_j + u_j D_y^* D_y = D_x g + h D_y \quad (j = 0, 1, 2, 3)$$

and $D_x g + h D_y$ is of class C^∞, each part u_j of f is of class C^∞ on Ω. □

Example 4. *Let r_j and s_j $(j = 0, 1, 2, 3)$ be positive numbers. Consider an ellipsoid*

$$\Omega = \Big\{(z, w) \in \mathbf{C}^2 \times \mathbf{C}^2 \mid \sum_{j=0}^{3}\Big(\frac{x_j^2}{r_j^2} + \frac{y_j^2}{s_j^2}\Big) < 1\Big\}.$$

Let g and h be $\mathbf{C}(2)$-valued functions satisfying $D_x^* h = g D_y^*$ on Ω. Then there exists a $\mathbf{C}(2)$-valued function f on Ω of class C^∞, which is a solution of (1) .

3. J-regular Functions with Values in C(4)

The four-dimensional matrix algebra $M(4; \mathbf{C})$ is on the field $\mathbf{C}$ of complex numbers generated by eight bases $\varepsilon_j (j = 0, \cdots, 7)$. Here put

$$\delta_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\varepsilon_0 = \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_0 \end{pmatrix}, \quad \varepsilon_1 = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} \delta_2 & 0 \\ 0 & \delta_2 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_3 \end{pmatrix},$$

$$\varepsilon_4 = \begin{pmatrix} 0 & \delta_0 \\ \delta_0 & 0 \end{pmatrix}, \quad \varepsilon_5 = \begin{pmatrix} 0 & \delta_1 \\ \delta_1 & 0 \end{pmatrix}, \quad \varepsilon_6 = \begin{pmatrix} 0 & \delta_2 \\ \delta_2 & 0 \end{pmatrix}, \quad \varepsilon_7 = \begin{pmatrix} 0 & \delta_3 \\ \delta_3 & 0 \end{pmatrix}.$$

Then the following rules are obtained:

$$\varepsilon_0^2 = \varepsilon_2^2 = \varepsilon_4^2 = \varepsilon_6^2 = I, \ \ \varepsilon_1^2 = \varepsilon_3^2 = \varepsilon_5^2 = \varepsilon_7^2 = -I,$$

$$\varepsilon_1\varepsilon_2 = \varepsilon_3 = \varepsilon_2\varepsilon_1, \ \ \varepsilon_2\varepsilon_4 = \varepsilon_6 = \varepsilon_4\varepsilon_2, \ \ \varepsilon_3\varepsilon_4 = \varepsilon_7 = \varepsilon_4\varepsilon_3, \ \ \varepsilon_1\varepsilon_6 = \varepsilon_7 = \varepsilon_6\varepsilon_1,$$

$$\varepsilon_2\varepsilon_5 = \varepsilon_7 = \varepsilon_5\varepsilon_2, \ \ \varepsilon_1\varepsilon_4 = \varepsilon_5 = \varepsilon_4\varepsilon_1, \ \ \varepsilon_5\varepsilon_6 = \varepsilon_3 = \varepsilon_6\varepsilon_5, \ \ \varepsilon_3\varepsilon_1 = -\varepsilon_2 = \varepsilon_1\varepsilon_3.$$

The element ε_0 is the identity and the element ε_1 is the imaginary unit $\sqrt{-1}$ in the $M(4; \mathbf{C})$ of complex numbers.

The algebra $\mathbf{C}(4) = \{z = \sum_{j=0}^{7} \varepsilon_j x_j \mid x_j \in \mathbf{R}\ (j = 0, 1, \cdots, 7)\}$ is a commutative subalgebra of $M(4; \mathbf{C})$ and represented by the following form:

$$\mathbf{C}(4) = \{z = \sum_{j=1}^{4} z_j \varepsilon_{2j-2} \mid z_j \in \mathbf{C}\ (j = 1, 2, 3, 4)\},$$

where $z_1 = \varepsilon_0 x_0 + \varepsilon_1 x_1$, $z_2 = \varepsilon_0 x_2 + \varepsilon_1 x_3$, $z_3 = \varepsilon_0 x_4 + \varepsilon_1 x_5$ and $z_4 = \varepsilon_0 x_6 + \varepsilon_1 x_7$. Here $\mathbf{C}(4)$ can be identified with $\mathbf{C}^4$.

Define the multiplication of $z = \sum_{j=0}^{7} \varepsilon_j x_j$ and $w = \sum_{j=0}^{7} \varepsilon_j y_j$ by

$$zw = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{pmatrix},$$

where $a_1 = z_1w_1 + z_2w_2 + z_3w_3 + z_4w_4$, $a_2 = z_1w_2 + z_2w_1 + z_3w_4 + z_4w_3$, $a_3 = z_1w_3 + z_2w_4 + z_3w_1 + z_4w_2$ and $a_4 = z_1w_4 + z_2w_3 + z_3w_2 + z_4w_1$ in $\mathbf{C}(4)$.

The conjugation z^* and the norm $\|z\|$ of z in $\mathbf{C}(4)$ are defined, respectively, by

$$\begin{aligned} z^* &= \sum_{j=0}^{7} (-1)^j \varepsilon_j x_j = \sum_{j=1}^{4} \overline{z_j} \varepsilon_{2j-2}, \\ \|z\| &= \frac{1}{2}\sqrt{\mathrm{tr}(zz^*)} = \sqrt{\sum_{j=0}^{7} x_j^2}, \end{aligned}$$

where $\overline{z_j}$ $(j = 1, 2, 3, 4)$ are usual conjugate numbers in $\mathbf{C}$.

Consider two differential operators:

$$D := \frac{1}{4}\left(\frac{\partial}{\partial z_1} + \varepsilon_2 \frac{\partial}{\partial z_2} + \varepsilon_4 \frac{\partial}{\partial z_3} + \varepsilon_6 \frac{\partial}{\partial z_4}\right) = \begin{pmatrix} D_1 & D_2 \\ D_2 & D_1 \end{pmatrix}$$

and

$$D^* = \frac{1}{4}\left(\frac{\partial}{\partial \overline{z_1}} + \varepsilon_2 \frac{\partial}{\partial \overline{z_2}} + \varepsilon_4 \frac{\partial}{\partial \overline{z_3}} + \varepsilon_6 \frac{\partial}{\partial \overline{z_4}}\right) = \begin{pmatrix} \overline{D_1} & \overline{D_2} \\ \overline{D_2} & \overline{D_1} \end{pmatrix},$$

where $\frac{\partial}{\partial z_k}$, $\frac{\partial}{\partial \overline{z_k}}$ $(k = 1, 2, 3, 4)$ are usual complex differential operators (see [2]) and

$$D_1 = \begin{pmatrix} \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} \end{pmatrix}, \quad D_2 = \begin{pmatrix} \frac{\partial}{\partial x_4} - i\frac{\partial}{\partial x_5} & \frac{\partial}{\partial x_6} - i\frac{\partial}{\partial x_7} \\ \frac{\partial}{\partial x_6} - i\frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_4} - i\frac{\partial}{\partial x_5} \end{pmatrix}.$$

Then,

$$F := \frac{1}{4}\mathrm{tr}(DD^*) = \sum_{j=0}^{7} \frac{\partial^2}{\partial x_j^2} = \frac{1}{16}\sum_{j=1}^{4} \frac{\partial^2}{\partial z_j \partial \overline{z_j}}.$$

Let Ω be an open set in $\mathbf{C}^4$ and $f(z) = \sum_{k=1}^{4} f_k(z)\varepsilon_{2k-2} = \sum_{j=0}^{7} \varepsilon_j u_j$ be a function defined on Ω with values in $\mathbf{C}(4)$, where $z = (z_1, z_2, z_3, z_4)$ and $f_k(z)$ $(k = 1, 2, 3, 4)$ are usual complex-valued functions and u_j $(j = 0, 1, \cdots, 7)$ are real-valued functions.

Definition 5. *Let Ω be an open set in $\mathbf{C}^4$. A function $f(z)$ is said to be J-regular on Ω with values in $\mathbf{C}(4)$ if the following two conditions are satisfied:*
(a) $f_k(z)$ $(k = 1, 2, 3, 4)$ are continuously differential functions on Ω, and
(b)

$$D^* f = 0 \quad \text{on} \quad \Omega. \tag{3}$$

The system (3) for the J-regular function $f(z)$ is equivalent to the following system of equations:

$$\begin{aligned} \frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}} + \frac{\partial f_3}{\partial \overline{z_3}} + \frac{\partial f_4}{\partial \overline{z_4}} &= 0, \quad \frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}} + \frac{\partial f_4}{\partial \overline{z_3}} + \frac{\partial f_3}{\partial \overline{z_4}} = 0, \\ \frac{\partial f_3}{\partial \overline{z_1}} + \frac{\partial f_4}{\partial \overline{z_2}} + \frac{\partial f_1}{\partial \overline{z_3}} + \frac{\partial f_2}{\partial \overline{z_4}} &= 0, \quad \frac{\partial f_4}{\partial \overline{z_1}} + \frac{\partial f_3}{\partial \overline{z_2}} + \frac{\partial f_2}{\partial \overline{z_3}} + \frac{\partial f_1}{\partial \overline{z_4}} = 0. \end{aligned} \tag{4}$$

Definition 6. *Let Ω be a domain in $\mathbf{C}^4$ and let*

$$f = (u_0, u_1, \cdots, u_7) : \Omega \to \mathbf{C}(4).$$

This mapping is said to be harmonic if all its components u_j $(j = 0, 1, \cdots, 7)$ of f are harmonic on Ω. The system (3) *is called a generalized Cauchy-Riemann system if every solution $f(z)$ has only harmonic components u_j $(j = 0, 1, \cdots, 7)$.*

Proposition 7. *Let Ω be a domain in $\mathbf{C}^4$ and $f(z)$ be a J-regular function on Ω. If $Ff(z) = 0$, then the system* (3) *is a generalized Cauchy-Riemann system on Ω.*

Proof. If $Ff(z) = 0$, then $\sum_{j=0}^{7} \frac{\partial^2 f}{\partial x_j^2} = 0$. Therefore, the components $u_0, u_1, \cdots, u_7$ are harmonic on Ω. That is, the system (3) is a generalized Cauchy-Riemann system on Ω. □

Let $f(z)$ be a J-regular function defined on Ω of $\mathbf{C}^4$. Define the derivative $f'(z)$ by $f'(z) = Df(z)$.

Theorem 8. *Let Ω be a domain in $\mathbf{C}^4$ and $f(z)$ be a J-regular function defined on Ω with values in $\mathbf{C}(4)$. Then,*

$$f'(z) = \frac{1}{4}\Big(\sum_{j=0}^{3} \varepsilon_{2j} \frac{\partial}{\partial x_{2j}}\Big) f(z) = -\frac{1}{4}\Big(\sum_{j=0}^{3} \varepsilon_{2j+1} \frac{\partial}{\partial x_{2j+1}}\Big) f(z). \tag{5}$$

Proof. Because the function $f(z)$ is J-regular on Ω,

$$f'(z) = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_2 & b_1 & b_4 & b_3 \\ b_3 & b_4 & b_1 & b_2 \\ b_4 & b_3 & b_2 & b_1 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_1 & c_4 & c_3 \\ c_3 & c_4 & c_1 & c_2 \\ c_4 & c_3 & c_2 & c_1 \end{pmatrix} = \frac{1}{4}\Big(\sum_{j=0}^{3} \varepsilon_{2j} \frac{\partial}{\partial x_{2j}}\Big) f(z),$$

where

$$\begin{aligned} b_1 &= \frac{1}{8}\Big\{\Big(\frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} + \frac{\partial u_4}{\partial x_4} + \frac{\partial u_5}{\partial x_5} + \frac{\partial u_6}{\partial x_6} + \frac{\partial u_7}{\partial x_7}\Big) \\ &\quad + i\Big(\frac{\partial u_1}{\partial x_0} - \frac{\partial u_0}{\partial x_1} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} + \frac{\partial u_5}{\partial x_4} - \frac{\partial u_4}{\partial x_5} + \frac{\partial u_7}{\partial x_6} - \frac{\partial u_6}{\partial x_7}\Big)\Big\}, \\ b_2 &= \frac{1}{8}\Big\{\Big(\frac{\partial u_2}{\partial x_0} + \frac{\partial u_3}{\partial x_1} + \frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_6}{\partial x_4} + \frac{\partial u_7}{\partial x_5} + \frac{\partial u_4}{\partial x_6} + \frac{\partial u_5}{\partial x_7}\Big) \\ &\quad + i\Big(\frac{\partial u_3}{\partial x_0} - \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} - \frac{\partial u_0}{\partial x_3} + \frac{\partial u_7}{\partial x_4} - \frac{\partial u_6}{\partial x_5} + \frac{\partial u_5}{\partial x_6} - \frac{\partial u_4}{\partial x_7}\Big)\Big\}, \\ b_3 &= \frac{1}{8}\Big\{\Big(\frac{\partial u_4}{\partial x_0} + \frac{\partial u_5}{\partial x_1} + \frac{\partial u_6}{\partial x_2} + \frac{\partial u_7}{\partial x_3} + \frac{\partial u_0}{\partial x_4} + \frac{\partial u_1}{\partial x_5} + \frac{\partial u_2}{\partial x_6} + \frac{\partial u_3}{\partial x_7}\Big) \\ &\quad + i\Big(\frac{\partial u_5}{\partial x_0} - \frac{\partial u_4}{\partial x_1} + \frac{\partial u_7}{\partial x_2} - \frac{\partial u_6}{\partial x_3} + \frac{\partial u_1}{\partial x_4} - \frac{\partial u_0}{\partial x_5} + \frac{\partial u_3}{\partial x_6} - \frac{\partial u_2}{\partial x_7}\Big)\Big\}, \\ b_4 &= \frac{1}{8}\Big\{\Big(\frac{\partial u_6}{\partial x_0} + \frac{\partial u_7}{\partial x_1} + \frac{\partial u_4}{\partial x_2} + \frac{\partial u_5}{\partial x_3} + \frac{\partial u_2}{\partial x_4} + \frac{\partial u_3}{\partial x_5} + \frac{\partial u_0}{\partial x_6} + \frac{\partial u_1}{\partial x_7}\Big) \\ &\quad + i\Big(\frac{\partial u_7}{\partial x_0} - \frac{\partial u_6}{\partial x_1} + \frac{\partial u_5}{\partial x_2} - \frac{\partial u_4}{\partial x_3} + \frac{\partial u_3}{\partial x_4} - \frac{\partial u_2}{\partial x_5} + \frac{\partial u_1}{\partial x_6} - \frac{\partial u_0}{\partial x_7}\Big)\Big\}, \end{aligned}$$

and

$$c_1 = \frac{1}{4}\Big(\frac{\partial f_1}{\partial x_0}+\frac{\partial f_2}{\partial x_2}+\frac{\partial f_3}{\partial x_4}+\frac{\partial f_4}{\partial x_6}\Big),\quad c_2 = \frac{1}{4}\Big(\frac{\partial f_2}{\partial x_0}+\frac{\partial f_1}{\partial x_2}+\frac{\partial f_4}{\partial x_4}+\frac{\partial f_3}{\partial x_6}\Big),$$
$$c_3 = \frac{1}{4}\Big(\frac{\partial f_3}{\partial x_0}+\frac{\partial f_4}{\partial x_2}+\frac{\partial f_1}{\partial x_4}+\frac{\partial f_2}{\partial x_6}\Big),\quad c_4 = \frac{1}{4}\Big(\frac{\partial f_4}{\partial x_0}+\frac{\partial f_3}{\partial x_2}+\frac{\partial f_2}{\partial x_4}+\frac{\partial f_1}{\partial x_6}\Big).$$

Similarly, the result can be proved for the right-hand side of (5). □

Theorem 9. *Let $f(z)$ be a J-regular function on a domain G of $\mathbf{C}^4$ and let*

$$\begin{aligned}\omega = {} & dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_2}\wedge d\overline{z_3}\wedge d\overline{z_4}\\ & -dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_3}\wedge d\overline{z_4}\varepsilon_2\\ & +dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_2}\wedge d\overline{z_4}\varepsilon_4\\ & -dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_2}\wedge d\overline{z_3}\varepsilon_6.\end{aligned}$$

Then, for any bounded domain $\Omega\subset G$ with a smooth boundary $b\Omega$, $\int_{b\Omega}\omega f = 0$.

Proof. Let

$$\begin{aligned}\omega_{(1)} &= dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_2}\wedge d\overline{z_3}\wedge d\overline{z_4},\\ \omega_{(2)} &= dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_3}\wedge d\overline{z_4},\\ \omega_{(3)} &= dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_2}\wedge d\overline{z_4},\\ \omega_{(4)} &= dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_2}\wedge d\overline{z_3}.\end{aligned}$$

Then,

$$\begin{aligned}\omega f &= (\omega_{(1)}-\omega_{(2)}\varepsilon_2+\omega_{(3)}\varepsilon_4-\omega_{(4)}\varepsilon_6)(f_1+f_2\varepsilon_2+f_3\varepsilon_4+f_4\varepsilon_6)\\ &= (f_1\omega_{(1)}-f_2\omega_{(2)}+f_3\omega_{(3)}-f_4\omega_{(4)})\\ &\quad +(f_2\omega_{(1)}-f_1\omega_{(2)}+f_4\omega_{(3)}-f_3\omega_{(4)})\varepsilon_2\\ &\quad +(f_3\omega_{(1)}-f_4\omega_{(2)}+f_1\omega_{(3)}-f_2\omega_{(4)})\varepsilon_4\\ &\quad +(f_4\omega_{(1)}-f_3\omega_{(2)}+f_2\omega_{(3)}-f_1\omega_{(4)})\varepsilon_6.\end{aligned}$$

Therefore,

$$\begin{aligned}d(\omega f) = {} & \Big(\frac{\partial f_1}{\partial\overline{z_1}}+\frac{\partial f_2}{\partial\overline{z_2}}+\frac{\partial f_3}{\partial\overline{z_3}}+\frac{\partial f_4}{\partial\overline{z_4}}\Big)dV+\Big(\frac{\partial f_2}{\partial\overline{z_1}}+\frac{\partial f_1}{\partial\overline{z_2}}+\frac{\partial f_4}{\partial\overline{z_3}}+\frac{\partial f_3}{\partial\overline{z_4}}\Big)dV\varepsilon_2\\ & +\Big(\frac{\partial f_3}{\partial\overline{z_1}}+\frac{\partial f_4}{\partial\overline{z_2}}+\frac{\partial f_1}{\partial\overline{z_3}}+\frac{\partial f_2}{\partial\overline{z_4}}\Big)dV\varepsilon_4+\Big(\frac{\partial f_4}{\partial\overline{z_1}}+\frac{\partial f_3}{\partial\overline{z_2}}+\frac{\partial f_2}{\partial\overline{z_3}}+\frac{\partial f_1}{\partial\overline{z_4}}\Big)dV\varepsilon_6,\end{aligned}$$

where $dV = dz\wedge d\overline{z}$. By the system (4), $d(\omega f) = 0$. By the Stokes' theorem,

$$\int_{b\Omega}\omega f = \int_{\Omega} d(\omega f) = 0.$$

□

Theorem 10. *Let Ω be a domain in $\mathbf{C}^4$. If the system* (3) *is a generalized Cauchy-Riemann system on Ω, then it is elliptic.*

Proof. Assume that there exist a non-zero vector $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_7)\in\mathbf{R}^8$ and a non-zero column vector

$\nu = (\nu_1, \nu_2, \nu_3, \nu_4)^t \in \mathbf{C}^4$ such that

$$\begin{aligned}\Big(\sum_{j=0}^{7} \lambda_j \varepsilon_j\Big)\nu &= \begin{pmatrix} \lambda_0 + i\lambda_1 & \lambda_2 + i\lambda_3 & \lambda_4 + i\lambda_5 & \lambda_6 + i\lambda_7 \\ \lambda_2 + i\lambda_3 & \lambda_0 + i\lambda_1 & \lambda_6 + i\lambda_7 & \lambda_4 + i\lambda_5 \\ \lambda_4 + i\lambda_5 & \lambda_6 + i\lambda_7 & \lambda_0 + i\lambda_1 & \lambda_2 + i\lambda_3 \\ \lambda_6 + i\lambda_7 & \lambda_4 + i\lambda_5 & \lambda_2 + i\lambda_3 & \lambda_0 + i\lambda_1 \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{pmatrix} \\ &= \begin{pmatrix} (\lambda_0 + i\lambda_1)\nu_1 + (\lambda_2 + i\lambda_3)\nu_2 + (\lambda_4 + i\lambda_5)\nu_3 + (\lambda_6 + i\lambda_7)\nu_4 \\ (\lambda_2 + i\lambda_3)\nu_1 + (\lambda_0 + i\lambda_1)\nu_2 + (\lambda_6 + i\lambda_7)\nu_3 + (\lambda_4 + i\lambda_5)\nu_4 \\ (\lambda_4 + i\lambda_5)\nu_1 + (\lambda_6 + i\lambda_7)\nu_2 + (\lambda_0 + i\lambda_1)\nu_3 + (\lambda_2 + i\lambda_3)\nu_4 \\ (\lambda_6 + i\lambda_7)\nu_1 + (\lambda_4 + i\lambda_5)\nu_2 + (\lambda_2 + i\lambda_3)\nu_3 + (\lambda_0 + i\lambda_1)\nu_4 \end{pmatrix} \\ &= 0. \end{aligned} \tag{6}$$

From the system (6) and by the rules of bases ε_j $(j = 0, 1, \cdots, 7)$ and $\lambda \neq 0$, a solution $f(z)$ of the system (3) on Ω is obtained:

$$f(z) = e^{\omega}\nu,$$

where $\omega = \sum_{j=0}^{7} \lambda_j x_j$. However, $Ff(z) \neq 0$. That is, the function $f(z)$ is not harmonic on Ω. It is a contradiction. Therefore, $\left|\sum_{j=0}^{7} \lambda_j \varepsilon_j\right| \neq 0$. That is, the system (3) is elliptic. □

References

[1] S. Gotô, K. Nôno, Regular Functions with Values in a Commutative Subalgebra $\mathbf{C}(\mathbf{C})$ of Matrix Algebra $M(4; \mathbf{R})$, Bull. Fukuoka Univ. Ed. 61 part III (2012) 9–15.

[2] L. Hörmander, An introduction to complex analysis in several variables, Elsevier, 1973.

[3] J. Kajiwara, X.D. Li, K.H. Shon, Regeneration in complex, quaternion and Clifford analysis, Proc. the 9th International Conf. on Finite or Infinite Dimensional Complex Analysis and Applications, Advances in Complex Analysis and Its Applications, Kluwer Academic Pub., Hanoi 2(9) (2004) 287–298.

[4] J. Kajiwara, X.D. Li, K.H. Shon, Function spaces in complex and Clifford analysis, Inhomogeneous Cauchy Riemann system of quaternion and Clifford analysis in ellipsoid, Proc. the 14th International Conf. on Finite or Infinite Dimensional Complex Analysis and Applications, National Univ. Pub., Hanoi, Vietnam, Hue Univ. 14 (2006) 127–155.

[5] H. Koriyama, H. Mae, K. Nôno, Hyperholomorphic fucntions and holomorphic functions in quaternionic analysis, Bull. Fukuoka Univ. Ed. 60 part III (2011) 1–9.

[6] D. Lian, J.Y. Chiang, The maximal rank of a kind of partial banded block matrix subject to linear equations, Filomat, 27(2) (2013) 381–389.

[7] S.J. Lim, K.H. Shon, Properties of hyperholomorphic functions in Clifford analysis, East Asian Math. J. 28(5) (2012) 553–559.

[8] S.J. Lim, K.H. Shon, Hyperholomorphic fucntions and hyperconjugate harmonic functions of octonion variables, J. Inequal. Appl. 77 (2013) 1–8.

[9] M. Naser, Hyperholomorphic functions, Silberian Math. J. 12 (1971) 959–968.

[10] K. Nôno, Hyperholomorphic functions of a quaternion variable, Bull. Fukuoka Univ. Ed. 32 (1983) 21–37.

[11] K. Nôno, Characterization of domains of holomorphy by the existence of hyper-conjugate harmonic functions, Rev. Roumaine Math. Pures Appl. 31(2) (1986) 159–161.

[12] C. Song, G. Chen, X. Zhang, An iterative solution to coupled quaternion matrix equations, Filomat, 26(4) (2012) 809–826.

[13] I. Stern, Boundary value problems for generalized Cauchy-Riemann systems in the space, Boundary value and Initial value problems in complex analysis. Pitman Rec. Notes in Math. Ser. 256, Longman Sci. Tech., Harlow 1 (Halle, 1988) (1991) 159–183.

[14] A. Sudbery, Quaternionic analysis, Math. Proc. Camb. Phil. Soc. 85(2) (1979) 199–225.