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An Elliptic Extension of the Genocchi Polynomials

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Abstract. We define an elliptic extension of the Genocchi polynomials and obtain the sums of products for the elliptic Genocchi polynomials. The formulas of sums of products for the Genocchi polynomials are also derived.

1. Introduction

The classical Genocchi polynomials $G_n(x)$ are defined by means of the following generating function (see, e.g., [16, 18, 19])

$$\frac{2ze^{xz}}{e^z+1}=\sum_{n=0}^{\infty}G_n(x)\frac{z^n}{n!}\qquad(|z|<\pi). \tag{1}$$

Let $G_n=G_n(0)$ be the Genocchi numbers, $G_{2n+1}=0$ $(n\geq 1)$, which several valuations are

$$G_0=0, G_1=1, G_2=-1, G_4=1, G_6=-3, G_8=17, G_{10}=-155, G_{12}=2073.$$

We define n-th Genocchi functions as follows:

$$\widehat{G}_n(x):=G_n(x)\ (0\leq x<1, n\in\mathbb{N}_0),\quad \widehat{G}_n(x+1)=-\widehat{G}_n(x), \tag{2}$$

which is called the *periodic Genocchi polynomials*. Any $x\in\mathbb{R}, r\in\mathbb{Z}$, we have

$$\widehat{G}_n(x)=(-1)^{[x]}G_n(\{x\}),\quad \widehat{G}_n(x+r)=(-1)^r\widehat{G}_n(x), \tag{3}$$

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where $\{x\}$ denotes the fractional part of x; $[x]$ denotes the greatest integer not exceeding x.

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$, each of degree n in x as well as in α, for a real or complex parameter α, are defined by means of the following generating functions (see, for details, [14, p. 25-32] and [20, p. 59-66]):

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} \qquad (|z| < 2\pi; 1^{\alpha} := 1) \tag{4}$$

and

$$\left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!} \qquad (|z| < \pi; 1^{\alpha} := 1) \tag{5}$$

respectively. The classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$, for $\alpha = 1$ in (4) and (5), are respectively defined by

$$\frac{ze^{xz}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} \qquad (|z| < 2\pi), \tag{6}$$

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} \qquad (|z| < \pi). \tag{7}$$

Obviously, the classical Bernoulli number $B_n := B_n(0)$ and Euler number $E_n := 2^n E_n\left(\frac{1}{2}\right)$ $(n \in \mathbb{N})$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, \ldots\}$.

Let $S_N(n; x_1, \ldots, x_N)$ denotes sums of products for the Bernoulli polynomials as follows:

$$S_N(n; x_1, \ldots, x_N) = \sum_{\substack{j_1,\ldots,j_N \geq 0 \\ j_1+\cdots+j_N=n}} \binom{n}{j_1, \ldots, j_N} B_{j_1}(x_1)\cdots B_{j_N}(x_N), \tag{8}$$

which summation takes place over all positive or zero integers $j_i \geq 0$ such that $j_1 + j_2 + \cdots + j_N = n$, where

$$\binom{n}{j_1, \ldots, j_N} := \frac{n!}{j_1! \ldots j_N!}$$

denote the multinomial coefficients.

By (4), (6) and (8), we can find the following relation:

$$B_k^{(N)}(y) = S_N(k; x_1, \ldots, x_N), \tag{9}$$

when $y = x_1 + \cdots + x_N$.

Dilcher obtained the following formula.

Theorem 1 ([3, p. 31, Lemma 4]). *Let $x_1, \ldots, x_N, y$ be complex numbers with $y = x_1 + \cdots + x_N$, for $n \geq N$ we have*

$$S_N(n; x_1, \ldots, x_N) = (-1)^{N-1} N \binom{n}{N} \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} B_k^{(N)}(y) \frac{B_{n-k}(y)}{n-k}.$$

Recently, Ivashkevich [10] and Machide [12] introduced the following elliptic extensions for the clcssical Bernoulli and Euler polynomials, i.e., so-called *elliptic Bernoulli functions and elliptic Euler functions* are defined by means of the following generating functions respectively:

$$\frac{1}{2\pi i} \sum\nolimits_e \frac{e(-\mu x' - \nu x)}{\xi + \mu\tau + \nu} = \sum_{n=0}^{\infty} B_n(x', x; \tau) \frac{(2\pi i \xi)^{n-1}}{n!} \tag{10}$$

and

$$-\frac{1}{\pi i}\sum\nolimits_e \frac{e(-\mu x'-\nu x-x/2)}{\xi+\mu\tau+\nu+1/2}=\sum_{n=0}^{\infty}E_n(x',x;\tau)\frac{(2\pi i\xi)^n}{n!},\tag{11}$$

$(x',x\in\mathbb{R};\ \tau\in H:=\{\tau\in\mathbb{C},\Im\tau>0\};\ \xi\in\mathbb{C},\ \mu,\nu\in\mathbb{Z})$

where symbol $\sum_e$ denotes the Eisenstein summation ([23, p. 14]) defined by

$$\sum\nolimits_e=\sum_{\nu}\nolimits_e\left(\sum_{\mu}\nolimits_e\right)=\lim_{N\to\infty}\sum_{\nu=-N}^{N}\left(\lim_{M\to\infty}\sum_{\mu=-M}^{M}\right).\tag{12}$$

Let N be a positive integer and n be a nonnegative integer. We set $\vec{x}_i=(x'_i,x_i)$ for $i=1,\dots N$. Let $S_N^\tau(n;\vec{x}_1,\dots,\vec{x}_N)$ and $T_N^\tau(n;\vec{x}_1,\dots,\vec{x}_N)$ denote the sums of products of elliptic Bernoulli functions and elliptic Euler functions respectively.

$$S_N^\tau(n;\vec{x}_1,\dots,\vec{x}_N)=\sum_{\substack{j_1,\dots,j_N\ge 0\\ j_1+\cdots+j_N=n}}\binom{n}{j_1,\dots,j_N}B_{j_1}(x'_1,x_1;\tau)\cdots B_{j_N}(x'_N,x_N;\tau)$$

and

$$T_N^\tau(n;\vec{x}_1,\dots,\vec{x}_N)=\sum_{\substack{j_1,\dots,j_N\ge 0\\ j_1+\cdots+j_N=n}}\binom{n}{j_1,\dots,j_N}E_{j_1}(x'_1,x_1;\tau)\cdots E_{j_N}(x'_N,x_N;\tau).$$

Machide obtained the following results.

Theorem 2 ([12, p. 824, Theorem 2 and p. 830, Theorem 15]). *Let n be an integer with $n\ge N$. For any $i=1,2,\dots,N$, let x'_i and x_i be real numbers with $x'_i\notin\mathbb{Z}$. Set*

$$\vec{x}_i=(x'_i,x_i)\qquad(i=1,2,\dots,N),\qquad(y',y)=(x'_1+\cdots+x'_N,x_1+\cdots+x_N).$$

Suppose that $y'\notin\mathbb{Z}$, we have

$$S_N^\tau(n;\vec{x}_1,\dots,\vec{x}_N)=(-1)^{N-1}N\binom{n}{N}\sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}S_N^\tau(k;\vec{x}_1,\dots,\vec{x}_N)\frac{B_{n-k}(y',y;\tau)}{n-k},\tag{13}$$

$$T_N^\tau(n;\vec{x}_1,\dots,\vec{x}_N)=\frac{2^{N-1}}{(N-1)!}\sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}S_N^\tau(k;\vec{x}_1,\dots,\vec{x}_N)E_{n+N-1-k}(y',y;\tau).\tag{14}$$

Theorem 3 ([12, p. 825, Lemma 4]). *We have*

(i) Let $x_1,\dots,x_N$ be real numbers and $x'_1,\dots,x'_N$ be complex numbers with $x'_1,\dots,x'_N\notin\mathbb{Z}$. Set $\vec{x}_i=(x'_i,x_i)$ for $i=1,\dots N$. If $0\le x_1,\dots,x_N<1$, we have

$$\lim_{x'_1\to-i\infty}\cdots\lim_{x'_N\to-i\infty}\lim_{\tau\to i\infty}S_N^\tau(n;\vec{x}_1,\dots,\vec{x}_N)=S_N(n;x_1,\dots,x_N).\tag{15}$$

(ii) Set $\vec{x}_i=(\frac{1}{2},0)$ for $i=1,\dots,N-1$, $\vec{x}_N=(x'_N,0)$, we have

$$\text{The coefficient of }(x'_N)^0(=1)\text{ of }\lim_{\tau\to i\infty}S_N^\tau(n;\vec{x}_1,\dots,\vec{x}_N)=S_N(n),\tag{16}$$

where

$$S_N(n)=\sum_{\substack{j_1,\dots,j_N\ge 0\\ j_1+\cdots+j_N=n}}\binom{2n}{2j_1,\dots,2j_N}B_{2j_1}\cdots B_{2j_N}.$$

In the recent past a lot of papers appeared providing the sums of products for the special numbers and polynomilas and the related investigations; see [4–7, 9, 15, 17, 18, 21, 22] and the references therein.

In this paper, we define the elliptic Genocchi functions by means of the Eisenstein summation and Jacobi's theta functions. We research the sums of products for the elliptic Genocchi functions. Some formulas of sums of products for the Genocchi polynomials and related results are also obtained.

2. The Definitions for the Elliptic Genocchi Functions

In this section we give formula for sums of products of elliptic Genocchi functions. Thereout we derive the corresponding formulas for sums of products of the Genocchi polynomials and numbers.

We will use some standard notations: $H := \{\tau \in \mathbb{C}, \Im\tau > 0\}, e(t) := \exp(2\pi it), q = e(\tau), z = e(\xi), w = e(x)$. The classical Jacobi's theta functions [14, p. 371] are

$$\vartheta_1(x;\tau) = \sum_{n\in\mathbb{Z}} e\left(\frac{1}{2}\left(n+\frac{1}{2}\right)^2\tau + \left(n+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)\right) \tag{17}$$

$$\vartheta_2(x;\tau) = i\sum_{n\in\mathbb{Z}} (-1)^n e\left(\frac{1}{2}\left(n+\frac{1}{2}\right)^2\tau + \left(n+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)\right) \tag{18}$$

Obviously, we have the following quasi periodicity:

$$\vartheta_1(x+1;\tau) = -\vartheta_1(x;\tau), \qquad \vartheta_1(x+\tau;\tau) = -e\left(-x-\frac{\tau}{2}\right)\vartheta_1(x;\tau) \tag{19}$$

$$\vartheta_2(x+1;\tau) = -\vartheta_2(x;\tau), \qquad \vartheta_2(x+\tau;\tau) = e\left(-x-\frac{\tau}{2}\right)\vartheta_2(x;\tau) \tag{20}$$

We consider the following function

$$F(x,\xi;\tau) = 1 - \frac{1}{1-z} - \frac{1}{1-w} - \sum_{m,n=1}^{\infty} (z^m w^n - z^{-m} w^{-n}) q^{mn} \tag{21}$$

$$(0 < \Im\xi < \Im\tau,\ 0 < \Im x < \Im\tau).$$

Zagier ([24]) showed that the function $F(x,\xi;\tau)$ can be continued to a meromorphic function with poles at divisors $x = m + n\tau$ and $\xi = m' + n'\tau$, and function $F(x,\xi;\tau)$ can be expressed in terms of the classical Jacobi's theta functions, by formula

$$F(x,\xi;\tau) = \frac{1}{2\pi i}\frac{\vartheta'_1(0;\tau)\vartheta_1(x+\xi;\tau)}{\vartheta_1(x;\tau)\vartheta_1(\xi;\tau)} \qquad (x,\xi \in \mathbb{C}\setminus\mathbb{Z}+\tau\mathbb{Z}), \tag{22}$$

where $\vartheta'_1(x;\tau) = \frac{\partial}{\partial x}\vartheta_1(x;\tau)$. For fixed $x \in \mathbb{C}\setminus\mathbb{Z}+\tau\mathbb{Z}$, the function $F(x,\xi;\tau)$ with respect to ξ is meromorphic with only simple poles on the lattice $\mathbb{Z}+\tau\mathbb{Z}$. The function $F(x,\xi;\tau)$ satisfies the following properties by (19) and (21)

$$F(\xi,\eta;\tau) = F(\eta,\xi;\tau), \quad F(x,\xi+1;\tau) = F(x,\xi;\tau), \quad F(x,\xi+\tau;\tau) = e(-x)F(x,\xi;\tau). \tag{23}$$

We recall a classical result: Suppose $L := \{\nu + \mu\tau \mid \mu,\nu \in \mathbb{Z}\}$ to denote the lattice generated by 1 and τ. Any $\eta \in \mathbb{C}$ determines a character χ_η on L as follows:

$$\chi_\eta(\xi) = e\left(\frac{\xi\overline{\eta} - \overline{\xi}\eta}{\tau - \overline{\tau}}\right).$$

The Kronecker's identity (see [11, p. 277], or [23, p. 70])

$$\sum_{w\in L}{}_{e}\frac{\chi_\eta(w)}{\xi+w}=2\pi i\, e\left(\xi\frac{\eta-\overline{\eta}}{\tau-\overline{\tau}}\right)F(\xi,\eta;\tau). \tag{24}$$

Let $H(x,\xi;\tau)=F\left(x,\xi+\frac{1}{2};\tau\right)$, $\mathscr{C}=\{\nu+\frac{1}{2}+\mu\tau\mid\mu,\nu\in\mathbb{Z}\}$. By (22), we obtain that

$$H(x,\xi;\tau)=\frac{1}{2\pi i}\frac{\vartheta'_1(0;\tau)\vartheta_2(x+\xi;\tau)}{\vartheta_1(x;\tau)\vartheta_2(\xi;\tau)}. \tag{25}$$

For fixed $x\in\mathbb{C}\setminus\mathbb{Z}+\frac{1}{2}+\tau\mathbb{Z}$, the function $H(x,\xi;\tau)$ with respect to ξ is meromorphic with only simple poles on $\mathscr{C}$. The function $H(x,\xi;\tau)$ satisfies the following properties by (19) and (20).

$$H(x,\xi+1;\tau)=H(x,\xi;\tau),\quad H(x,\xi+\tau;\tau)=e(-x)H(x,\xi;\tau) \tag{26}$$

and

$$H(x+1,\xi;\tau)=H(x,\xi;\tau),\quad H(x+\tau,\xi;\tau)=-e(-\xi)H(x,\xi;\tau). \tag{27}$$

Let $\mathfrak{H}(x',x;\xi;\tau)=-2e(x\xi)H(-x'+x\tau,\xi;\tau)$. The *elliptic Genocchi functions* are defined by means of the following generating function

$$\mathfrak{H}(x',x;\xi;\tau)=\sum_{n=0}^{\infty}G_n(x',x;\tau)\frac{(2\pi i\xi)^{n-1}}{n!}. \tag{28}$$

By (26), we see easily that

$$G_n(x'+1,x;\tau)=G_n(x',x;\tau),\quad G_n(x',x+1;\tau)=-G_n(x',x;\tau). \tag{29}$$

By (24), (25) and (28), when x' and x are real numbers with $-1<x<0$, we can derive the following another expression of the generating function of elliptic Genocchi functions

$$-\frac{2}{\pi i}\sum{}_{e}\frac{e(-\mu x'-\nu x-x/2)}{2\xi+2\mu\tau+2\nu+1}=\sum_{n=0}^{\infty}G_n(x',x;\tau)\frac{(2\pi i\xi)^{n-1}}{n!}, \tag{30}$$

$(x',x\in\mathbb{R};\ \tau\in H;\ \xi\in\mathbb{C})$.

Therefore, we have

$$G_n(x',x;\tau)=\begin{cases}0 & n=0,\\ \dfrac{2\cdot n!}{(\pi i)^n}\displaystyle\sum{}_{e}\frac{e(\mu x'+\nu x-x/2)}{(2\mu\tau+2\nu-1)^n} & n\ge 1.\end{cases}$$

3. Sums of Products for the Elliptic Genocchi Functions

In this section we give formula for sums of products of elliptic Genocchi functions. Thereout we derive the corresponding formulas for sums of products of the Genocchi polynomials and numbers.

We now define the function

$$\mathfrak{H}^{(\ell)}(x',x;\xi;\tau):=\frac{1}{(2\pi i)^\ell}\left(\frac{\partial}{\partial\xi}\right)^\ell\mathfrak{H}(x',x;\xi;\tau),\quad(\ell\ge 0),$$

especially, $\mathfrak{H}^{(0)}(x',x;\xi;\tau)=\mathfrak{H}(x',x;\xi;\tau)$. By (26), it is easy to show the following properties.

$$\mathfrak{H}(x',x;\xi+1;\tau)=e(x)\mathfrak{H}(x',x;\xi;\tau),\quad \mathfrak{H}(x',x;\xi+\tau;\tau)=e(x')\mathfrak{H}(x',x;\xi;\tau) \tag{31}$$

and

$$\mathfrak{H}^{(\ell)}(x',x;\xi+1;\tau)=e(x)\mathfrak{H}^{(\ell)}(x',x;\xi;\tau),\quad \mathfrak{H}^{(\ell)}(x',x;\xi+\tau;\tau)=e(x')\mathfrak{H}^{(\ell)}(x',x;\xi;\tau). \tag{32}$$

We differentiate both side of (28) with respect to the variable ξ, iterate $n-1$ times, yields the relationship between the function $\mathfrak{H}^{(\ell)}(x',x;\xi;\tau)$ and elliptic Genocchi functions $G_n(x',x;\tau)$ below.

$$\mathfrak{H}^{(\ell)}(x',x;\xi;\tau)=\sum_{n=0}^{\infty}G_{n+\ell+1}(x',x;\tau)\frac{(2\pi i\xi)^n}{(n+\ell+1)n!}. \tag{33}$$

We define sums of products for the elliptic Genocchi functions below.

$$R_N^{\tau}(n;\vec{x}_1,\dots,\vec{x}_N)=\sum_{\substack{j_1,\dots,j_N\geq 0\\ j_1+\cdots+j_N=n}}\binom{n}{j_1,\dots,j_N}G_{j_1}(x_1',x_1;\tau)\cdots G_{j_N}(x_N',x_N;\tau).$$

By (28), we obtain the generating function of $R_N^{\tau}(n;\vec{x}_1,\dots,\vec{x}_N)$

$$(2\pi i\xi)^N\prod_{i=1}^{N}\mathfrak{H}(x_i',x_i;\xi;\tau)=\sum_{n=0}^{\infty}R_N^{\tau}(n;\vec{x}_1,\dots,\vec{x}_N)\frac{(2\pi i\xi)^n}{n!}. \tag{34}$$

We need the following lemma.

Lemma 4. *Let n be an integer with $n\geq N$. For any $i=1,2,\dots,N$, let x_i' and x_i be real numbers with $x_i'\notin\mathbb{Z}$. Set*

$$\vec{x}_i=(x_i',x_i)\qquad (i=1,2,\dots,N),\qquad (y',y)=(x_1'+\cdots+x_N',x_1+\cdots+x_N).$$

Suppose that $y'\notin\mathbb{Z}$, we have

$$(N-1)!\prod_{i=1}^{N}\mathfrak{H}(x_i',x_i;\xi;\tau)=2^{N-1}\sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}S_N^{\tau}(k;\vec{x}_1,\dots,\vec{x}_N)\mathfrak{H}^{(N-1-k)}(y',y;\xi;\tau). \tag{35}$$

Proof. Let function $K(\xi)$ equals LHS of (35) minus RHS of (35). By (31) and (32) yields

$$K(\xi+1)=e(y)K(\xi),\quad K(\xi+\tau)=e(y')K(\xi). \tag{36}$$

Let ξ be a complex number near the origin. By (33), (34) and $R_N^{\tau}(n;\vec{x}_1,\dots,\vec{x}_N)=0$ when $0\leq n\leq N-1$ and $G_0(x',x;\tau)=0$, it is not difficult to show that

$$\begin{aligned}K(\xi)=&(N-1)!\sum_{n=N}^{\infty}R_N^{\tau}(n;\vec{x}_1,\dots,\vec{x}_N)\frac{(2\pi i\xi)^{n-N}}{n!}\\&-2^{N-1}\sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}S_N^{\tau}(k;\vec{x}_1,\dots,\vec{x}_N)\sum_{n=0}^{\infty}G_{n+N-k}(y',y;\tau)\frac{(2\pi i\xi)^n}{(n+N-k)n!}.\end{aligned} \tag{37}$$

From (37) we see that function $K(\xi)$ is holomorphic at $\xi=0$. By (25) we know function $H(x,\xi;\tau)$ with respect to ξ is meromorphic with only simple poles on $\mathscr{C}$, the possible poles of function $K(\xi)$ are on $\mathscr{C}$. By (36) and (37), we obtain that function $K(\xi)$ is a holomorphic function in ξ.

On the other hand, since $\left|e(y)\right|=\left|e(y')\right|=1$, $e(y')\neq 1$, and combining (36), we say that function $K(\xi)$ is a bounded function. Therefor, we obtain the $K(\xi)=0$ by applying Liouville's theorem. This completes the proof. □

Theorem 5 (Sums of products for the elliptic Genocchi functions). *Let n be an integer with $n \geq N$. For any $i = 1, 2, \ldots, N$, let x_i' and x_i be real numbers with $x_i' \notin \mathbb{Z}$. Set $\vec{x}_i = (x_i', x_i)$, $(y', y) = (x_1' + \cdots + x_N', x_1 + \cdots + x_N)$, $y' \notin \mathbb{Z}$, we have*

$$R_N^\tau(n; \vec{x}_1, \ldots, \vec{x}_N) = 2^{N-1} N \binom{n}{N} \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} S_N^\tau(k; \vec{x}_1, \ldots, \vec{x}_N) \frac{G_{n-k}(y', y; \tau)}{n-k}. \tag{38}$$

Proof. By (33), (34) and (35), we arrive at formula (38). □

Next we give the formulas of sums of products of the Genocchi polynomials and numbers. Set

$$R_N(n; x_1, \ldots, x_N) = \sum_{\substack{j_1, \ldots, j_N \geq 0 \\ j_1 + \cdots + j_N = n}} \binom{n}{j_1, \ldots, j_N} G_{j_1}(x_1) \cdots G_{j_N}(x_N)$$

and

$$R_N(n) = \sum_{\substack{j_1, \ldots, j_N \geq 0 \\ j_1 + \cdots + j_N = n}} \binom{2n}{2j_1, \ldots, 2j_N} G_{2j_1} \cdots G_{2j_N}.$$

Lemma 6. *Let x be a real number and x' a complex number with $x' \notin \mathbb{Z}$, we have*

$$\lim_{\tau \to i\infty} G_n(x', x; \tau) = \begin{cases} (-1)^{[x]} \dfrac{e(x') + 1}{e(x') - 1} & n = 1, x \in \mathbb{Z}, \\ \widehat{G}_n(x) & \textit{otherwise}. \end{cases} \tag{39}$$

Proof. By (29), we have $G_n(x', x; \tau) = (-1)^{[x]} G_n(x', \{x\}; \tau)$ for any $x \in \mathbb{R}$. Suppose $0 \leq x < 1$, function $H(x, \xi; \tau)$ has the following expression by (21)

$$H(x, \xi; \tau) = 1 + \frac{1}{e(x) - 1} - \frac{1}{e(\xi) + 1} - \sum_{j=1}^\infty \frac{(-q)^j}{e(-x) - q^j} e(j\xi) + \sum_{j=1}^\infty \frac{(-q)^j}{e(x) - q^j} e(-j\xi)$$

$$(0 < \Im\xi < \Im\tau, \ 0 < \Im x < \Im\tau).$$

By (1) and (28), via a simple computation, we obtain

$$\begin{aligned} G_n(x', x; \tau) =& G_n(x) - 2n \Bigg[x^{n-1} \frac{e(-x' + x\tau)}{e(-x' + x\tau) - 1} - \sum_{j=1}^\infty (x + j)^{n-1} \frac{e(x\tau)(-q)^j}{e(x') - e(x\tau) q^j} \\ & + \sum_{j=1}^\infty (x - j)^{n-1} \frac{e(-x\tau)(-q)^j}{e(-x') - e(-x\tau) q^j} \Bigg]. \end{aligned} \tag{40}$$

For $j \in \mathbb{N}$, we have

$$\lim_{\tau \to i\infty} e(x\tau)(-q)^j = \lim_{\tau \to i\infty} e(-x\tau)(-q)^j = 0,$$

$$\lim_{\tau \to i\infty} x^{n-1} \frac{e(-x' + x\tau)}{e(-x' + x\tau) - 1} = \begin{cases} \dfrac{1}{1 - e(x')} & n = 1, x = 0, \\ 0 & \textit{otherwise}, \end{cases}$$

in conjunction with (40), we obtain the desired (39). This proof is completed. □

Lemma 7. *We have*

(i) *Let $x_1, \dots, x_N$ be real numbers and $x'_1, \dots, x'_N$ be complex numbers with $x'_1, \dots, x'_N \notin \mathbb{Z}$. Set $\vec{x}_i = (x'_i, x_i)$ for $i = 1, \dots, N$. If $0 \le x_1, \dots, x_N < 1$, we have*

$$\lim_{x'_1 \to -i\infty} \cdots \lim_{x'_N \to -i\infty} \lim_{\tau \to i\infty} R_N^\tau(n; \vec{x}_1, \dots, \vec{x}_N) = R_N(n; x_1, \dots, x_N). \tag{41}$$

(ii) *Set $\vec{x}_i = (\frac{1}{2}, 0)$ for $i = 1, \dots, N-1$, $\vec{x}_N = (x'_N, 0)$, N is any positive integers, we have*

$$\text{The coefficient of } (x'_N)^0 (= 1) \text{ of } \lim_{\tau \to i\infty} R_N^\tau(n; \vec{x}_1, \dots, \vec{x}_N) = R_N(n). \tag{42}$$

Proof. For $0 \le x < 1$, by (39), we have

$$\lim_{x' \to -i\infty} \lim_{\tau \to i\infty} G_n(x', x; \tau) = G_n(x),$$

which implies (41).

From (39), noticing that $G_{2n+1} = 0$ $(n \ge 1)$ and $\lim_{\tau \to i\infty} G_1(\frac{1}{2}, 0; \tau) = 0$, we have

$$\lim_{\tau \to i\infty} R_N^\tau(n; \vec{x}_1, \dots, \vec{x}_N) = R_N(n) + \frac{e(x'_N) + 1}{e(x'_N) - 1} \sum_{\substack{j_1, \dots, j_{N-1} \ge 0, j_N = 1 \\ j_1 + \cdots + j_{N-1} = n-1}} \binom{2n}{2j_1, \dots, 2j_{N-1}, 2} G_{2j_1} \cdots G_{2j_{N-1}}.$$

For any N positive integer, $\frac{e(x'_N)+1}{e(x'_N)-1}$ is an odd finction, we derive (42) immediately. □

Theorem 8 (Sums of products of Genocchi polynomials). *Let $x_1, \dots, x_N, y$ be complex numbers with $y = x_1 + \cdots + x_N$. For $n \ge N$, we have*

$$\begin{aligned} R_N(n; x_1, \dots, x_N) &= 2^{N-1} N \binom{n}{N} \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} B_k^{(N)}(y) \frac{G_{n-k}(y)}{n-k} &(43)\\ &= 2^{N-1} N \binom{n}{N} \sum_{k=0}^{N-1} (-1)^k \left\{ \sum_{j=0}^{k} \binom{N-k-1+j}{j} s(N, N-k+j) y^j \right\} \frac{G_{n-k}(y)}{n-k}, &(44) \end{aligned}$$

where $s(n, k)$ denotes the Stirling numbers of the first kind.

Proof. By analytic continuation, from (15), (38), (39) and (41), and noting that fact $B_k^{(N)}(y) = S_N(k; x_1, \dots, x_N)$, we deduce (43) for any complex numbers $x_1, \dots, x_N$ with $y = x_1 + \cdots + x_N$. Applying identity [8, 52.2.21] after appropriate substitutions

$$\binom{N-1}{k} B_k^{(N)}(y) = \sum_{j=0}^{k} \binom{N-k-1+j}{j} s(N, N-k+j) y^j. \tag{45}$$

The formula (44) follows directly from (43). □

Corollary 9. *For $n \ge N$, we have*

$$\begin{aligned} \sum_{\substack{j_1, \dots, j_N \ge 0 \\ j_1 + \cdots + j_N = n}} \binom{n}{j_1, \dots, j_N} G_{j_1} \cdots G_{j_N} &= 2^{N-1} N \binom{n}{N} \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} B_k^{(N)} \frac{G_{n-k}}{n-k} &(46)\\ &= 2^{N-1} N \binom{n}{N} \sum_{k=0}^{N-1} (-1)^k s(N, N-k) \frac{G_{n-k}}{n-k}. &(47) \end{aligned}$$

Proof. Taking $y = 0$ in (43) and (44), we deduce (46) and (47) respectively. □

Theorem 10. *For $n \geq N$, we have*

$$R_N(n) = 2^{N-1}N\binom{2n}{N}\sum_{k=0}^{[(N-1)/2]}\binom{N-1}{2k}S_N(k)\frac{G_{2n-2k}}{2n-2k}. \tag{48}$$

Proof. If $\vec{x}_i = (\frac{1}{2}, 0)$ for $i = 1, \ldots, N-1$, $\vec{x}_N = (x'_N, 0)$, $(y', y) = \left(\frac{N-1}{2} + x'_N, 0\right)$ for any positive integer N (odd or even), by (38) and (39), we obtain that

$$\begin{aligned}\lim_{\tau\to i\infty} R_N^\tau(n;\vec{x}_1,\ldots,\vec{x}_N) =& 2^{N-1}N\binom{2n}{N}\sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}\\ &\times\left\{\frac{G_{2n-k}}{2n-k} + \frac{(-1)^{N-1}e(x'_N)+1}{(-1)^{N-1}e(x'_N)-1}\right\}\lim_{\tau\to i\infty} S_N^\tau(k;\vec{x}_1,\ldots,\vec{x}_N).\end{aligned}$$

For any positive integers N, $\frac{(-1)^{N-1}e(x'_N)+1}{(-1)^{N-1}e(x'_N)-1}$ is an odd function. Combining (16) and (42), we obtain the desired (48). □

In particular, taking $N = 2$ in (38), we can get the convolution identity for the elliptic Genocchi polynomials

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}G_k(x'_1,x_1;\tau)G_{n-k}(x'_2,x_2;\tau) &= 2(n-1)G_n(x'_1+x'_2,x_1+x_2;\tau)\\ &\quad -2n[B_1(x'_1,x_1;\tau)+B_1(x'_2,x_2;\tau)]G_{n-1}(x'_1+x'_2,x_1+x_2;\tau),\end{aligned} \tag{49}$$

which is an elliptic extension of the convolution identity (see e.g., [8, Ch. 50])

$$\sum_{k=0}^{n}\binom{n}{k}G_k(x_1)G_{n-k}(x_2) = 2(n-1)G_n(x_1+x_2) - 2n(x_1+x_2-1)G_{n-1}(x_1+x_2). \tag{50}$$

By (39), (49) and noticing that $\lim_{x'\to -i\infty}\lim_{\tau\to i\infty} B_1(x',x;\tau) = B_1(x)$, we obtain (50) in a different way.

4. Further Remarks

Remark 11. *We still use the notation of [11]. The theta function $\theta(\xi,\tau)$ should be written as the following form in [11, p. 267]*

$$\theta(\xi,\tau) = \sum_{j=-\infty}^{\infty}(-1)^j e\left(\frac{1}{2}\left(j+\frac{1}{2}\right)^2\tau + \left(j+\frac{1}{2}\right)\xi\right),$$

it follows that, by [24, p. 455-456, Theorem (vii)], the equation (2) of [11, p. 273] should be corrected as

$$F(\xi,\eta,\tau) = \frac{1}{2\pi i}\frac{\theta'(0,\tau)\theta(\xi+\eta,\tau)}{\theta(\xi,\tau)\theta(\eta,\tau)}.$$

Remark 12. *Equation (18) of [13, p. 1065] should be corrected as follows:*

$$2\pi i\frac{\partial}{\partial\xi}\underline{\Lambda}(\xi,\tau;-2\pi i x',-2\pi i x) = \underline{F}(x',x;\xi;\tau),$$

where $\underline{F}(x',x;\xi;\tau) = e(x\xi)F_M(-x'+x\tau,\xi;\tau)$, $F_M(x,\xi;\tau)$ denotes the function $F(x,\xi;\tau)$ of [12] and [13].

Remark 13. *In [12], Lemma 4 (ii) and Lemma 16 (ii) are involved in Lemma 4 (iii) and Lemma 16 (iii) respectively. Because the function* $\frac{1+e(x'_N)}{1-e(x'_N)}$ *is an odd function for any positive integer N, i.e., we only need the Lemma 4 (iii) and Lemma 16 (iii), we can complete these proofs of (21) and (39) or (41) in [12] respectively. Another thing is to replace "*$\lim_{x'\to -i\infty}\lim_{\tau\to i\infty}$*" by "*$\lim_{x'_1\to -i\infty}\cdots\lim_{x'_N\to -i\infty}\lim_{\tau\to i\infty}$*" in Lemma 4 (i) and Lemma 16(i) respectively.*

Remark 14. *H. M. Srivastava and Á. Pintér [21] obtained the following relationships between the Bernoulli and Euler polynomials, i.e., Srivastava-Pintér's addition theorem:*

$$B_n^{(\alpha)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}\left[B_k^{(\alpha)}(y)+\frac{k}{2}B_{k-1}^{(\alpha-1)}(y)\right]E_{n-k}(x), \tag{51}$$

$$E_n^{(\alpha)}(x+y)=\sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\left[E_{k+1}^{(\alpha-1)}(y)-E_{k+1}^{(\alpha)}(y)\right]B_{n-k}(x). \tag{52}$$

when $\alpha=1$ *we have*

$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}\left[B_k(y)+\frac{k}{2}y^{k-1}\right]E_{n-k}(x), \tag{53}$$

$$E_n(x+y)=\sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\left[y^{k+1}-E_{k+1}(y)\right]B_{n-k}(x). \tag{54}$$

A question is: how we obtain the elliptic analogues of Srivastava-Pintér's addition theorem from the elliptic Bernoulli and elliptic Euler functions?

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