Filomat 30:7 (2016), 1931–1939
DOI 10.2298/FIL1607931C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on Fractional Integral Operator Associated with Multiindex Mittag-Leffler Functions

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Abstract. Recently Kiryakova and several other ones have investigated so-called multiindex Mittag-Leffler functions associated with fractional calculus. Here, in this paper, we aim at establishing a new fractional integration formula (of pathway type) involving the generalized multiindex Mittag-Leffler function $E_{\gamma,\kappa}[(\alpha_j,\beta_j)_m;z]$. Some interesting special cases of our main result are also considered and shown to be connected with certain known ones.

1. Introduction and Preliminaries

In recent years, the fractional calculus has become one of the most rapidly growing research subject of mathematical analysis due to its numerous applications in various parts of science as well as mathematics. The present far-reaching development of the fractional calculus has been shown by the remarkably large number of contributions (see, *e.g.*, [1, 7, 10, 11, 13, 16, 18, 20, 22, 23, 31, 33–35] and the related references therein). Recently, Kiryakova [14] established the relationship between multiple (multiindex) Mittag-Leffler functions and generalized fractional calculus, which has mainly motivated our present investigation. Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^+$, $\mathbb{N}$ be the sets of complex numbers, real and positive real numbers, nonpositive integers, and positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For the present investigation, we consider the following definitions and earlier works.

Definition 1. *The generalized multiindex Mittag-Leffler function is defined and studied by Saxena and Nishimoto [26] in the following manner* (*see also [27]*) :

$$E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z] = E_{\gamma,\kappa}\left[(\alpha_j,\beta_j)^m_{j=1};z\right] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\prod\limits_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)}.\frac{z^n}{n!}$$

$$(\alpha_j,\ \beta_j,\ \gamma,\ \kappa,\ z \in \mathbb{C},\ \Re(\beta_j) > 0\ (j = 1,\ldots,m);\ \Re\Big(\sum_{j=1}^{m}\alpha_j\Big) > \max\{0,\ \Re(\kappa)-1\}), \tag{1.1}$$

2010 *Mathematics Subject Classification*. Primary 26A33, 33E12; Secondary 33C60, 33E20
Keywords. Pathway fractional integral operators, generalized multiindex Mittag-Leffler functions, generalized Wright function.
Received: 28 October 2014; Accepted: 01 April 2015
Communicated by Hari M. Srivastava
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where $(\gamma)_n$ is the Pochhammer symbol defined, for $\gamma \in \mathbb{C}$, as follows (see, *e.g.*, [29, p. 2 and p. 5]):

$$(\gamma)_n = \begin{cases} 1, & n = 0, \\ \gamma(\gamma+1)\cdots(\gamma+n-1), & n \in \mathbb{N} \end{cases}$$
$$= \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \quad \left(\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-\right)$$

and Γ being the familiar Gamma function (see, *e.g.*, [28, Section 1.1] and [29, Section 1.1]).

Here, it is noted that $E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]$ is a special case of Wright generalized hypergeometric function (see [38, 39]), the H-function [19] as well as the generalized multiindex Bessel function $J^{(\alpha_j)_m,\gamma}_{(\beta_j)_m,\kappa}[z]$, which is introduced here. A complete description of (1.1) can be found in the review article [6] (see also [5]). Some important special cases of this function are enumerated below:

(i) When $m = 1$ with $\left(\{\Re(\beta)\} > 0; \Re\left(\alpha_j\right) > \max\{0, \Re(\kappa) - 1\}\right)$, the function (1.1) reduces to the one that has been considered by Srivastava and Tomovski [32, p. 200, Eq. (1.13)]:

$$E^{\gamma,\kappa}_{\alpha,\beta}[z] = E_{\gamma,\kappa}[\alpha,\beta;z] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)} . \frac{z^n}{n!}. \tag{1.2}$$

The special case of (1.2) when $\kappa = 1$ was studied by Kilbas *et al.* [9, p. 32, Eq. (1.3)].

(ii) The special case of (1.1) when γ, $\kappa = 1$ yields the $2m$-parametric Mittag-Leffler type function due to Al-Bassam and Luchko [2]:

$$E^{1,1}_{(\alpha_j,\beta_j)_m}[z] = E_{1,1}\left[(\alpha_j,\beta_j)^m_{j=1}; z\right] = \sum_{n=0}^{\infty} \frac{1}{\prod\limits_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} . \frac{z^n}{n!}$$
$$\left(\alpha_j,\ \beta_j,\ z \in \mathbb{C},\ \Re(\beta_j) > 0,\ \Re\left(\alpha_j\right) > 0\ (j = 1, \ldots, m)\right). \tag{1.3}$$

Further setting $m = 2$ in (1.3) gives another interesting generalization of Mittag-Leffler function due to Djrbashyan [3].

(iii) When $\gamma, \kappa = 1$ and α_j is replaced by $\frac{1}{\alpha_j} (j = 1, \ldots, m)$, (1.1) reduces to the multiindex Mittag–Leffler function defined by Kriyakova [15]

$$E^{1,1}_{\left(\frac{1}{\alpha_j},\beta_j\right)_m}[z] = E_{1,1}\left[\left(\frac{1}{\alpha_j},\beta_j\right)^m_{j=1}; z\right] = \sum_{n=0}^{\infty} \frac{1}{\prod\limits_{j=1}^{m} \Gamma\left(\frac{n}{\alpha_j}+\beta_j\right)} . \frac{z^n}{n!}$$
$$\left(\alpha_j,\ \beta_j,\ z \in \mathbb{C},\ \Re(\beta_j) > 0,\ \Re\left(\alpha_j\right) > 0\ (j = 1, \ldots, m)\right). \tag{1.4}$$

(iv) If we set $m = \kappa = \gamma = 1$ with $\min\{\Re(\alpha), \Re(\beta)\} > 0$, then (1.1) reduces to the generalized Mittag-Leffler function considered by Wiman [37]. The case when $m = \kappa = \gamma = \beta = 1$ can be found in [4].

In recent years, the interest to this function has grown due to its applications in some reaction-diffusion problems and their various generalizations appearing in the solutions of fractional order differential and integral equations. The theory of this class of functions has been developed in a series of articles by Kiryakova *et al.* [12–14, 16, 17] (see also [8] and [24]).

Definition 2. *The H-function is defined in terms of a Mellin-Barnes integral in the following manner* (*see, e.g., [19]*) :

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\ \left(b_j,\beta_j\right)_{1,q}\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\cdots,\left(a_p,\alpha_p\right)\\ (b_1,\beta_1),\cdots,\left(b_q,\beta_q\right)\end{matrix}\right.\right]$$
$$=\frac{1}{2\pi i}\int_{\mathfrak{L}}\Theta(s)\,z^{-s}\mathrm{d}s, \tag{1.5}$$

where

$$\Theta(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j s\right)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma\left(1-b_j-\beta_j s\right)}, \tag{1.6}$$

and $m,\ n,\ p,\ q\in\mathbb{N}_0$ *with* $0\le m\le q,\ 0\le n\le p$, *and the parameters* $a_i,\ b_i\in\mathbb{C}$ *and* $\alpha_i,\ \beta_j\in\mathbb{R}^+$ $(i=1,\ldots,p;\ \ j=1,\ldots,q)$ *with the contour* $\mathfrak{L}$ *suitably chosen, and an empty product, if it occurs, is taken to be unity. The theory of the H-function is well explained in the book of Srivastava et al. [30, Chapter 1].*

Definition 3. *The generalized Wright's function is defined as follows* (*see, e.g., [38, 39]*) :

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,\,A_1),\,\ldots,\,\left(\alpha_p,\,A_p\right);\\ (\beta_1,\,B_1),\,\ldots,\,\left(\beta_q,\,B_q\right);\end{matrix}\ z\right]=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j+A_j\,k\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j+B_j\,k\right)}\frac{z^k}{k!}, \tag{1.7}$$

where the coefficients $A_1,\,\ldots,\,A_p\in\mathbb{R}^+$ *and* $B_1,\,\ldots,\,B_q\in\mathbb{R}^+$ *with*

$$1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geqq 0. \tag{1.8}$$

Definition 4. *Let* $\alpha_j,\ \beta_j,\ \gamma\in\mathbb{C}$ $(j=1,\ldots,m)$ *be such that* $\sum\limits_{j=1}^{m}\Re(\alpha)_j>\max\{0;\Re(\kappa)-1\};\kappa>0,\Re(\beta)>-1$ *and* $\Re(\gamma)>0$. *Then the generalized multiindex Bessel function* $J_{(\beta_j)_m,\kappa}^{(\alpha_j)_m,\gamma}[z]$ *is defined as follows:*

$$J_{(\beta_j)_m,\kappa}^{(\alpha_j)_m,\gamma}[z]=\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\prod\limits_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}.\frac{(-z)^n}{n!}\quad(m\in\mathbb{N}). \tag{1.9}$$

Remark 1. *If we set* $m=1,\kappa=0,\alpha_1=1,\beta_1=\nu$ *and replace* z *by* $\frac{z^2}{4}$ *in* (1.9), *we get*

$$J_{\nu,0}^{1,\gamma}\left[\frac{z^2}{4}\right]=\left(\frac{2}{z}\right)^{\nu}J_{\nu}[z], \tag{1.10}$$

where $J_\nu[z]$ *is a well known Bessel function of the first kind defined by* (*see [10, p. 32, Eq. (1.7.1)]*):

$$J_{\nu}[z]=\sum_{k=o}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)}\quad\left(z\in\mathbb{C}/(-\infty,0];\nu\in\mathbb{C}\right). \tag{1.11}$$

For more details about the Bessel functions, the reader may be referred (*for example*) *to the earlier extensive works by Erdélyi et al. [4, Vol. 2] and Watson [36].*

Here, in this paper, we aim at establishing a (presumably) new fractional integration formula (of pathway type) involving the generalized multi-index Mittag-Leffler function $E_{\gamma,\kappa}[(\alpha_j,\beta_j)_m;z]$. Some interesting special cases of our main result are also considered. Our main result is obtained by applying the generalized multiindex Mittag-Leffler function to the pathway type fractional integral operator given in (1.12). So we continue to recall the following definition.

Definition 5. *Let $f(x) \in L(a,b)$, $\eta \in \mathbb{C}$, $\Re(\eta) > 0$, $a > 0$ and let us take a pathway parameter $\alpha < 1$. Then the pathway fractional integration operator is defined and represented as follows (see [21, p. 239]):*

$$\left(P_{0^+}^{(\eta,\alpha,a)} f\right)(t) = t^\eta \int_0^{\frac{t}{a(1-\alpha)}} \left[1 - \frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}} f(\tau)\,\mathrm{d}\tau, \tag{1.12}$$

where $L(a,b)$ is the set of Lebesgue measurable functions defined on $(a,\, b)$.

Let $[a,\, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real line $\mathbb{R}$. The left-sided and right-sided Riemann-Liouville fractional integrals $I_{a+}^{\eta} f$ and $I_{b-}^{\eta} f$ of order $\eta \in \mathbb{C}$ ($\Re(\eta) > 0$) are defined, respectively, by

$$\left(I_{a+}^{\eta} f\right)(x) := \frac{1}{\Gamma(\eta)} \int_a^x \frac{f(t)\,\mathrm{d}t}{(x-t)^{1-\eta}} \quad (x > a;\ \Re(\eta) > 0) \tag{1.13}$$

and

$$\left(I_{b-}^{\eta} f\right)(x) := \frac{1}{\Gamma(\eta)} \int_x^b \frac{f(t)\,\mathrm{d}t}{(t-x)^{1-\eta}} \quad (x < b;\ \Re(\eta) > 0), \tag{1.14}$$

where $f \in C_\mu$ $(\mu \geq -1)$ (see, *e.g.*, [10, p. 69]).

Remark 2. *The special case of the pathway fractional integration operator $\left(P_{0^+}^{(\eta,\alpha,a)} f\right)(t)$ in (1.12) when $\alpha = 0$, $a = 1$, and $\eta \to \eta - 1$ reduces immediately to the left-sided Riemann-Liouville fractional integrals as follows:*

$$\left(P_{0^+}^{(\eta-1,0,1)} f\right)(t) = \int_0^t (t-\tau)^{\eta-1} f(\tau)\,\mathrm{d}\tau = \Gamma(\eta) \left(I_{0+}^{\eta} f\right)(t) \quad (\Re(\eta) > 0). \tag{1.15}$$

Further one of the Erdélyi-Kober type fractional integrals (see [10, p. 105, Eq. (2.6.1)]) defined by

$$\begin{gathered}\left(I_{a+;\sigma,\alpha}^{\eta} f\right)(t) := \frac{\sigma\, t^{-\sigma(\eta+\alpha)}}{\Gamma(\eta)} \int_a^t \frac{\tau^{\sigma\,\alpha+\sigma-1}\, f(\tau)\,\mathrm{d}\tau}{(t^\sigma - \tau^\sigma)^{1-\eta}} \\ (0 \leqq a < t < b \leqq \infty;\ \Re(\eta) > 0;\ \sigma > 0;\ \alpha \in \mathbb{C})\end{gathered} \tag{1.16}$$

appears to be closely related to the pathway fractional integration operator (1.12). It is found that one integral cannot contain the other one as a purely special case. Yet it is easy to see that some special cases of the two integrals have, for example, the following relationship:

$$\left(P_{0^+}^{(\eta-1,0,1)} f\right)(t) = \Gamma(\eta)\, t^\eta \left(I_{0+;1,0}^{\eta} f\right)(t). \tag{1.17}$$

Setting $f(t) = t^{\beta-1}$ in (1.12), we obtain the following formula (see [21, Eq. (12)]):

$$\begin{gathered}P_{0^+}^{(\eta,\alpha)} \left\{t^{\beta-1}\right\} = \frac{t^{\eta+\beta}}{[a(1-\alpha)]^\beta}\, \frac{\Gamma(\beta)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\beta+1\right)} \\ (\alpha < 1;\ \Re(\eta) > 0;\ \Re(\beta) > 0)\,.\end{gathered} \tag{1.18}$$

Indeed, setting $f(t) = t^{\beta-1}$ in (1.12) and then changing $u = \frac{a(1-\alpha)\tau}{t}$, some algebra gives us that

$$P_{0^+}^{(\eta,\alpha)}\left\{t^{\beta-1}\right\} = \frac{t^{\eta+\beta}}{[a(1-\alpha)]^{\beta}}\, B\left(\frac{\eta}{1-\alpha}+1,\, \beta\right),$$

where $B(\alpha,\beta)$ is the well-known Beta function which is closely related to the Gamma function as follows (see, *e.g.*, [28, pp. 9-11] and [29, pp. 7-10]):

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \left(\alpha,\, \beta \in \mathbb{C}\setminus\mathbb{Z}_0^-\right). \tag{1.19}$$

2. Pathway Fractional Integration of Generalized Multiindex Mittag-Leffler Functions

In this section we consider composition of the pathway fractional integral $P_{0^+}^{(\eta,\alpha)}$ given by (1.12) with the multiindex Mittag-Leffler (1.1). We begin by stating the following theorem.

Theorem 1. *Let $\alpha < 1$, the parameters $\eta,\ \gamma,\ \kappa,\ \rho,\ \alpha_j,\ \beta_j \in \mathbb{C}$ $(j = 1, \ldots, m;\ m \in \mathbb{N})$, $a > 0$, and $\sigma > 0$ be such that*

$$\Re(\gamma) > 0,\ \Re(\eta) > 0,\ \Re(\kappa) > 0,\ \Re\left(\alpha_j-\kappa\right) \geq -1\ (j = 1, \ldots, m),$$
$$\Re(\rho+\sigma) > 0, \quad \text{and} \quad \Re\left(1+\frac{\eta}{1-\alpha}\right) > \max\{0,\, -\Re(\rho)\}.$$

Then we have the following relation:

$$\begin{aligned} P_{0^+}^{(\eta,\alpha)}\left\{x^{\rho-1}E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[cx^{\sigma}]\right\} &= \frac{x^{\rho+\eta}\,\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)\,(a(1-\alpha))^{\rho}} \\ &\times {}_2\Psi_{m+1}\left[\begin{matrix} (\gamma,\kappa),\ (\rho,\sigma); \\ \left(\beta_j,\ \alpha_j\right)_1^m,\ \left(1+\rho+\frac{\eta}{1-\alpha},\ \sigma\right); \end{matrix}\ c\left(\frac{x}{a(1-\alpha)}\right)^{\sigma}\right]. \end{aligned} \tag{2.1}$$

Proof. Let the left-hand side of the formula (2.1) be denoted by $\mathcal{I}$. Applying (1.1) and using the definition (1.12) to (2.1), we get

$$\begin{aligned} \mathcal{I} &= P_{0^+}^{(\eta,\alpha)}\left\{x^{\rho-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j)}\frac{[cx^{\sigma}]^n}{n!}\right\} \\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j)}\frac{c^n}{n!}P_{0^+}^{(\eta,\alpha)}\left\{x^{\rho+\sigma n-1}\right\}. \end{aligned}$$

Here, applying (1.18) with β replaced by $(\rho+\sigma n)$ to the pathway integral, after a little simplification, we obtain the following expression

$$\begin{aligned} \mathcal{I} &= \sum_{n=0}^{\infty}\frac{\Gamma(\gamma+\kappa n)}{\Gamma(\gamma)\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j)}\frac{\Gamma(\rho+\sigma n)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(1+\frac{\eta}{1-\alpha}+\rho+\sigma n\right)}\frac{c^n x^{\rho+\sigma n+\eta}}{[a(1-\alpha)]^{\rho+n\sigma}n!} \\ &= \frac{x^{\rho+\eta}\,\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)\,(a(1-\alpha))^{\rho}} \\ &\quad\times\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+\kappa n)}{\Gamma(\gamma)\prod_{j=1}^m\Gamma(\alpha_j n+\beta_j)}\frac{\Gamma(\rho+\sigma n)}{\Gamma\left(1+\frac{\eta}{1-\alpha}+\rho+\sigma n\right)}\frac{c^n x^{\sigma n}}{[a(1-\alpha)]^{n\sigma}n!}, \end{aligned}$$

whose last summation, in view of (1.7), is easily seen to arrive at the expression in (2.1). This completes the proof. □

Indeed, by suitably specializing the values of the parameters α_j, β_j, γ, $\kappa \in \mathbb{C}$ ($j = 1, \ldots, m$), one can deduce numerous fractional calculus results involving the various types of Mittag-Leffler functions as the corollary of our main result. Further we can present a large number of special cases of our main result (2.1). Here we give only two examples of this type.

Setting $m = 1$ in the result of Theorem 1 yields the following result.

Corollary 1. *Let $\alpha < 1$, the parameters η, γ, κ, ρ, α, $\beta \in \mathbb{C}$ $a > 0$, and $\sigma > 0$ be such that*

$$\Re(\gamma) > 0,\ \Re(\eta) > 0,\ \Re(\kappa) > 0,\ \Re\,(\alpha - \kappa) \geq -1,$$
$$\Re(\rho + \sigma) > 0, \quad \text{and} \quad \Re\left(1 + \frac{\eta}{1-\alpha}\right) > \max\{0, -\Re(\rho)\}.$$

Then we have the following relation:

$$\begin{aligned} P_{0^+}^{(\eta,\alpha)}\left\{x^{\rho-1}E_{(\alpha,\beta)}^{\gamma,\kappa}[cx^{\sigma}]\right\} &= \frac{x^{\rho+\eta}\,\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)\,(a(1-\alpha))^{\rho}} \\ &\times {}_2\Psi_2\left[\begin{matrix} (\gamma,\kappa),\ (\rho,\sigma); \\ \left(\beta,\ \alpha\right),\ \left(1+\rho+\dfrac{\eta}{1-\alpha},\ \sigma\right); \end{matrix}\ c\left(\frac{x}{a(1-\alpha)}\right)^{\sigma}\right]. \end{aligned} \tag{2.2}$$

If we set $\gamma, \kappa = 1$ and replaced α_j by $\frac{1}{\alpha_j}$ ($j = 1, \ldots, m$) in (2.1), Theorem 1 reduces to the following corollary.

Corollary 2. *Let $\alpha < 1$, the parameters η, ρ, α_j, $\beta_j \in \mathbb{C}$ ($j = 1, \ldots, m$; $m \in \mathbb{N}$), $a > 0$, $\sigma > 0$ and $\Re(\eta) > 0$, $\Re\left(\alpha_j\right) > 0$ ($j = 1, \ldots, m$), $\Re(\rho) > 0$. Then the following relation holds true:*

$$\begin{aligned} P_{0^+}^{(\eta,\alpha)}\left\{x^{\rho-1}E_{(\frac{1}{\alpha_j},\beta_j)_m}[cx^{\sigma}]\right\} &= \frac{x^{\rho+\eta}\,\Gamma(1+\frac{\eta}{1-\alpha})}{(a(1-\alpha))^{\rho}} \\ &\times {}_2\Psi_{m+1}\left[\begin{matrix} (1,1),\ (\rho,\sigma); \\ \left(\beta_j,\ \dfrac{1}{\alpha_j}\right)_1^m,\ \left(1+\rho+\dfrac{\eta}{1-\alpha},\ \sigma\right); \end{matrix}\ c\left(\frac{x}{a(1-\alpha)}\right)^{\sigma}\right]. \end{aligned} \tag{2.3}$$

Remark 3. *Setting $m = 1$, $\kappa = 1$ and taking $\alpha_1 = \sigma$, $\beta_1 = \rho$ in the main result* (2.1)*, the resulting formula is seen to become the known result given by Nair [21, p. 244, Eq. (24)].*

3. Further Special Cases and Concluding Remarks

By setting $\alpha = 0$, $a = 1$ and $\eta \to \eta - 1$ in (2.1), (2.2) and (2.3), respectively, and applying (1.15), we obtain three (presumably) new fractional integral formulas involving left-sided Riemann–Liouville fractional integral operators stated in Corollaries 3, 4 and 5 below.

Corollary 3. *Let η, γ, κ, ρ, α_j, $\beta_j \in \mathbb{C}$ ($j = 1, \ldots, m$; $m \in \mathbb{N}$), $\sigma > 0$ and $\Re(\gamma) > 0$, $\Re(\eta) > 0$, $\Re(\kappa) > 0$, $\Re(\rho) > 0$, $\Re\left(\alpha_j\right) \geq 0$ ($j = 1, \ldots, m$). Then the following relation holds:*

$$\begin{aligned} I_{0^+}^{\eta}\left\{x^{\rho-1}E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa}[cx^{\sigma}]\right\} &= \frac{x^{\rho+\eta-1}}{\Gamma(\gamma)} \\ &\times {}_2\Psi_{m+1}\left[\begin{matrix} (\gamma,\kappa),\ (\rho,\sigma); \\ \left(\beta_j,\ \alpha_j\right)_1^m,\ \left(\rho+\eta,\ \sigma\right); \end{matrix}\ c\,(x)^{\sigma}\right]. \end{aligned} \tag{3.1}$$

Corollary 4. *Let η, γ, κ, ρ, α, $\beta \in \mathbb{C}$, $\sigma > 0$ $\Re(\gamma) > 0$, $\Re(\eta) > 0$, $\Re(\kappa) > 0$, $\Re(\alpha) \geq 0$, $\Re(\rho) > 0$, $\Re(\rho) > 0$. Then the following relation holds:*

$$I_{0^+}^{\eta}\left\{x^{\rho-1}E_{(\alpha,\beta)}^{\gamma,\kappa}(cx^{\sigma})\right\} = \frac{x^{\rho+\eta-1}}{\Gamma(\gamma)} \times {}_2\Psi_2\left[\begin{matrix}(\gamma,\kappa),\,(\rho,\sigma);\\ \left(\beta,\,\alpha\right),\,\left(\rho+\eta,\,\sigma\right);\end{matrix}\,c\,(x)^{\sigma}\right]. \tag{3.2}$$

Corollary 5. *Let η, ρ, α_j, $\beta_j \in \mathbb{C}$ $(j = 1, \ldots, m;\ m \in \mathbb{N})$, $\sigma > 0$, $\Re(\eta) > 0$, $\Re\left(\alpha_j\right) > 0$ $(j = 1, \ldots, m)$, $\Re(\rho) > 0$. Then we have the following relation:*

$$I_{0^+}^{\eta}\left\{x^{\rho-1}E_{(\frac{1}{\alpha_j},\beta_j)_m}[cx^{\sigma}]\right\} = x^{\rho+\eta-1} \times {}_2\Psi_{m+1}\left[\begin{matrix}(1,1),\,(\rho,\sigma);\\ \left(\beta_j,\,\dfrac{1}{\alpha_j}\right)_1^m,\,\left(\rho+\eta,\,\sigma\right);\end{matrix}\,c\,(x)^{\sigma}\right]. \tag{3.3}$$

If we replace σ by $\frac{1}{\alpha_1}$ and ρ by β_1 in (3.3), we obtain the following result due to Saxena *et al.* [25, p. 372, Eq. (4.1)]:

Corollary 6. *Let η, α_j, $\beta_j \in \mathbb{C}$ $(j = 1, \ldots, m;\ m \in \mathbb{N})$, $\Re(\eta) > 0$, $\Re\left(\alpha_j\right) > 0$ $(j = 1, \ldots, m)$,. Then we have the following relation:*

$$I_{0^+}^{\eta}\left\{x^{\beta_1-1}E_{(\frac{1}{\alpha_j},\beta_j)_m}[cx^{\frac{1}{\alpha_1}}]\right\} = x^{\rho+\eta-1}E_{(\frac{1}{\alpha_j}),(\beta_1+\eta,\beta_2,\ldots,\beta_m)}[cx^{\frac{1}{\alpha_1}}]. \tag{3.4}$$

It is noted that if we set $\alpha = 0$, $a = 1$, and $f(t)$ is replaced by ${}_2F_1\left(\eta+\beta, -\gamma; \eta; 1-\frac{t}{x}\right)f(t)$, (1.12) yields the Saigo fractional integral operator. Thus we can obtain the generalizations of left-sided fractional integrals, like Saigo, Erdélyi-Kober (see [31]; see also [10]), and so on, by suitable substitutions. Therefore the results presented here are easily shown to be converted to those corresponding to the above well known fractional operators.

Several further consequences of Theorem 1 and Corollaries 1-5 can easily be derived by using some known and new relationship between generalized multiindex Mittag-Leffler function $E_{\gamma,\kappa}[(\alpha_j, \beta_j)_m; z]$ and Fox H-function, generalized multiindex Bessel function and generalized Wright function (as given in Definitions 2, 3 and 4, respectively), after some suitable parametric replacements. These relatively simpler fractional integral formulas for pathway fractional integral operator (1.12) can be deduced from Theorem 1, and Corollaries 1-5 by appropriately applying the following relationships:

$$E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa}[z] = \frac{1}{\Gamma(\gamma)}H_{1,m+1}^{1,1}\left[-z\left|\begin{matrix}(1-\gamma,\kappa)\\(0,1),(1-\beta_1,\alpha_1),\ldots,(1-\beta_m,\alpha_m)\end{matrix}\right.\right]. \tag{3.5}$$

$$E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa}[z] = \frac{1}{\Gamma(\gamma)}\,{}_1\Psi_m\left[\begin{matrix}(\gamma,\kappa)\\(\beta_j,\alpha_j)_1^m\end{matrix}\,\Bigg|z\right]. \tag{3.6}$$

We also have

$$E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa}[z] = J_{(\beta_j-1)_m,\kappa}^{(\alpha_j)_m,\gamma}\left[-z\right]. \tag{3.7}$$

It is further noted that the generalized multiindex Mittag-Leffler function $E_{\gamma,\kappa}[(\alpha_j, \beta_j)_m; z]$ (see [26, 27]) is an elegant unification of various special functions. Similarly the pathway fractional integral operator (1.12) involving in our main result will enable us to derive a number of results covering a wide range of distributions due to presence of parameter α.

Acknowledgements This research was, in part, supported by the *Basic Science Research Program* through the *National Research Foundation of Korea* funded by the Ministry of Education, Science and Technology of the Republic of Korea (Grant No. 2010-0011005). Also this research was supported by Dongguk University Research Fund.

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