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De-Haan Type Conditions for Max Domains of Attraction

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Abstract. Motivated by de Haan's pioneering work, this paper gives some general sufficient and necessary conditions that functions are regularly varying and Γ°-varying, respectively. Specially, these criteria can be employed to determine whether a given distribution belongs to one of the max-domains of attractions of extreme value distributions.

1. Introduction

For a distribution function (df) F, if there exists some constants $a_n > 0$ and b_n such that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x)$$

for all continuity points x of G, where G is a non-degenerate df, then we say that F belongs to the max-domain of attraction of G, abbreviated as $F \in D(G)$. It is well-known that G must belong to one of the following three classes

$$\begin{array}{lll} \text{Type I} & \text{Gumbel:} & \Lambda(x) = \exp\Big(-\exp(-x)\Big), \quad x \in \mathbb{R}; \\ \text{Type II} & \text{Fréchet:} & \Phi_\alpha(x) = \begin{cases} 0, & x < 0, \\ \exp\Big(-x^{-\alpha}\Big), & x \geq 0 \end{cases} \quad \text{for } \alpha > 0; \\ \text{Type III} & \text{Weibull:} & \Psi_\alpha(x) = \begin{cases} \exp\Big(-(-x)^{\alpha}\Big), & x < 0, \\ 1, & x \geq 0 \end{cases} \quad \text{for } \alpha > 0. \end{array}$$

Standard monographs of extreme value theory are de Haan [1], Leadbetter et al. [7], Resnick [10], Reiss [9], Embrechts et al. [3], Kotz and Nadarajah [6], de Haan and Ferreira [2], Falk et al. [4]. Other complementing references dealing with conditions for $F \in D(G)$ are Geluk [5] and Peng et al. [8]. An interesting necessary and sufficient condition for $F \in D(G)$ is the following one refered to as de Haan MDA condition, [cf. de Haan [1], pages 100-103, Theorem 2.6.1, Theorem 2.6.2 and its remark], namely

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Theorem 1.1. *(de Haan MDA criteria) For df F, define* $\overline{F}(x) = 1 - F(x)$ *the survival function, and let* $x_0 := \sup\{x : F(x) < 1\}$ *represent the upper endpoint of F. Then,*

(1) $F \in D(\Lambda)$ *if and only if for some* $p \in \mathbb{R}$,

$$\lim_{x \to x_0} \frac{x^p \overline{F}(x) \int_x^{x_0} \int_y^{x_0} t^p \overline{F}(t) dt dy}{\left(\int_x^{x_0} t^p \overline{F}(t)\, dt\right)^2} = 1$$

and all the integrals in the preceding expression are finite.

(2) $F \in D(\Phi_\alpha)$ *if and only if for* $\alpha - p > 2$,

$$\lim_{x \to \infty} \frac{x^p \overline{F}(x) \int_x^{\infty} \int_y^{\infty} t^p \overline{F}(t) dt dy}{\left(\int_x^{\infty} t^p \overline{F}(t) dt\right)^2} = \frac{\alpha - p - 1}{\alpha - p - 2}$$

and all the integrals in the preceding expression are finite.

(3) $F \in D(\Psi_\alpha)$ *if and only if* $x_0 < \infty$ *and for any constant* $p \in \mathbb{R}$

$$\lim_{x \to x_0} \frac{x^p \overline{F}(x) \int_x^{x_0} \int_y^{x_0} t^p \overline{F}(t) dt dy}{\left(\int_x^{x_0} t^p \overline{F}(t) dt\right)^2} = \frac{\alpha + 1}{\alpha + 2}.$$

Motivated by the de Haan MDA criteria (Theorem 1.1), in this paper we present some necessary and sufficient conditions for $F \in D(G)$, which are in the spirit of de Haan [1]. Note that $F \in D(\Lambda)$ iff $\overline{F} \in \Gamma$-class and $F \in D(\Phi_\alpha)$ iff $\overline{F} \in RV_{-\alpha}$, see [1] and [10]. Proposition 1.13 in [10] also shows that $F \in D(\Psi_\alpha)$ iff $x_0 < \infty$ and $\overline{F}(x_0 - x^{-1}) \in RV_{-\alpha}$. The new criteria in this paper are formulated for general function H while we get the corresponding results to MDA as $H = \overline{F}$. The paper is organized as follows: In Section 2 we discuss $\Gamma^\circ(f)$-class, a variation of Γ-class given by [1] and [10]. Section 3 is for RV.

2. The Class $\Gamma^\circ(f)$

An ultimate positive and measurable function H defined on an interval $(x_l, x_0]$ is in the class $\Gamma^\circ(f)$ if it satisfies $\lim_{x \to x_0} H(x) = 0$ and

$$\lim_{x \to x_0} \frac{H(x + yf(x))}{H(x)} = e^{-y}, \quad \forall y \in \mathbb{R}.$$

The function f is an auxiliary function and f satisfies $f(x)/x \to 0$ and $f(x + yf(x))/f(x) \to 1, \forall y$. Obviously, $H \in \Gamma^\circ(f)$ iff $1/H \in \Gamma$ defined by de Haan [1]. Note that $H \in \Gamma^\circ(f)$ if and only if for some (all) $\alpha \in \mathbb{R}, \beta > 0$, we have $x^\alpha H^\beta(x) \in \Gamma^\circ(g)$, and then $g(x) = f(x)/\beta$. The following result is a result of de Haan.

Proposition 2.1. *a) The following are equivalent*

(i) $H \in \Gamma^\circ(f)$.

(ii) There exist functions a, b, c *such that*

$$H(x) = c(x) \exp\left(-\int_{a^\circ}^{x} \frac{a(z)}{b(z)} dz\right), \quad x \geq a^\circ,$$

and $c(x) \to c > 0$, $a(x) \to 1$ *and* $b'(x) \to 0$ *as* $x \to x_0$.

(iii) We have

$$\lim_{x\to x_0}\frac{H(x)\int_x^{x_0}\int_y^{x_0}H(z)dzdy}{(\int_x^{x_0}H(z)dz)^2}=1.$$

b) If $H\in\Gamma^\circ(f)$, $\alpha\in\mathbb{R}$, then $\int_x^{x_0}z^\alpha H(z)dz\sim x^\alpha H(x)f(x)\in\Gamma^\circ(f)$. Conversely, if H is nonincreasing and if $\int_x^{x_0}z^\alpha H(z)dz\in\Gamma^\circ(f)$, then $H\in\Gamma^\circ(f)$.

Remark 2.2. *If $H\in\Gamma^\circ(f)$, the result implies that $f(x)\sim b(x)\sim\int_x^{x_0}H(z)dz/H(x)$.*

From Proposition 2.1, we can obtain the following corollary.

Corollary 2.3. *Suppose that $H\in\Gamma^\circ(f)$. Then*

(i) for all $\alpha>0$,

$$\lim_{x\to x_0}\frac{H(x)\int_x^{x_0}\left(\int_y^{x_0}H(z)dz\right)^\alpha dy}{\left(\int_x^{x_0}H(z)dz\right)^{1+\alpha}}=\frac{1}{\alpha}.$$

(ii) for all $\alpha>\beta$, $p\in\mathbb{R}$,

$$\lim_{x\to x_0}\frac{x^{p\beta}H^\beta(x)\int_x^{x_0}z^{p\alpha}H^\alpha(z)dz}{x^{p\alpha}H^\alpha(x)\int_x^{x_0}z^{p\beta}H^\beta(z)dz}=\frac{\beta}{\alpha}.$$

Proof. (i) From Proposition 2.1 b), it follows that

$$\int_x^{x_0}H(z)dz\sim f(x)H(x)\in\Gamma^\circ(f)$$

and then

$$\left(\int_x^{x_0}H(z)dz\right)^\alpha\sim f^\alpha(x)H^\alpha(x)\in\Gamma^\circ(f/\alpha).$$

It follows that

$$\int_x^{x_0}\left(\int_y^{x_0}H(z)dz\right)^\alpha dy\sim\left(\int_x^{x_0}H(z)dz\right)^\alpha f(x)/\alpha.$$

Now result (*i*) follows.

(ii) From Proposition 2.1 b), we have

$$\int_x^{x_0}z^{p\alpha}H^\alpha(z)dz\sim x^{p\alpha}H^\alpha(x)f(x)/\alpha$$

and

$$\int_x^{x_0}z^{p\beta}H^\beta(z)dz\sim x^{p\beta}H^\beta(x)f(x)/\beta,$$

then the result follows. □

Remark 2.4. *Note that in Corollary 2.3 (ii), for all $\alpha > 0$, we have $\int_x^{x_0} z^{p\alpha}H^\alpha(z)dz \in \Gamma^\circ(f/\alpha)$.*

The converse result is also true.

Theorem 2.5. *Each of the following statements implies that $H \in \Gamma^\circ(f)$:*

(i) For some $\alpha > 0$,

$$\lim_{x\to x_0} \frac{H(x)\int_x^{x_0}\left(\int_y^{x_0} H(z)dz\right)^\alpha dy}{\left(\int_x^{x_0} H(z)dz\right)^{1+\alpha}} = \frac{1}{\alpha}.$$

(ii) For some $\alpha > \beta, p \in \mathbb{R}$,

$$\lim_{x\to x_0} \frac{x^{p\beta}H^\beta(x)\int_x^{x_0} z^{p\alpha}H^\alpha(z)dz}{x^{p\alpha}H^\alpha(x)\int_x^{x_0} z^{p\beta}H^\beta(z)dz} = \frac{\beta}{\alpha}.$$

(iii) If H is nonincreasing and for some $p \in \mathbb{R}$, $\int_x^{x_0} z^p H(z)dz \in \Gamma^\circ(f)$.

Proof. (i) Define $A(x)$ and $R(x)$ as follows:

$$\begin{aligned} A(x) &= \int_x^{x_0} H(z)dz,\\ R(x) &= \frac{A^{1+\alpha}(x)}{\int_x^{x_0} A^\alpha(y)dy}. \end{aligned}$$

In the case of (i), we have that $R(x)/H(x) \to \alpha$. Taking the derivative of $R(x)$, we have

$$\begin{aligned} R'(x) &= \frac{-(1+\alpha)A^\alpha(x)H(x)\int_x^{x_0} A^\alpha(y)dy + A^{1+\alpha}(x)A^\alpha(x)}{\left(\int_x^{x_0} A^\alpha(y)dy\right)^2}\\ &= \frac{-H(x)A^\alpha(x)}{\int_x^{x_0} A^\alpha(y)dy}\left(1+\alpha-\frac{R(x)}{H(x)}\right). \end{aligned}$$

Note that $-R'(x)$ is positive for large values of x. Now we have

$$\frac{R'(x)}{R(x)} = -\frac{R(x)}{A(x)}\left((1+\alpha)\frac{H(x)}{R(x)} - 1\right) = -\frac{a(x)}{b(x)}$$

where $a(x) = \alpha((1+\alpha)H(x)/R(x) - 1)$ and $b(x) = \alpha A(x)/R(x)$. Note that $a(x) \to 1$ and that

$$b(x) = \alpha\frac{\int_x^{x_0} A^\alpha(y)dy}{A^\alpha(x)}.$$

It is easy to see that

$$\begin{aligned} b'(x) &= \alpha\frac{-A^{2\alpha}(x) + \alpha A^{\alpha-1}(x)H(x)\int_x^{x_0} A^\alpha(y)dy}{A^{2\alpha}(x)}\\ &= \alpha\left(-1+\alpha\frac{H(x)}{R(x)}\right) \to 0. \end{aligned}$$

Taking integrals, we have

$$R(x) = C \exp\left(-\int_{x^\circ}^{x} \frac{a(z)}{b(z)}dz\right)$$

and we have (cf. Proposition 2.1, (ii)) that $R \in \Gamma^\circ(b)$. Since $H(x) \sim R(x)/\alpha$, we also get that $H \in \Gamma^\circ(b)$.

(ii) To prove (ii) we define function $r(x)$ and $R(x)$ as follows:

$$r(x) = \frac{x^{p\beta}H^{\beta}(x)\int_x^{x_0} z^{p\alpha}H^{\alpha}(z)dz}{x^{p\alpha}H^{\alpha}(x)\int_x^{x_0} z^{p\beta}H^{\beta}(z)dz}$$

and

$$R(x) = \frac{\int_x^{x_0} z^{p\alpha}H^{\alpha}(z)dz}{\int_x^{x_0} z^{p\beta}H^{\beta}(z)dz}.$$

By assumption we have $r(x) \to \beta/\alpha$ and $R(x) \sim (\beta/\alpha)x^{p(\alpha-\beta)}H^{(\alpha-\beta)}(x)$. Taking the derivative of $R(x)$, we find

$$\begin{aligned} R'(x) &= \frac{-\left(\int_x^{x_0} z^{p\beta}H^{\beta}(z)dz\right)\times x^{p\alpha}H^{\alpha}(x) + \left(\int_x^{x_0} z^{p\alpha}H^{\alpha}(z)dz\right)\times x^{p\beta}H^{\beta}(x)}{\left(\int_x^{x_0} z^{p\beta}H^{\beta}(z)dz\right)^2} \\ &= \frac{x^{p\alpha}H^{\alpha}(x)}{\int_x^{x_0} z^{p\beta}H^{\beta}(z)dz}(r(x)-1). \end{aligned}$$

It follows that

$$\frac{R'(x)}{R(x)} = -(1-r(x))\frac{x^{p\alpha}H^{\alpha}(x)}{\int_x^{x_0} z^{p\alpha}H^{\alpha}(z)dz} = -\frac{a(x)}{b(x)},$$

where

$$a(x) = (1-r(x))\frac{x^{p\alpha}H^{\alpha}(x)}{R^{\alpha/(\alpha-\beta)}(x)}$$

and

$$b(x) = R^{-\alpha/(\alpha-\beta)}(x)\int_x^{x_0} z^{p\alpha}H^{\alpha}(z)dz = R^{-\beta/(\alpha-\beta)}(x)\int_x^{x_0} z^{p\beta}H^{\beta}(z)dz.$$

First consider $a(x)$. Using $r(x) \to \beta/\alpha$ and $R(x) \sim (\beta/\alpha)x^{p(\alpha-\beta)}H^{(\alpha-\beta)}(x)$, we obtain that

$$a(x) \to \delta = \left(1-\frac{\beta}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^{-\alpha/(\alpha-\beta)} > 0.$$

For $b(x)$ we find

$$\begin{aligned} -b'(x) &= \frac{\beta}{\alpha-\beta}R^{-\alpha/(\alpha-\beta)}(x)R'(x)\int_x^{x_0} z^{p\beta}H^{\beta}(z)dz + R^{-\beta/(\alpha-\beta)}(x)x^{p\beta}H^{\beta}(x) \\ &= I + II. \end{aligned}$$

First consider I. Using the expression for $R'(x)$ and then using $R(x) \sim (\beta/\alpha)x^{p(\alpha-\beta)}H^{(\alpha-\beta)}(x)$, we have

$$\begin{aligned} I &= \frac{\beta}{\alpha-\beta}R^{-\alpha/(\alpha-\beta)}(x)x^{p\alpha}H^{\alpha}(x)(r(x)-1) \\ &\to \frac{\beta}{\alpha-\beta}\left(\frac{\beta}{\alpha}\right)^{-\alpha/(\alpha-\beta)}\left(\frac{\beta}{\alpha}-1\right) = -\left(\frac{\beta}{\alpha}\right)^{-\beta/(\alpha-\beta)}. \end{aligned}$$

For II we find

$$II \to \left(\frac{\beta}{\alpha}\right)^{-\beta/(\alpha-\beta)}.$$

It follows that $b'(x) \to 0$. Taking integrals, we get that

$$R(x) = C\exp\left(-\int_{x^\circ}^{x} \frac{a(z)}{b(z)}dz\right)$$

and using Proposition 2.1 (ii), we find that $R \in \Gamma^\circ(b/\delta)$. From here it follows that $x^p H(x) \in \Gamma^\circ$, and hence also that $H \in \Gamma^\circ$.

(iii) This is Proposition 2.1 b). □

3. The Class $RV_{-\alpha}$

An ultimate positive and measurable function H is in the class $RV_{-\gamma}, \gamma > 0$, if it satisfies

$$\lim_{x\to\infty} \frac{H(xy)}{H(x)} = y^{-\gamma},$$

see, de Haan [1] and Resnick [10].

If $\gamma > 1$, Karamata's Theorem shows that $H \in RV_{-\gamma}$ implies

$$\int_x^\infty H(y)dy \sim \frac{xH(x)}{\gamma - 1} \in RV_{1-\gamma}.$$

Conversely, if $\int_x^\infty H(y)dy \in RV_{1-\gamma}$, and if H is nonincreasing, then $H \in RV_{-\gamma}$.

Similar to Corollary 2.3, we have the following result.

Corollary 3.1. *Suppose that $H \in RV_{-\gamma}$. Then*

(i) Suppose that $\gamma > 1$ and α so that $\alpha(\gamma - 1) > 1$. We have

$$\lim_{x\to\infty} \frac{H(x)\int_x^\infty \left(\int_y^\infty H(z)dz\right)^\alpha dy}{\left(\int_x^\infty H(z)dz\right)^{1+\alpha}} = \frac{\gamma - 1}{\alpha(\gamma - 1) - 1}.$$

(ii) Suppose α, q are so that $q\gamma > 1$, $\alpha(\gamma q - 1) > 1$. We have

$$\lim_{x\to\infty} \frac{H^q(x)\int_x^\infty \left(\int_y^\infty H^q(z)dz\right)^\alpha dy}{\left(\int_x^\infty H^q(z)dz\right)^{1+\alpha}} = \frac{q\gamma - 1}{\alpha(q\gamma - 1) - 1}.$$

(iii) For all $\alpha > \beta > 0$, $\alpha(\gamma - p) > 1$, $\beta(\gamma - p) > 1$, $p \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{x^{p\beta}H^\beta \int_x^\infty z^{p\alpha}H^\alpha(z)dz}{x^{p\alpha}H^\alpha(x)\int_x^\infty z^{p\beta}H^\beta(z)dz} = \frac{\beta(\gamma - p) - 1}{\alpha(\gamma - p) - 1}.$$

Proof. (i) This is a simple consequence of Karamata's Theorem.

(ii) This follows from (i) since $H^q \in RV_{-q\gamma}$.

(iii) If $H \in RV_{-\gamma}$, we have $x^{p\alpha}H^\alpha(x) \in RV_{-\alpha(\gamma-p)}$ and

$$\int_x^\infty z^{p\alpha}H^\alpha(z)dz \sim \frac{x^{p\alpha+1}H^\alpha(x)}{\alpha(\gamma-p)-1}.$$

In a similar way we have

$$\int_x^\infty z^{p\beta}H^\beta(z)dz \sim \frac{x^{p\beta+1}H^\beta(x)}{\beta(\gamma-p)-1}.$$

The result follows. □

We also have the converse results. In the next result we assume that $H(x) = \bar{F}(x)$, the tail distribution.

Theorem 3.2. *(i) Suppose that all integrals exist and that*

$$\lim_{x\to\infty} \frac{H(x)\int_x^\infty \left(\int_y^\infty H(z)dz\right)^\alpha dy}{\left(\int_x^\infty H(z)dz\right)^{1+\alpha}} = \delta,$$

where $a\delta > 1$. Then $H \in RV_{-\gamma}$ where $\gamma = \delta/(\alpha\delta - 1) + 1$.

(ii) Suppose that all integrals exist and that for some $\alpha > \beta > 0$, $p \in \mathbb{R}$ we have

$$\lim_{x\to\infty} \frac{x^{p\beta}H^\beta(x)\int_x^\infty z^{p\alpha}H^\alpha(z)dz}{x^{p\alpha}H^\alpha(x)\int_x^\infty z^{p\beta}H^\beta(z)dz} = \delta,$$

where $\delta < \beta/\alpha$. Then $H \in RV_{-\gamma}$, where $\gamma = p + (1-\delta)/(\beta - \delta\alpha)$.

Proof. (i) Define $A(x)$ and $R(x)$, $h(x)$ as follows:

$$A(x) = \int_x^\infty H(z)dz, \quad R(x) = \frac{\int_x^\infty A^\alpha(y)dy}{A^\alpha(x)}, \quad h(x) = \frac{H(x)R(x)}{A(x)}.$$

In the case of (i), we have that $h(x) \to \delta$. Taking the derivative of $R(x)$, we have

$$\begin{aligned} R'(x) &= \frac{-A^{2\alpha}(x) + \alpha\left(\int_x^\infty A^\alpha(y)dy\right)A^{\alpha-1}(x)H(x)}{A^{2\alpha}(x)} \\ &= -1 + \alpha h(x). \end{aligned}$$

Since $h(x) \to \delta$, we have $R'(x) \to -1 + \alpha\delta$ and then we obtain that $R(x)/x \to \alpha\delta - 1$. It follows that

$$\frac{x}{R(x)} \to \frac{1}{\alpha\delta - 1}$$

and then

$$\frac{xH(x)}{A(x)} \to \frac{\delta}{\alpha\delta - 1}.$$

Karamata's theorem shows that $H \in RV_{-\gamma}$ with $\gamma = \delta/(\alpha\delta - 1) + 1$. Note that $\delta = (\gamma - 1)/(\alpha(\gamma - 1) - 1)$.

(ii) We proceed as in the proof of Theorem 2.5. Define functions $r(x)$ and $R(x)$ as follows:

$$r(x) = \frac{x^{p\beta}H^\beta(x)\int_x^\infty z^{p\alpha}H^\alpha(z)dz}{x^{p\alpha}H^\alpha(x)\int_x^\infty z^{p\beta}H^\beta(z)dz}$$

and

$$R(x) = \frac{\int_x^\infty z^{p\alpha} H^\alpha(z)dz}{\int_x^\infty z^{p\beta} H^\beta(z)dz}.$$

By assumption we have $r(x) \to \delta$ and $R(x) \sim \delta x^{p(\alpha-\beta)} H^{\alpha-\beta}(x)$. As in Theorem 2.5, we find

$$R'(x) = \frac{x^{p\alpha} H^\alpha(x)}{\int_x^\infty z^{p\beta} H^\beta(z)dz}(r(x) - 1)$$

and

$$\frac{R'(x)}{R(x)} = -(1 - r(x))\frac{x^{p\alpha} H^\alpha(x)}{\int_x^\infty z^{p\alpha} H^\alpha(z)dz} = -\frac{a(x)}{b(x)},$$

where

$$a(x) = (1 - r(x))\frac{x^{p\alpha} H^\alpha(x)}{R^{\alpha/(\alpha-\beta)}(x)}$$

and

$$b(x) = R^{-\beta/(\alpha-\beta)}(x)\int_x^\infty z^{p\beta} H^\beta(z)dz.$$

First consider $a(x)$. Using $r(x) \to \delta$ and $R(x) \sim \delta x^{p(\alpha-\beta)} H^{\alpha-\beta}(x)$, we obtain that

$$a(x) \to (1 - \delta)\delta^{-\alpha/(\alpha-\beta)}.$$

For $b(x)$ we find

$$\begin{aligned} -b'(x) &= \frac{\beta}{\alpha-\beta} R^{-\alpha/(\alpha-\beta)}(x)R'(x)\int_x^\infty z^{p\beta} H^\beta(z)dz + R^{-\beta/(\alpha-\beta)}(x)x^{p\beta} H^\beta(x) \\ &= I + II. \end{aligned}$$

First consider I. Using the expression for $R'(x)$ and then using $R(x) \sim \delta x^{p(\alpha-\beta)} H^{\alpha-\beta}(x)$, we have

$$\begin{aligned} I &= \frac{\beta}{\alpha-\beta} R^{-\alpha/(\alpha-\beta)}(x)x^{p\alpha} H^\alpha(x)(r(x) - 1) \\ &\to \frac{\beta}{\alpha-\beta}\delta^{-\alpha/(\alpha-\beta)}(\delta - 1). \end{aligned}$$

For II we find

$$II \to \delta^{-\beta/(\alpha-\beta)}.$$

It follows that

$$-b'(x) \to \frac{\beta}{\alpha-\beta}\delta^{-\alpha/(\alpha-\beta)}(\delta - 1) + \delta^{-\beta/(\alpha-\beta)} = d.$$

Using $\alpha/(\alpha - \beta) - \beta/(\alpha - \beta) = 1$, we have

$$\begin{aligned} d &= \frac{\beta}{\alpha-\beta}\delta^{-\alpha/(\alpha-\beta)}(\delta - 1) + \delta\delta^{-\alpha/(\alpha-\beta)} \\ &= \delta^{-\alpha/(\alpha-\beta)}\left(\delta - (1 - \delta)\frac{\beta}{\alpha-\beta}\right) \\ &= \delta^{-\alpha/(\alpha-\beta)}\frac{\delta\alpha - \beta}{\alpha-\beta}. \end{aligned}$$

It follows that

$$-\frac{b(x)}{x} \to \delta^{-\alpha/(\alpha-\beta)} \frac{\delta\alpha - \beta}{\alpha - \beta}$$

and as a consequence also that

$$\frac{xR'(x)}{R(x)} = -\frac{xa(x)}{b(x)} \to \frac{(1-\delta)(\alpha-\beta)}{\delta\alpha - \beta}.$$

It follows that $R \in RV_{-\gamma}$, where

$$\gamma = \frac{(1-\delta)(\alpha-\beta)}{\beta - \delta\alpha}.$$

Since $R(x) \sim \delta x^{p(\alpha-\beta)} H^{\alpha-\beta}(x)$, we obtain that $H \in RV_{-\gamma}$, where

$$\gamma = p + \frac{1-\delta}{\beta - \delta\alpha}.$$

Note that the last expression implies that

$$\delta = \frac{\beta(\gamma - p) - 1}{\alpha(\gamma - p) - 1}.$$

The proof is complete. □

Remark 3.3. *As we mentioned that $\overline{H} \in D(\Psi_\alpha)$ iff $x_0 < \infty$ and $H(x_0 - x^{-1}) \in RV_{-\alpha}$. By Corollary 3.1 and Theorem 3.2, we can derive corresponding results for $\overline{H} \in D(\Psi_\alpha)$.*

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