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A New Identity via Partial Fraction Decomposition

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Abstract. In this paper, we obtain a new identity using the partial fraction decomposition. As applications, some interesting binomial identities are also derived.

1. Introduction

The well-known the following binomial identity: for $n \in \mathbb{N} = \{1, 2, \ldots\}$ and all $\theta > 0$, we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\theta}{\theta + k} = \prod_{k=1}^{n} \frac{k}{\theta + k}. \tag{1}$$

Recently, Jonathon [14] gave a simple and interesting probabilistic proof of the above binomial identity who also further extended this binomial identity based on probabilistic method. We state his result as follows: for all $r, n \in \mathbb{N}$; $\theta > 0$, we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{\theta}{\theta + k}\right)^r = \left(\prod_{k=1}^{n} \frac{k}{\theta + k}\right) \left(1 + \sum_{j=1}^{r-1} \sum_{1 \le k_1 \le \cdots \le k_j \le n} \frac{\theta^j}{(\theta + k_1)(\theta + k_2) \cdots (\theta + k_j)}\right). \tag{2}$$

A. Sofo and H. M. Srivastava [26, 27] investigated the products of the shifted harmonic numbers and the reciprocal binomial coefficients and gave some identities for the harmonic numbers and binomial coefficients. H. M. Srivastava [29] gave a direct and unified approach to two binomial-coefficient identities. Verde-Star and H. M. Srivastava [33] found some the binomial formulas by the generating-function of the

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generalized Appell form for a sequence of Newton polynomials. R. Srivastava [32] showed several general families of combinatorial and other series identities.

The methods and techniques of the partial fraction decomposition are applied in the study of the harmonic numbers and binomial identity by W. Chu *et al* [2–8], H. Prodinger *et al* [19–22]. In the recent past a lot of papers appeared providing the results of the harmonic numbers and binomial identity and related investigations; see [1, 10, 12, 13, 16–18, 23–25, 28, 30, 31, 34–37] and the references therein.

The aim of the present paper is to further generalize the binomial identity (2) by using the method of the partial fraction decomposition. Some known and new binomial identities are also obtained.

2. Main Result and Proof

We first give the following lemma.

Lemma 1. *For $m,n,s,r \in \mathbb{N}_0$, $0 \le m \le s \le n, \theta > 0, \beta \neq 0$; z be any complex numbers, $a_0, a_1, \ldots; a_n$ and $b_{m+1}, b_{m+2}, \ldots, b_s$ be any different numbers, we have*

$$\frac{(z+b_{m+1})(z+b_{m+2})\cdots(z+b_s)}{(z-a_0)(z-a_1)\cdots(z-a_n)}\left(\frac{\beta}{z+\theta}\right)^r = \sum_{i=0}^{n}\frac{\prod_{l=m+1}^{s}(a_i+b_l)}{\prod_{j=0;i\neq j}^{n}(a_i-a_j)}\left(\frac{\beta}{a_i+\theta}\right)^r\frac{1}{z-a_i}+\frac{\lambda}{(z+\theta)^r}+\cdots+\frac{\mu}{z+\theta}. \quad (3)$$

Proof. By means of the standard partial fraction decomposition, we have

$$f(z)=\frac{(z+b_{m+1})(z+b_{m+2})\cdots(z+b_s)}{(z-a_0)(z-a_1)\cdots(z-a_n)}\left(\frac{\beta}{z+\theta}\right)^r=\sum_{i=0}^{n}\frac{A_i}{z-a_i}+\frac{\lambda}{(z+\theta)^r}+\cdots+\frac{\mu}{z+\theta},$$

where the coefficients A_i remain to be determined.

$$\begin{aligned}A_i &= \lim_{z\to a_i}(z-a_i)f(z)\\ &= \lim_{z\to a_i}\frac{(z+b_{m+1})(z+b_{m+2})\cdots(z+b_s)}{(z-a_0)(z-a_1)\cdots(z-a_{i-1})(z-a_{i+1})\cdots(z-a_n)}\left(\frac{\beta}{z+\theta}\right)^r\\ &= \frac{\prod_{l=m+1}^{s}(a_i+b_l)}{\prod_{j=0;i\neq j}^{n}(a_i-a_j)}\left(\frac{\beta}{a_i+\theta}\right)^r.\end{aligned}$$

This completes the proof of Lemma 1. □

Our main result is the following theorem.

Theorem 2. *Suppose $m,n,s,r \in \mathbb{N}_0$, $0 \le m \le s \le n, \theta > 0, \beta \neq 0$; $a_0, a_1, \ldots, a_n$ and $b_{m+1}, b_{m+2}, \ldots, b_s$ be any different numbers.*

When $\theta = b_h, h \notin \{m+1, m+2, \ldots, s\}$ and $r > 1$, we have

$$\begin{aligned}&\sum_{i=0}^{n}\frac{\prod_{l=m+1}^{s}(a_i+b_l)}{\prod_{j=0;i\neq j}^{n}(a_i-a_j)}\left(\frac{\beta}{a_i+\theta}\right)^r\\ &=(-1)^n\left(\frac{\Pi_{l=m+1}^{s}(b_l-\theta)}{\Pi_{j=0}^{n}(a_j+\theta)}\right)\left(\frac{\beta^r}{(a_0+\theta)^{r-1}}+\sum_{j=1}^{r-1}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\beta^r}{(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\right.\\ &\quad\left.+\sum_{i+j=r-1}\sum_{i=1}^{s-m}\sum_{m+1\le k_1<\cdots<k_i\le s}\sum_{0\le s_1\le\cdots\le s_j\le n}\frac{\beta^r}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\right). \quad (4)\end{aligned}$$

When $\theta = b_h, h \in \{m+1, m+2, \ldots, s\}$ and $r > 2$, we have

$$\sum_{i=0}^{n} \frac{\prod_{l=m+1}^{s}(a_i+b_l)}{\prod_{j=0;i\neq j}^{n}(a_i-a_j)}\left(\frac{\beta}{a_i+\theta}\right)^r$$
$$= (-1)^n\left(\frac{\prod_{l=m+1,l\neq h}^{s}(b_l-\theta)}{\prod_{j=0}^{n}(a_j+\theta)}\right)\left(\frac{\beta^r}{(a_0+\theta)^{r-2}} + \sum_{j=1}^{r-2}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\beta^r}{(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\right.$$
$$\left. + \sum_{i+j=r-2}\sum_{i=1}^{s-m-1}\sum_{\substack{m+1\le k_1<\cdots<k_i\le s\\ k_1,\ldots,k_i\neq h}}\sum_{0\le s_1\le\cdots\le s_j\le n}\frac{\beta^r}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\right). \tag{5}$$

Proof. Multiplying z on both sides of (3), and then let $z\to\infty$, we obtain

$$\sum_{i=0}^{n} \frac{\prod_{l=m+1}^{s}(a_i+b_l)}{\prod_{j=0;i\neq j}^{n}(a_i-a_j)}\left(\frac{\beta}{a_i+\theta}\right)^r + \mu = 0.$$

When $\theta = b_h, h \notin \{m+1, m+2, \ldots, s\}$ and $r > 1$, we have

$$\mu = [(z+\theta)^{-1}]\frac{(z+b_{m+1})(z+b_{m+2})\cdots(z+b_s)}{(z-a_0)(z-a_1)\cdots(z-a_n)}\left(\frac{\beta}{z+\theta}\right)^r$$
$$= \beta^r[(z+\theta)^{r-1}]\frac{(z+b_{m+1})(z+b_{m+2})\cdots(z+b_s)}{(z-a_0)(z-a_1)\cdots(z-a_n)}$$
$$= \beta^r[z^{r-1}]\frac{(z+b_{m+1}-\theta)(z+b_{m+2}-\theta)\cdots(z+b_s-\theta)}{(z-\theta-a_0)(z-\theta-a_1)\cdots(z-\theta-a_n)}$$
$$= (-1)^{n+1}\beta^r\left(\frac{\prod_{l=m+1}^{s}(b_l-\theta)}{\prod_{j=0}^{n}(a_j+\theta)}\right)[z^{r-1}]\frac{(1+\frac{z}{b_{m+1}-\theta})(1+\frac{z}{b_{m+2}-\theta})\cdots(1+\frac{z}{b_s-\theta})}{(1-\frac{z}{\theta+a_0})(1-\frac{z}{\theta+a_1})\cdots(1-\frac{z}{\theta+a_n})}$$
$$= (-1)^{n+1}\beta^r\left(\frac{\prod_{l=m+1}^{s}(b_l-\theta)}{\prod_{j=0}^{n}(a_j+\theta)}\right)[z^{r-1}]\left(1+\sum_{i=1}^{s-m}\sum_{m+1\le k_1<\cdots<k_i\le s}\frac{z^i}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)}\right)$$
$$\times\sum_{j\ge0}\left(\frac{z}{a_0+\theta}\right)^j\sum_{j\ge0}\left(\frac{z}{a_1+\theta}\right)^j\cdots\sum_{j\ge0}\left(\frac{z}{a_n+\theta}\right)^j$$
$$= (-1)^{n+1}\beta^r\left(\frac{\prod_{l=m+1}^{s}(b_l-\theta)}{\prod_{j=0}^{n}(a_j+\theta)}\right)[z^{r-1}]\left(1+\sum_{i=1}^{s-m}\sum_{m+1\le k_1<\cdots<k_i\le s}\frac{z^i}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)}\right)$$
$$\times\sum_{0\le s_1\le s_2\le\cdots\le s_j\le n}\frac{z^j}{(a_{s_1}+\theta)(a_{s_2}+\theta)\cdots(a_{s_j}+\theta)}$$
$$= (-1)^{n+1}\beta^r\left(\frac{\prod_{l=m+1}^{s}(b_l-\theta)}{\prod_{j=0}^{n}(a_j+\theta)}\right)\left(\sum_{0\le s_1\le\cdots\le s_{r-1}\le n}\frac{1}{(a_{s_1}+\theta)\cdots(a_{s_{r-1}}+\theta)}\right.$$
$$\left.+\sum_{i+j=r-1}\sum_{i=1}^{s-m}\sum_{m+1\le k_1<\cdots<k_i\le s}\sum_{0\le s_1\le\cdots\le s_j\le n}\frac{1}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\right)$$
$$= (-1)^{n+1}\left(\frac{\prod_{l=m+1}^{s}(b_l-\theta)}{\prod_{j=0}^{n}(a_j+\theta)}\right)\left(\frac{\beta^r}{(a_0+\theta)^{r-1}}+\sum_{j=1}^{r-1}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\beta^r}{(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\right.$$
$$\left.+\sum_{i+j=r-1}\sum_{i=1}^{s-m}\sum_{m+1\le k_1<\cdots<k_i\le s}\sum_{0\le s_1\le\cdots\le s_j\le n}\frac{\beta^r}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\right).$$

When $\theta = b_h, h \in \{m+1, m+2, \dots, s\}$ and $r > 2$, we have

$$\begin{aligned}
\mu &= [(z+\theta)^{-1}]\frac{(z+b_{m+1})(z+b_{m+2})\cdots(z+b_s)}{(z-a_0)(z-a_1)\cdots(z-a_n)}\left(\frac{\beta}{z+\theta}\right)^r \\
&= \beta^r[(z+\theta)^{r-1}]\frac{(z+b_{m+1})(z+b_{m+2})\cdots(z+b_s)}{(z-a_0)(z-a_1)\cdots(z-a_n)} \\
&= \beta^r[(z+\theta)^{r-1}]\frac{(z+b_{m+1})(z+b_{m+2})\cdots(z+b_s)}{(z-a_0)(z-a_1)\cdots(z-a_n)} \\
&= \beta^r[z^{r-1}]\frac{(z+b_{m+1}-\theta)\cdots(z+b_{h-1}-\theta)z(z+b_{h+1}-\theta)\cdots(z+b_s-\theta)}{(z-\theta-a_0)(z-\theta-a_1)\cdots(z-\theta-a_n)} \\
&= (-1)^{n+1}\beta^r\left(\frac{\Pi_{l=m+1,l\neq h}^{s}(b_l-\theta)}{\Pi_{j=0}^{n}(a_j+\theta)}\right)[z^{r-2}]\frac{(1+\frac{z}{b_{m+1}-\theta})\cdots(1+\frac{z}{b_{h-1}-\theta})(1+\frac{z}{b_{h+1}-\theta})\cdots(1+\frac{z}{b_s-\theta})}{(1-\frac{z}{\theta+a_0})(1-\frac{z}{\theta+a_1})\cdots(1-\frac{z}{\theta+a_n})} \\
&= (-1)^{n+1}\beta^r\left(\frac{\Pi_{l=m+1,l\neq h}^{s}(b_l-\theta)}{\Pi_{j=0}^{n}(a_j+\theta)}\right)[z^{r-2}]\Bigg(1+\sum_{i=1}^{s-m-1}\sum_{\substack{m+1\leq k_1<\cdots<k_i\leq s,\\ k_1,\dots,k_i\neq h}}\frac{z^i}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)}\Bigg) \\
&\quad\times\sum_{j\geq 0}\left(\frac{z}{a_0+\theta}\right)^j\sum_{j\geq 0}\left(\frac{z}{a_1+\theta}\right)^j\cdots\sum_{j\geq 0}\left(\frac{z}{a_n+\theta}\right)^j \\
&= (-1)^{n+1}\beta^r\left(\frac{\Pi_{l=m+1,l\neq h}^{s}(b_l-\theta)}{\Pi_{j=0}^{n}(a_j+\theta)}\right)[z^{r-2}]\Bigg(1+\sum_{i=1}^{s-m-1}\sum_{\substack{m+1\leq k_1<\cdots<k_i\leq s,\\ k_1,\dots,k_i\neq h}}\frac{z^i}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)}\Bigg) \\
&\quad\times\sum_{0\leq s_1\leq s_2\leq\cdots\leq s_j\leq n}\frac{z^j}{(a_{s_1}+\theta)(a_{s_2}+\theta)\cdots(a_{s_j}+\theta)} \\
&= (-1)^{n+1}\beta^r\left(\frac{\Pi_{l=m+1,l\neq h}^{s}(b_l-\theta)}{\Pi_{j=0}^{n}(a_j+\theta)}\right)\Bigg(\sum_{0\leq s_1\leq\cdots\leq s_{r-2}\leq n}\frac{1}{(a_{s_1}+\theta)\cdots(a_{s_{r-2}}+\theta)} \\
&\quad+\sum_{i+j=r-2}\sum_{i=1}^{s-m-1}\sum_{\substack{m+1\leq k_1<\cdots<k_i\leq s\\ k_1,\dots,k_i\neq h}}\sum_{0\leq s_1\leq\cdots\leq s_j\leq n}\frac{1}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\Bigg) \\
&= (-1)^{n+1}\left(\frac{\Pi_{l=m+1,l\neq h}^{s}(b_l-\theta)}{\Pi_{j=0}^{n}(a_j+\theta)}\right)\Bigg(\frac{\beta^r}{(a_0+\theta)^{r-2}}+\sum_{j=1}^{r-2}\sum_{1\leq s_1\leq\cdots\leq s_j\leq n}\frac{\beta^r}{(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)} \\
&\quad+\sum_{i+j=r-2}\sum_{i=1}^{s-m-1}\sum_{\substack{m+1\leq k_1<\cdots<k_i\leq s\\ k_1,\dots,k_i\neq h}}\sum_{0\leq s_1\leq\cdots\leq s_j\leq n}\frac{\beta^r}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\Bigg).
\end{aligned}$$

In the above proof process, we apply the relation

$$\sum_{0\leq s_1\leq\cdots\leq s_{r-1}\leq n}\frac{\beta^r}{(a_{s_1}+\theta)\cdots(a_{s_{r-1}}+\theta)} = \frac{\beta^r}{(a_0+\theta)^{r-1}}+\sum_{j=1}^{r-1}\sum_{1\leq s_1\leq\cdots\leq s_j\leq n}\frac{\beta^r}{(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}. \tag{6}$$

The proof is complete. □

3. Some Applications

In this section, as some applications we obtain some interesting binomial identities from Theorem 2.

Corollary 3. *For $n \in \mathbb{N}_0, \theta > 0$; $a_0, a_1, \ldots, a_n$ be any different numbers.*
When $\theta = a_t, t \notin \{1, 2, \ldots, n\}$ and $r > 1$, we have

$$\sum_{i=0}^{n} \frac{\prod_{j=1}^{n}(a_i + a_j)}{\prod_{j=0;i\neq j}^{n}(a_i - a_j)} \left(\frac{\theta}{a_i + \theta}\right)^r$$
$$= (-1)^n \left(\frac{\Pi_{j=1}^{n}(a_j - \theta)}{\Pi_{j=0}^{n}(a_j + \theta)}\right)\left(\frac{\theta^r}{(a_0 + \theta)^{r-1}} + \sum_{j=1}^{r-1} \sum_{1\le s_1\le\cdots\le s_j\le n} \frac{\theta^r}{(a_{s_1} + \theta)\cdots(a_{s_j} + \theta)}\right.$$
$$\left. + \sum_{i+j=r-1} \sum_{i=1}^{n} \sum_{1\le k_1<\cdots<k_i\le n} \sum_{0\le s_1\le\cdots\le s_j\le n} \frac{\theta^r}{(a_{k_1} - \theta)\cdots(a_{k_i} - \theta)(a_{s_1} + \theta)\cdots(a_{s_j} + \theta)}\right). \tag{7}$$

When $\theta = a_t, t \in \{1, 2, \ldots, n\}$ and $r > 2$, we have

$$\sum_{i=0}^{n} \frac{\prod_{j=1}^{n}(a_i + a_j)}{\prod_{j=0;i\neq j}^{n}(a_i - a_j)} \left(\frac{\theta}{a_i + \theta}\right)^r$$
$$= (-1)^n \left(\frac{\Pi_{j=1,j\neq t}^{n}(a_j - \theta)}{\Pi_{j=0}^{n}(a_j + \theta)}\right)\left(\frac{\theta^r}{(a_0 + \theta)^{r-2}} + \sum_{j=1}^{r-1} \sum_{1\le s_1\le\cdots\le s_j\le n} \frac{\theta^r}{(a_{s_1} + \theta)\cdots(a_{s_j} + \theta)}\right.$$
$$\left. + \sum_{i+j=r-2} \sum_{i=1}^{n-1} \sum_{\substack{1\le k_1<\cdots<k_i\le n,\\ t\neq k_1,\ldots,k_i}} \sum_{0\le s_1\le\cdots\le s_j\le n} \frac{\theta^r}{(a_{k_1} - \theta)\cdots(a_{k_i} - \theta)(a_{s_1} + \theta)\cdots(a_{s_j} + \theta)}\right). \tag{8}$$

Proof. Taking $\beta = \theta, s = n, m = 0, b_l = a_j$ $(j = 1, 2, \ldots, n)$ in (4) and (5) of Theorem 2, respectively, we get the corresponding (7) and (8) of . □

Corollary 4. *For $n, d \geq 1$; $a_0, a_1, \ldots, a_n$ be any different numbers.*
When $t \notin \{1, 2, \ldots, n\}$, we have

$$\sum_{i=0}^{n} \frac{\prod_{j=m+1}^{n}(a_i + a_j)}{\prod_{j=0;i\neq j}^{n}(a_i - a_j)} \frac{1}{(a_i + a_t)^{d+1}}$$
$$= (-1)^n \left(\frac{\Pi_{j=m+1,}^{n}(a_j - a_t)}{\Pi_{j=0}^{n}(a_j + a_t)}\right)\left(\frac{1}{(a_0 + a_t)^d} + \sum_{j=1}^{d} \sum_{1\le s_1\le\cdots\le s_j\le n} \frac{1}{(a_{s_1} + a_t)\cdots(a_{s_j} + a_t)}\right.$$
$$\left. + \sum_{i+j=d} \sum_{i=1}^{n-m} \sum_{m+1\le k_1<\cdots<k_i\le n} \sum_{0\le s_1\le\cdots\le s_j\le n} \frac{1}{(a_{k_1} - a_t)\cdots(a_{k_i} - a_t)(a_{s_1} + a_t)\cdots(a_{s_j} + a_t)}\right). \tag{9}$$

When $t \in \{1, 2, \ldots, n\}$, we have

$$\sum_{i=0}^{n} \frac{\prod_{j=m+1}^{n}(a_i + a_j)}{\prod_{j=0;i\neq j}^{n}(a_i - a_j)} \frac{1}{(a_i + a_t)^{d+1}}$$
$$= (-1)^n \left(\frac{\Pi_{j=m+1,j\neq t}^{n}(a_j - a_t)}{\Pi_{j=0}^{n}(a_j + a_t)}\right)\left(\frac{1}{(a_0 + a_t)^{d-1}} + \sum_{j=1}^{d} \sum_{1\le s_1\le\cdots\le s_j\le n} \frac{1}{(a_{s_1} + a_t)\cdots(a_{s_j} + a_t)}\right.$$
$$\left. + \sum_{i+j=d-1} \sum_{i=1}^{n-m-1} \sum_{\substack{m+1\le k_1<\cdots<k_i\le n,\\ t\neq k_1,\ldots,k_i}} \sum_{0\le s_1\le\cdots\le s_j\le n} \frac{1}{(a_{k_1} - a_t)\cdots(a_{k_i} - a_t)(a_{s_1} + a_t)\cdots(a_{s_j} + a_t)}\right). \tag{10}$$

Proof. Taking $s = n, b_l = a_j, \beta = 1, \theta = a_t, r = d + 1$ in (4) and (5) of Theorem 2, respectively, we get the corresponding (9) and (10) of . □

Corollary 5. *Let $n, r \in \mathbb{N}_0, r > 1, \theta > 0$; $a_0, a_1, \ldots, a_n$ be any different numbers,, we have*

$$\sum_{i=0}^{n} \frac{1}{\prod_{j=0;i\neq j}^{n}(a_i - a_j)} \left(\frac{\theta}{a_i + \theta}\right)^r = \frac{(-1)^n}{\prod_{j=0}^{n}(a_j + \theta)} \sum_{0\le s_1\le\cdots\le s_{r-1}\le n} \frac{\theta^r}{(a_{s_1} + \theta)\cdots(a_{s_{r-1}} + \theta)} \tag{11}$$

$$= \frac{(-1)^n}{\prod_{j=0}^{n}(a_j + \theta)} \left(1 + \sum_{j=1}^{r-1} \sum_{1\le s_1\le\cdots\le s_j\le n} \frac{\theta^j}{(a_{s_1} + \theta)\cdots(a_{s_j} + \theta)}\right). \tag{12}$$

Proof. Taking $m = s, \beta = \theta$ in (4) of Theorem 2 and define that $\prod_{j=n+1}^{n} = 1, \sum_{i=1}^{0} = 0$, follows. □

Remark 6. *Obviously, the identity* (10) *is just a new representation of Mansour's result in [17, p. 138, Theorem 2.5]. In other words we say that Theorem 2 is an extension of Mansour's result.*

We below give some interesting binomial identities from Theorem 2.

Theorem 7. *For $n, m, r \in \mathbb{N}_0, 0 \le m \le n, \theta > 0$.*
When $\theta \notin \{m + 1, m + 2, \ldots, n\}$ and $r > 1$, we have

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{m+k}\left(\frac{\theta}{\theta + k}\right)^r \\ &= \left(\prod_{k=1}^{n} \frac{k}{\theta + k}\right)\left(\prod_{k=m+1}^{n} \frac{k - \theta}{k - m}\right)\left(1 + \sum_{j=1}^{r-1} \sum_{1\le s_1\le s_2\le\cdots\le s_j\le n} \frac{\theta^j}{(\theta + s_1)(\theta + s_2)\cdots(\theta + s_j)}\right. \\ &\quad \left. + \sum_{i+l=r-1} \sum_{i=1}^{n-m} \sum_{m+1\le k_1<\cdots<k_i\le n} \sum_{0\le s_1\le\cdots\le s_l\le n} \frac{\theta^{r-1}}{(k_1 - \theta)\cdots(k_i - \theta)(\theta + s_1)\cdots(\theta + s_l)}\right).\end{aligned} \tag{13}$$

When $\theta \in \{m + 1, m + 2, \ldots, n\}$ and $r > 2$, we have

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{m+k}\left(\frac{\theta}{\theta + k}\right)^r \\ &= \frac{1}{(n-m)!}\left(\prod_{k=1}^{n} \frac{k}{\theta + k}\right)\left(\prod_{k=m+1,k\neq\theta}^{n} (k - \theta)\right)\left(1 + \sum_{j=1}^{r-2} \sum_{1\le s_1\le\cdots\le s_j\le n} \frac{\theta^j}{(\theta + s_1)(\theta + s_2)\cdots(\theta + s_j)}\right. \\ &\quad \left. + \sum_{i+l=r-2} \sum_{i=1}^{n-m-1} \sum_{m+1\le k_1<\cdots k_i\le n, k_1,\ldots,k_i\neq\theta} \sum_{0\le s_1\le\cdots\le s_l\le n} \frac{\theta^{r-1}}{(k_1 - \theta)\cdots(k_i - \theta)(\theta + s_1)\cdots(\theta + s_l)}\right).\end{aligned} \tag{14}$$

Proof. Taking $s = n, \beta = \theta, a_i = i, b_l = l$ and changing the corresponding variable index in (4) and (5) of Theorem 2, and noting that the relation

$$\sum_{0\le s_1\le s_2\le\cdots\le s_{r-1}\le n} \frac{\theta^r}{(s_1 + \theta)(s_2 + \theta)\cdots(s_{r-1} + \theta)} = 1 + \sum_{j=1}^{r-1} \sum_{1\le s_1\le s_2\le\cdots\le s_j\le n} \frac{\theta^j}{(s_1 + \theta)(s_2 + \theta)\cdots(s_j + \theta)}, \tag{15}$$

we get the corresponding binomial identities (13) and (14) respectively. □

Corollary 8 ([20]). *For $l \le m \le n$, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{m+k}\frac{m}{(m+k)^{r+1}}=\binom{m+n}{m}^{-1}\Bigg(1+\sum_{j=1}^{r}\sum_{0\le s_1\le\cdots\le s_j\le n}\frac{m^{j-r}}{(s_1+m)(s_2+m)\cdots(s_j+m)}$$
$$+\sum_{i+j=r}\sum_{i=1}^{n-m}\sum_{m+1\le k_1<\cdots<k_i\le n}\sum_{0\le s_1\le\cdots\le s_j\le n}\frac{1}{(k_1-m)\cdots(k_i-m)(s_1+m)\cdots(s_j+m)}\Bigg), \tag{16}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{m+k}\left(\frac{p}{p+k}\right)^{r+1}$$
$$=\left(\prod_{k=1}^{n}\frac{k}{p+k}\right)\left(\prod_{k=m+1}^{n}\frac{k-p}{k-m}\right)\Bigg(1+\sum_{j=1}^{r}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{p^j}{(p+s_1)\cdots(p+s_j)}$$
$$+\sum_{i+l=r}\sum_{i=1}^{n-m}\sum_{m+1\le k_1<\cdots<k_i\le n}\sum_{0\le s_1\le\cdots\le s_l\le n}\frac{p^r}{(k_1-p)\cdots(k_i-p)(p+s_1)\cdots(p+s_l)}\Bigg). \tag{17}$$

Proof. Taking $\beta=\theta, s=n, r\longmapsto r+1, a_i=i(i=0,1,\ldots,n), b_l=l(l=m+1,m+2,\ldots,n)$, and taking $\theta=m$ and $\theta=p$ in (4) of Theorem 2, respectively, we get the binomial identities (16) and (17). □

Remark 9. *Obviously,* (16) *and* (17) *are just new representation of Prodinger's result in [20].*

Corollary 10 ([15, p. 2, Corollary 2.2]). *For $n, d \in \mathbb{N}_0$.*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{K}{K+k}\right)^r=\binom{n+K}{K}^{-1}\Bigg(1+\sum_{j=1}^{r-1}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{K^j}{(K+s_1)\cdots(K+s_j)}\Bigg). \tag{18}$$

Proof. Taking $n=m$ and $\theta=K$ in (13), we get (18) immediately. □

Remark 11. *Obviously,* (18) *is just a new representation of Kirschenhofer's result in [15, p. 4, Corollary 2.2].*

Taking $m=0$ in Theorem 7, we have

Corollary 12. *For $n\in\mathbb{N}$, $\theta>0$.*
When $\theta\notin\{1,2,\ldots,n\}$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\left(\frac{\theta}{\theta+k}\right)^r$$
$$=\left(\prod_{k=1}^{n}\frac{k-\theta}{\theta+k}\right)\Bigg(1+\sum_{j=1}^{r-1}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\theta^j}{(\theta+s_1)(\theta+s_2)\cdots(\theta+s_j)}$$
$$+\sum_{i+l=r-1}\sum_{i=1}^{n}\sum_{1\le k_1<\cdots<k_i\le n}\sum_{0\le s_1\le\cdots\le s_l\le n}\frac{\theta^{r-1}}{((k_1-\theta)\cdots(k_i-\theta)(\theta+s_1)\cdots(\theta+s_l)}\Bigg). \tag{19}$$

When $\theta\in\{1,2,\ldots,n\}$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\left(\frac{\theta}{\theta+k}\right)^r$$
$$=\left(\prod_{k=1}^{n}\frac{1}{\theta+k}\right)\left(\prod_{k=1,k\ne\theta}(k-\theta)\right)\Bigg(\theta+\sum_{j=1}^{r-2}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\theta^j}{(\theta+s_1)(\theta+s_2)\cdots(\theta+s_j)}$$
$$+\sum_{i+l=r-2}\sum_{i=1}^{n-1}\sum_{1\le k_1<\cdots<k_i\le n,k_1,\ldots,k_i\ne\theta}\sum_{0\le s_1\le\cdots\le s_l\le n}\frac{\theta^{r-1}}{((k_1-\theta)\cdots(k_i-\theta)(\theta+s_1)\cdots(\theta+s_l)}\Bigg). \tag{20}$$

Corollary 13 (see [14]). *For $r, n \in \mathbb{N}$; $\theta > 0$, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{\theta}{\theta+k}\right)^r = \left(\prod_{k=1}^{n}\frac{k}{\theta+k}\right)\left(1+\sum_{j=1}^{r-1}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\theta^j}{(\theta+s_1)(\theta+s_2)\cdots(\theta+s_j)}\right). \tag{21}$$

Proof. Taking $m = n, \beta = \theta$ in (13) of Theorem 14, it is must that $\theta \notin \{m+1, m+2, \ldots, n\}$ and define that $\prod_{k=n+1}^{n} = 1, \sum_{i=1}^{0} = 0$, follows. □

Setting $r = 1$ in (19), define the suitable empty sum $\sum_{j=1}^{0} = 0$, we have the following new binomial identity.

Corollary 14. *For all $n \in \mathbb{N}$ and $\theta > 0$*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{\theta}{\theta+k} = \prod_{k=1}^{n}\frac{k-\theta}{\theta+k}. \tag{22}$$

Setting $r = 2$ in , define the suitable empty sum $\sum_{j=1}^{0} = 0$, we have the following new binomial identity.

Corollary 15. *For all $n \in \mathbb{N}$ and $\theta > 0$*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\left(\frac{\theta}{\theta+k}\right)^2 = \left(\prod_{k=1}^{n}\frac{k-\theta}{\theta+k}\right)\left(1+\sum_{k=1}^{n}\frac{\theta}{\theta+k}\right). \tag{23}$$

Setting $r = 1$ in (19), define the suitable empty sum $\sum_{j=1}^{0} = 0$, we have the following new binomial identity.

Corollary 16. *For $n \in \mathbb{N}, m \in \mathbb{N}_0$, $0 \le m \le n$; $\theta > 0$.*
When $\theta \notin \{m+1, m+2, \ldots, n\}$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{m+k}\frac{\theta}{\theta+k} = \left(\prod_{k=1}^{n}\frac{k}{\theta+k}\right)\left(\prod_{k=m+1}^{n}\frac{k-\theta}{k-m}\right). \tag{24}$$

When $\theta \in \{m+1, m+2, \ldots, n\}$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{m+k}\frac{\theta}{\theta+k} = \frac{1}{(n-m)!}\left(\prod_{k=1}^{n}\frac{k}{\theta+k}\right)\left(\prod_{k=m+1,k\neq\theta}^{n}(k-\theta)\right). \tag{25}$$

In a similar manner as the proof in Theorem 2, we can also get the following results.

Theorem 17. *Suppose $n, r \in \mathbb{N}, m, s \in \mathbb{N}_0$ $0 \le m \le s, \beta \neq 0, \theta > 0$; $a_1, \ldots, a_n$ and $b_{m+1}, b_{m+2}, \ldots, b_s$ be any different numbers.*
When $\theta = b_h, h \notin \{m+1, m+2, \ldots, s\}$ and $r > 1$, we have

$$\begin{aligned}&\sum_{i=1}^{n}\frac{\prod_{l=m+1}^{s}(a_i+b_l)}{\prod_{j=1;i\neq j}^{n}(a_i-a_j)}\left(\frac{\beta}{a_i+\theta}\right)^r\\ &=(-1)^{n+1}\left(\frac{\prod_{l=m+1}^{s}(b_l-\theta)}{\prod_{j=1}^{n}(a_j+\theta)}\right)\left(\sum_{1\le s_1\le\cdots\le s_{r-1}\le n}\frac{\beta^r}{(a_{s_1}+\theta)\cdots(a_{s_{r-1}}+\theta)}\right.\\ &\quad\left.+\sum_{i+j=r-1}\sum_{i=1}^{s-m}\sum_{m+1\le k_1<\cdots<k_i\le s}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\beta^r}{(b_{k_1}-\theta)\cdots(b_{k_i}-\theta)(a_{s_1}+\theta)\cdots(a_{s_j}+\theta)}\right).\end{aligned} \tag{26}$$

When $\theta = b_h, h \in \{m+1, m+2, \ldots, s\}$ and $r > 2$, we have

$$\sum_{i=1}^{n} \frac{\prod_{l=m+1}^{s}(a_i + b_l)}{\prod_{j=1;i\neq j}^{n}(a_i - a_j)} \left(\frac{\beta}{a_i + \theta}\right)^r$$
$$= (-1)^{n+1} \left(\frac{\prod_{l=m+1,l\neq h}^{s}(b_l - \theta)}{\prod_{j=1}^{n}(a_j + \theta)}\right)\Bigg(\sum_{1\le s_1\le\cdots\le s_{r-2}\le n} \frac{\beta^r}{(a_{s_1} + \theta)\cdots(a_{s_{r-2}} + \theta)}$$
$$+ \sum_{i+j=r-2} \sum_{i=1}^{s-m-1} \sum_{\substack{m+1\le k_1<\cdots<k_i\le s\\ k_1,\ldots,k_i\neq h}} \sum_{1\le s_1\le\cdots\le s_j\le n} \frac{\beta^r}{(b_{k_1} - \theta)\cdots(b_{k_i} - \theta)(a_{s_1} + \theta)\cdots(a_{s_j} + \theta)}\Bigg). \tag{27}$$

Corollary 18. *For $a_1, a_2, \ldots, a_n$ be any different numbers, and $n, d \geq 1$, we have*

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n}(a_i + a_j)}{\prod_{j=1;i\neq j}^{n}(a_i - a_j)} \frac{1}{a_i^{d+1}}$$
$$= (-1)^{n+1} \left(\frac{\prod_{l=m+1}^{s}(b_l - \theta)}{\prod_{j=1}^{n}(a_j + \theta)}\right)\Bigg(\sum_{1\le s_1\le\cdots\le s_{r-1}\le n} \frac{\beta^r}{(a_{s_1} + \theta)\cdots(a_{s_{r-1}} + \theta)}$$
$$+ \sum_{i+j=r-1} \sum_{i=1}^{s-m} \sum_{m+1\le k_1<\cdots<k_i\le s} \sum_{1\le s_1\le\cdots\le s_j\le n} \frac{\beta^r}{(b_{k_1} - \theta)\cdots(b_{k_i} - \theta)(a_{s_1} + \theta)\cdots(a_{s_j} + \theta)}\Bigg). \tag{28}$$

Proof. Taking $s = n$, $m = 0$, $b_l = a_j$, $j = 1, 2, \ldots, n$, $\beta = 1$, $\theta = 0$, $r = d + 1$ in (26) of Theorem 17, follows. □

Remark 19. *Obviously,* (28) *is just new representation of Mansour's result in [17, p. 136, Theorem 2.1].*

Corollary 20. *For $n, r \in \mathbb{N}, m \in \mathbb{N}_0, 0 \le m \le n, \theta > 0$.*
When $\theta \notin \{m+1, m+2, \ldots, n\}$ and $r > 1$, we have

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1}\binom{n+k}{m+k}\left(\frac{\theta}{\theta + k}\right)^r$$
$$= \frac{1}{n(n-m)!}\left(\prod_{k=1}^{n}\frac{k}{\theta + k}\right)\left(\prod_{k=m+1}^{n}\frac{k-\theta}{k-m}\right)\Bigg(\sum_{j=1}^{r-1}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\theta^j}{(\theta + s_1)\cdots(\theta + s_j)}$$
$$+ \sum_{i+l=r-1}\sum_{i=1}^{n-m}\sum_{m+1\le k_1<\cdots<k_i\le n}\sum_{1\le s_1\le\cdots\le s_l\le n}\frac{\theta^{r-1}}{((k_1 - \theta)\cdots(k_i - \theta)(\theta + s_1)\cdots(\theta + s_l)}\Bigg). \tag{29}$$

When $\theta \in \{m+1, m+2, \ldots, n\}$ and $r > 2$, we have

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1}\binom{n+k}{m+k}\left(\frac{\theta}{\theta + k}\right)^r$$
$$= \frac{1}{n(n-m)!}\left(\prod_{k=1}^{n}\frac{k}{\theta + k}\right)\left(\prod_{k=m+1,k\neq\theta}^{n}(k-\theta)\right)\Bigg(\sum_{j=1}^{r-2}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\theta^j}{(\theta + s_1)\cdots(\theta + s_j)}$$
$$+ \sum_{i+l=r-2}\sum_{i=1}^{n-m-1}\sum_{\substack{m+1\le k_1<\cdots k_i\le n\\ k_1,\ldots,k_i\neq\theta}}\sum_{1\le s_1\le\cdots\le s_l\le n}\frac{\theta^{r-1}}{((k_1 - \theta)\cdots(k_i - \theta)(\theta + s_1)\cdots(\theta + s_l)}\Bigg). \tag{30}$$

Proof. Taking $s = n, \beta = \theta, a_i = i, b_l = l$ and changing the corresponding variable index in (26) and (27) of Theorem 17, respectively, we get (31) and (32) of . □

Corollary 21. *For $n, r \in \mathbb{N}, \theta > 0$.*
When $\theta \notin \{1, 2, \ldots, n\}$ and $r > 1$ we have

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\binom{n+k}{k}\left(\frac{\theta}{\theta+k}\right)^r$$
$$= \frac{1}{n}\left(\prod_{k=1}^{n}\frac{k-\theta}{k(k+\theta)}\right)\left(\sum_{j=1}^{r-1}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\theta^j}{(\theta+s_1)\cdots(\theta+s_j)}\right.$$
$$\left.+\sum_{i+l=r-1}\sum_{i=1}^{n}\sum_{1\le k_1<\cdots<k_i\le n}\sum_{1\le s_1\le\cdots\le s_l\le n}\frac{\theta^{r-1}}{((k_1-\theta)\cdots(k_i-\theta)(\theta+s_1)\cdots(\theta+s_l))}\right). \tag{31}$$

When $\theta \in \{1, 2, \ldots, n\}$ and $r > 2$, we have

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\binom{n+k}{k}\left(\frac{\theta}{\theta+k}\right)^r$$
$$= \frac{1}{n}\left(\prod_{k=1}^{n}\frac{1}{k+\theta}\right)\left(\prod_{k=1,k\neq\theta}^{n}(k-\theta)\right)\left(\sum_{j=1}^{r-2}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{\theta^j}{(\theta+s_1)\cdots(\theta+s_j)}\right.$$
$$\left.+\sum_{i+l=r-2}\sum_{i=1}^{n-1}\sum_{\substack{1\le k_1<\cdots k_i\le n,\\ k_1,\ldots,k_i\neq\theta}}\sum_{1\le s_1\le\cdots\le s_l\le n}\frac{\theta^{r-1}}{((k_1-\theta)\cdots(k_i-\theta)(\theta+s_1)\cdots(\theta+s_l))}\right). \tag{32}$$

Proof. Taking $m = 0, s = n, \beta = \theta, a_i = i, b_l = l$ and changing the corresponding variable index in (26) and (27) of Theorem 17, respectively, we get (31) and (32) of . □

Corollary 22. *For all $n, d \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{n+k}{k}\frac{1}{k^d} = \sum_{i+j=d}\sum_{1\le k_1<\cdots<k_i\le n}\frac{1}{k_1k_2\cdots k_i}\sum_{1\le s_1\le\cdots\le s_j\le n}\frac{1}{s_1s_2\cdots s_j}. \tag{33}$$

Proof. Taking $s = n, m = 0, \beta = 1, \theta = 0, r = d, b_l = a_j = j, a_i = i$ in (26) of Theorem 17, follows. □

Remark 23. *Obviously,* (33) *is just new representation of Prodinger's result in [20, p. 292, Theorem 1]*

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