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Optimality Conditions for Invex Interval Valued Nonlinear Programming Problems Involving Generalized *H*-Derivative

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Abstract. In this paper, some interval valued programming problems are discussed. The solution concepts are adopted from Wu [7] and Chalco-Cano et al. [34]. By considering generalized Hukuhara differentiability and generalized convexity (viz. η-preinvexity, η-invexity etc.) of interval valued functions, the KKT optimality conditions for obtaining (*LS* and *LU*) optimal solutions are elicited by introducing Lagrangian multipliers. Our results generalize the results of Wu [7], Zhang et al. [11] and Chalco-Cano et al. [34]. To illustrate our theorems suitable examples are also provided.

1. Introduction

Among many types of methodologies usually used to solve optimization models, the interval valued optimization problems have been of much interest in recent past and thus explored the extent of optimality conditions and duality applicability in different areas. Consequently, the parameters of optimization problems like differentiability, convexity have been generalized in different directions by many scientists in order to widen the application domain of interval valued optimization problems. Various generalizations of convex functions can be seen in Hanson [19], Vial [15], Hanson and Mond [20], Jeyakumar and Mond [32], Hanson et al. [21], Liang et al. [35], Gulati et al. [29], Zalmai [5], Antczak [28], Mandal and Nahak [23], Ahmad [9]. The extension of some of these to interval valued functions can be seen in Moore [24], Ishibuchi and Tanaka [6], Wu [28], Bhurjee and Panda [1], Zhang et al. [11], Cahlco-Cano et al. [34], Li et al. [16]. Also for various types of differentiability of interval valued functions one is referred to Hukuhara [18], Banks and Jacobs [8], De Blasi [4], Aubin and Cellina [12], Aubin and Franskowska [13], [14], Ibrahim [2], Cahlco-Cano et al. [33].

In particular for interval valued optimization problems, Wu [7] proposed the concept of *LU*, *UC* convexity and *LU*, *UC* pseudoconvexity, Chalco-Cano et al. [34] have given the concept of *LS*-convexity, and has

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derived KKT optimality conditions using H-differentiability and gH-differentiability respectively. Zhang et al. [11] extended the concept of preinvexity, invexity, pseudo-invexity, quasi-invexity to interval valued functions and has studied for KKT conditions under the assumption of H-differentiability. Moreover, the relation between interval valued optimization and variational like inequalities have been explored there in. Ahmad et al. [10] used the concept of $(p, r) - \rho - (\eta, \theta)$-invexity to study sufficient optimality conditions and duality theorems of Wolfe and Mond-Wier type duals of interval valued optimization problems. More recently, Li et al. [16] defined interval valued univex function and studied the KKT conditions and duality theorems under the assumption of gH-differentiability of interval valued functions. In this paper, we are interested in interval valued programming problems and we study KKT conditions under the assumptions of η-preinvexity, η-invexity and gH-differentiability. The paper is structured as:

In section 2, we provide some arithmetic of intervals and then give the concept of gH-differentiability of interval valued functions. Section 3 deals some solution concepts following from Wu [7] and Chalco-Cano et al. [34]. Further in section 4, we propose the concept of invexity of interval valued functions in both LU and LS sense and study its properties. Finally, in section 5, we derive KKT optimal conditions for invex interval valued programming problems involving gH-differentiability. Moreover by using the gradient of interval valued functions the same KKT conditions are discussed. To illustrate our theorems suitable examples are also provided. We conclude in section 6.

2. Preliminaries

2.1. Arithmetic of intervals

Let $\mathcal{I}_c$ denote the class of all closed and bounded intervals in R. i.e.,

$$\mathcal{I}_c = \{[a, b] : a, b \in R \ \ and \ \ a \leq b\}.$$

And $b - a$ is the width of the interval $[a, b] \in \mathcal{I}_c$. Then for $A \in \mathcal{I}_c$ we adopt the notation $A = [a^L, a^U]$, where a^L and a^U are respectively the lower and upper bounds of A. Let $A = [a^L, a^U], B = [b^L, b^U] \in \mathcal{I}_c$ and $\lambda \in R$. Then we have the following operations.

(i)

$$A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$$

(ii)

$$\lambda A = \lambda[a^L, a^U] = \begin{cases} [\lambda a^L, \lambda a^U] \ if \ \lambda \geq 0, \\ [\lambda a^U, \lambda a^L] \ if \ \lambda < 0. \end{cases}$$

In view of (i) and (ii) we see that

$$-B = -[b^L, b^U] = [-b^U, -b^L] \text{ and } A - B = A + (-B) = [a^L - b^L, a^U - b^L].$$

Also the real number $a \in R$ can be regarded as a closed interval $A_a = [a, a]$, then we have for $B \in \mathcal{I}_c$

$$a + B = A_a + B = [a + b^L, a + b^U].$$

Note that the space $\mathcal{I}_c$ is not a linear space with respect to the operations (i) and (ii), since it does not contain inverse elements (see, Assev [26], Aubin and Cellina [12]).

Further the generalized Hukuhara difference(gH-difference) of intervals A and B introduced in Stefanini and Bede [17] is defined as follows

$$A \ominus_g B = C \Leftrightarrow \begin{cases} \quad (i) A = B + C, \\ or \quad (ii) B = A + (-1)C. \end{cases}$$

The advantage of this definition is that the gH-difference of two intervals $A = [a, b]$ and $B = [c, d]$, always exists and is equal to

$$A \ominus_g B = [\min\{a - c, b - d\}, \max\{a - c, b - d\}].$$

Note that the gH-difference of two intervals is generalization of their H-difference whenever it exists.

2.2. Differentiation of interval valued functions.

Let X be a nonempty subset of R^n. A function $f : X \to \mathcal{I}_c$ is called an interval valued function. In this case we have

$$f(x) = [f^L(x), f^U(x)], \tag{2.1}$$

such that

$$f^L, f^U : X \to R, \tag{2.2}$$

satisfying $f^L(x) \leq f^U(x)$, for all $x \in X$.

Wu[7] introduced a straight forward concept of differentiability of interval valued functions as follows.

Definition 2.1. *[7] (2.1) is said to be weekly continuously differentiable at $x^* \in X$ if (2.2) is continuously differentiable at x^* (in usual sense).*

Further based on H-difference of two intervals, the H-derivative of interval valued functions was introduced by Hukuhara [18]. This definition of differentiability was used by Wu [7] in order to investigate optimization problems with interval valued objective functions. The same definition is further used by many other authors. However both the above derivatives have some limitations. For example consider a simple interval valued function $f : R \to \mathcal{I}_c$ defined by

$$f(x) = [-1, 1]|x|. \tag{2.3}$$

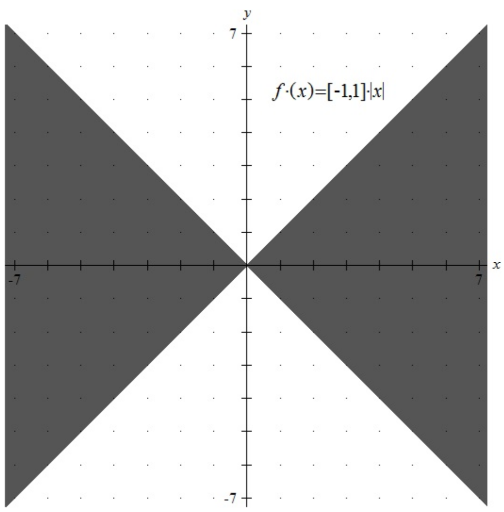

Figure 1: $f(x)$

The behavior of $f(x)$ can be seen in fig. 1. Since weakly continuously differentiability is established with respect to the differentiation of end point functions, therefore (2.1) is not weakly continuously differentiable. Also if $f : (a, b) \to \mathcal{I}_c$ defined by

$$f(x) = [-1, 1]x^2.$$

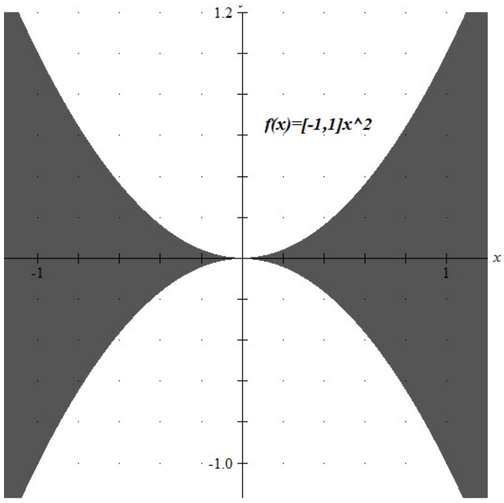

Figure 2: $f(x)$

The behavior of $f(x)$ can be seen in fig. 2. Then it is easy to see that f is not H-differentiable at $x = 0$. In fact in Bede and Gal [3], it has been shown that the function $f(x) = Ph(x), x \in (a, b)$, where $P \in \mathcal{I}_c$ and $h : (a, b) \to R^+$ with $h'(x^*) < 0$ is not H-differentiable at x^*.

Definition 2.2. *[17] Let $x^* \in (a, b)$. Then the gH-derivative of an interval valued function $f : (a, b) \to \mathcal{I}_c$ is*

$$f'(x^*) = \lim_{h \to 0} \frac{f(x^* + h) \ominus_g f(x^*)}{h},$$

provided $f'(x^*)$ exists in $\mathcal{I}_c$.

Remark 2.3. *We remark that the gH-differentiability of interval valued functions has the advantage to overcome the weakness of weak differentiability and H-differentiability. In particular*

(a) The function (2.3) is continuously gH-differentiable at $x^* = 0$ and $f'(x^*) = [-1, 1]$ for all $x \in (a, b)$.

(b) Since H-difference of $A = [a^L, a^U]$ and $B = [b^L, b^U]$ exists if $a^L - b^L \leq a^U - b^U$ [7], but gH-difference of A and B always exists and is generalization of H-difference provided it exists [17].

Therefore we can say that gH-differentiability of interval valued functions is preferable over weak and H-differentiability. Moreover the π-differentiability [33] and gH-differentiability coincide.

Theorem 2.4. *[33] Let $f : (a, b) \to \mathcal{I}_c$ such that f^L and f^U are differentiable at $x^* \in (a, b)$. Then f is gH-differentiable at x^* and*

$$f'(x^*) = \left[\min\left\{(f^L)'(x^*), (f^U)'(x^*)\right\}, \max\left\{(f^L)'(x^*), (f^U)'(x^*)\right\}\right].$$

The converse of above theorem is not true (see, Chalco-Cano et al. [33]). However we have the following result.

Theorem 2.5. *[33] Let $f : (a, b) \to \mathcal{I}_c$. Then f is gH-differentiable at $x^* \in (a, b)$ iff one of the following cases holds.*

(a) f^L and f^U are differentiable at x^*.

(b) The derivatives $(f^L)'_-(x^*)$, $(f^L)'_+(x^*)$, $(f^U)'_-(x^*)$ and $(f^U)'_+(x^*)$ exists and satisfy

$$(f^L)'_-(x^*) = (f^U)'_+(x^*) \text{ and } (f^L)'_+(x^*) = (f^U)'_-(x^*).$$

Proposition 2.6. *[34] Let $f : (a, b) \to \mathcal{I}_c$ be gH-differentiable at $x^* \in (a, b)$. Then $f^L + f^U$ is a differentiable function at x^*.*

Next Wu [7] proposed Hausdorff metric between two closed intervals A and B as follows

$$H(A, B) = \max\{|a^L - b^L|, |a^U - b^U|\}.$$

It is clear that $(\mathcal{I}_c, H)$ is a metric space. Therefore an interval valued function f defined on $X \subseteq R^n$ is continuous at x^* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|x - x^*\| > \delta$ implies $H(f(x), f(x^*)) < \epsilon$.

Proposition 2.7. *[12] Let $x^* \in X$. Then (2.1) is continuous at x^* iff (2.2) are continuous at x^*.*

Definition 2.8. *[34] Let $x^* = (x_1^*, x_2^*, ..., x_n^*)$ be fixed in X.*

(1) We consider the interval valued function $h_i(x_i) = f\left(x_1^*, x_2^*, ..., x_{i-1}^*, x_i^*, x_{i+1}^*, ..., x_n^*\right)$. If h_i is gH-differentiable at x_i^*, then we say that f has the i^{th} partial gH-derivative at x^* $\left(\text{denoted by } \left(\frac{\partial f}{\partial x_i}\right)_g (x^*)\right)$ and $\left(\frac{\partial f}{\partial x_i}\right)_g (x^*) = (h_i)'(x_i^*)$.

(2) We say that (2.1) is continuously gH-differentiable at x^* if all the partial gH-derivatives $\left(\frac{\partial f}{\partial x_i}\right)(x^*)$, $i = 1, 2, ..., n$ exists on some neighborhood of x^* and are continuous at x^* (in the sense of interval valued function).

Proposition 2.9. *[34] If (2.1) is continuously gH-differentiable at $x^* \in X$. Then $f^L + f^U$ is continuously differentiable at x^*.*

3. Solution concept

Consider the following interval valued optimization problem:

(IVP1)

$$\min f(x) = [f^L(x), f^U(x)],$$

subject to $x \in X \subseteq R^n$.

Since f is closed interval in R i.e., $f(x) \in \mathcal{I}_c, x \in X$, we follow the similar solution concept proposed in [7]. A partial ordering $\precsim_{LU}$ was invoked between two closed intervals in [7] as follows.

Let $A, B \in \mathcal{I}_c$. Then we say that $A \precsim_{LU} B$ iff $a^L \leq b^L$ and $a^U \leq b^U$, and $A \prec_{LU} B$ iff $A \precsim_{LU} B$ and $A \neq B$, or $A \prec_{LU} B$ iff one of the following conditions hold.

(a1) $a^L \leq b^L$ and $a^U < b^U$,

(a2) $a^L < b^L$ and $a^U < b^U$,

(a3) $a^L < b^L$ and $a^U \leq b^U$.

Definition 3.1. *[7] We say that $x^* \in X$ is an LU-solution of (IVP1) if there exists no $\hat{x} \in X$ such that $f(\hat{x}) \prec_{LU} f(x^*)$.*

Next we follow another solution concept introduced in [34].

Let $A \in \mathcal{I}_c$. Then the width (spread) of A is defined by $w(A) = a^S = a^U - a^L$. Let $A, B \in \mathcal{I}_c$, Chalco-Cano et al. [34] proposed the ordering relation between A and B by considering the minimization and maximization problem separately.

(i) For maximization, we write. $A \succsim_{LS} B$ iff $a^U \geq b^U$ and $a^S \leq b^S$ the width of interval can be regarded as uncertainty (noise, risk or a type variance). Therefore, the interval with smaller width (i.e., the uncertainty) and large upper bound is considered better).

(ii) For minimization, we write $A \precsim_{LS} B$ iff $a^L \leq b^L$ and $a^S \leq b^S$. In this case, the interval with smaller width (i.e., uncertainty) and smaller lower bound is considered better.

In this paper we shall consider only the minimization problem without loss of generality. In this sense, we write $A \precsim_{LS} B$ iff $a^L \leq b^L$ and $a^S \leq b^S$, and $A \prec_{LS} B$ iff $A \precsim_{LS} B$ and $A \neq B$, or $A \prec_{LS} B$ iff one of the following conditions hold.

(b1) $a^L \leq b^L$ and $a^S < b^S$,

(b2) $a^L < b^L$ and $a^S < b^S$,

(b3) $a^L < b^L$ and $a^S \leq b^S$.

Definition 3.2. *[34] We say that $x^* \in X$ is an LS-solution of (IVP1) if there exists no $\hat{x} \in X$ such that $f(\hat{x}) \prec_{LS} f(x^*)$.*

Proposition 3.3. *[34] Let A, B be two intervals in $\mathcal{I}_c$. If $A \precsim_{LS} B$, then $A \precsim_{LU} B$.*

Note that the converse of Proposition 3.3 is not valid.

Theorem 3.4. *[34] If $x^* \in X$ is a LU-solution of (IVP1), then x^* is an LS-solution of (IVP1) but not conversely.*

4. Concept of invexity of interval valued functions

Convexity plays key role in optimization theory (e.g., see, Bazaraa et al. [22]) and has been generalized in several directions. Weir and Mond [30] and Weir and Jeyakumar [31] introduced an important generalization of convex functions namely preinvex functions.

Definition 4.1. *([30], [31]) A set $K \subseteq R^n$ is said to be invex if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that $x, y \in K$, $\lambda \in [0, 1]$ implies $y + \lambda\eta(x, y) \in K$. We also say that K is an η-invex set.*

Next, consider the real valued function f, then for the definitions of preinvex, invex, pseudo-invex, quasi-invex, one is referred to ([19], [25], [30], [31]). Also for (2.1), the definition of LU-convexity and LS-convexity one is referred to [7] and [34] respectively. Further in the rest of this paper we shall denote by $\mathcal{I}_k$ the class of interval valued functions defined on η-invex set K.

Now Zhang et al. [11] extended the concepts of preinvexity, invexity, pseudo-invexity and quasi-invexity to interval valued functions in LU-sense as follows.

Definition 4.2. *[11] Let $f \in \mathcal{I}_k$. Then we say that f is*

(i) *LU-preinvex at x^* with respect to η if $f(x + \lambda\eta(x^*, x)) \precsim_{LU} \lambda f(x^*) + (1 - \lambda)f(x)$, for every $\lambda \in [0, 1]$ and each $x \in K$. We also say that is f is $LU - \eta$-preinvex function at x^*.*
(ii) *invex (η-invex) at x^* if the real valued functions f^L and f^U are η-invex at x^*. In this case we also say that f is $LU - \eta$-invex function at x^*.*
(iii) *pseudo-invex at x^* if the real valued functions f^L, f^U and $\lambda^L f^L + \lambda^U f^U$ are η-pseudo-invex at x^*, where $0 < \lambda^L, \lambda^U \in R$. In this case we also say that is f is $LU - \eta$-pseudo-invex function at x^*.*
(iv) *quasi-invex at x^* if the real valued functions f^L, f^U and $\lambda^L f^L + \lambda^U f^U$ are η-quasi-invex at x^*, where $0 < \lambda^L, \lambda^U \in R$. In this case we also say that is f is $LU - \eta$-quasi-invex function at x^*.*

Further we extend the above concepts to interval valued functions in LS-sense as follows.

Definition 4.3. *Let $f \in \mathcal{I}_k$. Then we say that f is*

(i) *$LS - \eta$-preinvex at x^* if $f(x + \lambda\eta(x^*, x)) \precsim_{LS} \lambda f(x^*) + (1 - \lambda)f(x)$, for every $\lambda \in [0, 1]$ and each $x \in K$.*
(ii) *$LS - \eta$-invex at x^* if the real valued functions f^L and f^S are η-invex at x^*.*

(iii) *$LS-\eta$-pseudo-invex at x^* if the real valued functions f^L, f^S and $\lambda^L f^L + \lambda^S f^S$ are η-pseudo-invex at x^*, where $0 < \lambda^L, \lambda^S \in R$.*

(iv) *$LS-\eta$-quasi-invex at x^* if the real valued functions f^L, f^S and $\lambda^L f^L + \lambda^U f^S$ are η-quasi-invex at x^*, where $0 < \lambda^L, \lambda^S \in R$.*

Proposition 4.4. *Let $f \in \mathcal{I}_k$ be an interval valued function defined on convex set $X \subseteq R^n$ and $x^* \in X$. Then the following statements hold true.*

(i) *f is $LU-\eta$-preinvex at x^* iff f^L and f^U are η-preinvex at x^* [11].*

(ii) *f is $LS-\eta$-preinvex at x^* iff f^L and f^S are η-preinvex at x^*.*

(iii) *If f is $LS-\eta$-preinvex at x^*. Then f is $LU-\eta$-preinvex at x^*.*

Proof. (ii) follows from Definition 4.3 immediately and (iii) is the consequence of Proposition 3.3. □

Proposition 4.5. *Let $f \in \mathcal{I}_k$ be $LS-\eta$-preinvex function. If x^* is unique LS-minimizer of f. Then f is LS-convex at x^*.*

Proof. From Definition 4.3 and Definition 3.2 the result follows immediately. □

Remark 4.6. (i) *The class of LU-convex interval valued functions is strictly contained in the class of LU-preinvex interval valued functions if $\eta(x, y) = x - y$, $x, y \in X$ [11].*

(ii) *The class of LS-convex interval valued functions is strictly contained in the class of LS-preinvex interval valued functions if $\eta(x, y) = x - y$, $x, y \in X$*

The converse of Remark 4.6 is not true as shown in following example.

Example 4.7. *For the converse of Remark 4.6 (i) Zhang et al. [11] have shown that the interval valued function $f(x) = -[1, 2]|x|$, $x \in R$ is not LU-convex. The behavior of $f(x)$ can be seen in fig. 3.*

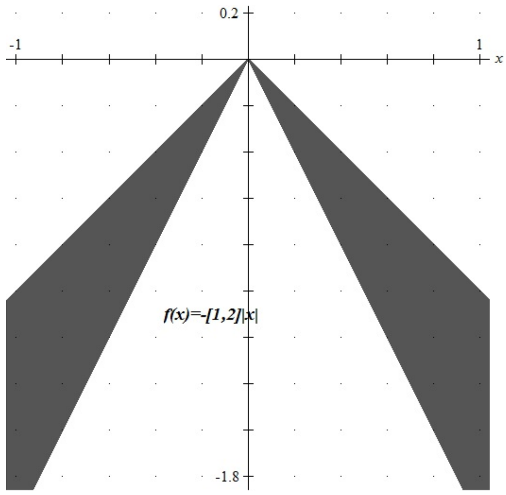

Figure 3: $f(x)$

However $f(x)$ is $LU-\eta$-preinvex, where η is given by

$$\eta(x, y) = \begin{cases} x - y \; if \; x, y \geq 0, \; or \; x, y \leq 0, \\ y - x, otherwise. \end{cases}$$

Now for the converse of Remark 4.6 (ii) we use the fact that if is LS-convex then f is LU-convex [34]. Since f is not LU-convex therefore f is not LS-convex.

Now we show f is $LS-\eta$-preinvex. From above we have $f^L(x) = -2|x|, f^U(x) = -|x|$, therefore $f^S(x) = |x|$, $x \in R$. Let $x, y \geq 0$ and $\lambda \in [0,1]$. Then we have

$$f^S(y + \lambda\eta(x,y)) = |y + \lambda\eta(x,y)| \leq \lambda x + (1-\lambda)y = \lambda f^S(x) + (1-\lambda)f^S(y).$$

For $x, y \leq 0$ the result follows similarly. Now let $x < 0$, $y > 0$ and $\lambda \in [0,1]$. Then we must have

$$f^S(y + \lambda\eta(x,y)) \leq \lambda f^S(x) + (1-\lambda)f^S(y).$$

For the case $x > 0$, $y < 0$ the similar argument holds. Therefore from Definition 4.3 (i), f is $LS-\eta$-preinvex.

Proposition 4.8. *Let $f, g \in \mathcal{I}_k$ be*

(i) *$LU-\eta$-preinvex functions. Then $kf, k > 0$ and $f+g$ are also $LU-\eta$-preinvex functions [11].*

(ii) *$LS-\eta$-preinvex functions. Then $kf, k > 0$ and $f+g$ are also $LS-\eta$-preinvex functions.*

Proof. (ii) Let f be $LS-\eta$-preinvex function. Then it is easy to see that $kf, k > 0$ is also $LS-\eta$-preinvex function by Proposition 4.4 (ii). Now let f, g be $LS-\eta$-preinvex functions. Then by Definition 4.3 (i) we have for $x, y \in K$ and $\lambda \in [0,1]$.

$$f(y + \lambda\eta(x,y)) \preceq_{LS} \lambda f(x) + (1-\lambda)f(y),$$

and

$$g(y + \lambda\eta(x,y)) \preceq_{LS} \lambda g(x) + (1-\lambda)g(y).$$

Therefore we have

$$(f+g)(y + \lambda\eta(x,y)) \preceq_{LS} \lambda(f+g)(x) + (1-\lambda)(f+g)(y).$$

Therefore by Definition 4.3 we see that $(f+g)$ is $LS-\eta$-preinvex function. □

Proposition 4.9. *Let $f, g \in \mathcal{I}_k$ be*

(i) weakly continuously differentiable and $LU-\eta$-preinvex function. Then f is $LU-\eta$-invex function but not conversely [11].

(ii) weakly continuously differentiable and $LS-\eta$-preinvex function, Then f is $LS-\eta$-invex function but not conversely.

(iii) continuously gH-differentiable and $LU-\eta$-preinvex function, Then $f^L + f^U$ is also η-invex function.

Proof. (ii) By using Proposition 4.4 and Definition 2.1 we see that f^L and f^S are η-preinvex and differentiable functions and hence are η-invex (since a differentiable preinvex real valued function is invex with respect to the same η [25]).

Further consider the interval valued function $f(x) = [a,b]e^x$, $a, b, x \in R$, $b < 0$, then for $\eta(x,y) = 1$, it is easy to see that f is $LS-\eta$-invex but not $LS-\eta$-preinvex.

(iii) Since f is $LU-\eta$-preinvex, then by Proposition 4.4 (i), f^L and f^U are η-preinvex. Therefore we have

$$f^L(x + \lambda\eta(x^*,x)) \leq \lambda f^L(x^*) + (1-\lambda)f^L(x),$$

and

$$f^U(x + \lambda\eta(x^*, x)) \leq \lambda f^U(x^*) + (1 - \lambda)f^U(x).$$

This gives

$$(f^L + f^U)(x + \lambda\eta(x^*, x)) \leq \lambda(f^L + f^U)(x^*) + (1 - \lambda)(f^L + f^U)(x).$$

This implies that $f^L + f^U$ is also η-preinvex. Therefore $f^L + f^U$ is η-preinvex and differentiable real valued function hence $f^L + f^U$ is η-invex function (since a differentiable preinvex real valued function is invex with respect to the same η [25]).
□

Condition C. [27] Consider the vector valued function $\eta : R^n \times R^n \to R^n$. Then the function η satisfy the condition C for $x, y \in R^n$ and $\lambda \in [0, 1]$ if

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y),$$

$$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

Proposition 4.10. *Let $f \in \mathcal{I}_k$ be continuously gH-differentiable and $LU - \eta$-invex function such that η satisfy condition C. Then f is also $LU - \eta$-preinvex function.*

Proof. Since K is η-invex set then for $x, y \in K$ we have $z = y + \lambda\eta(x, y) \in K$. Also since $f$is $LU - \eta$-invex, by Definition 4.2 (ii) f^L and f^U are η-invex. Therefore we have for x, z

$$(a1) \quad f^L(x) - f^L(z) \geq \eta^T(x, z)\nabla f^L(z),$$

$$(b1) \quad f^U(x) - f^U(z) \geq \eta^T(x, z)\nabla f^U(z),$$

and for y, z

$$(a2) \quad f^L(y) - f^L(z) \geq \eta^T(y, z)\nabla f^L(z),$$

$$(b2) \quad f^U(y) - f^U(z) \geq \eta^T(y, z)\nabla f^U(z).$$

Therefore from (a1), (a2) and (b1), (b2), we have

$$\lambda f^L(x) + (1 - \lambda)f^L(y) - f^L(z) \geq (\lambda\eta^T(x, z) + (1 - \lambda)\eta^T(y, z))\nabla f^L(z),$$

and

$$\lambda f^U(x) + (1 - \lambda)f^U(y) - f^U(z) \geq (\lambda\eta^T(x, z) + (1 - \lambda)\eta^T(y, z))\nabla f^U(z).$$

Now by applying condition C we see that

$$\lambda\eta^T(x, z) + (1 - \lambda)\eta^T(y, z) = 0.$$

Therefore we have

$$f^L(y + \lambda\eta(x, y)) \leq \lambda f^L(x) + (1 - \lambda)f^L(y),$$

and

$$f^U(y + \lambda\eta(x, y)) \leq \lambda f^U(x) + (1 - \lambda)f^U(y).$$

This shows by definition that f^L and f^U are η-preinvex and hence by Proposition 4.4 (i) f is $LU - \eta$-preinvex. □

5. Optimality conditions of type KKT

Consider the following optimization problem.

(P)

$\min f(x) = f(x_1, x_2, ..., x_n),$

subject to $g_i(x) \leq 0, i = 1, 2, ..., m,$

where f and $g_i, i = 1, 2, ..., m$ are real valued functions. In [19], the following result has been obtained for problem (P).

Theorem 5.1. *[19] Let $K \subseteq R^n$ be η-invex set, $x^* \in K$ and one of the following conditions is satisfied:*

(a) $f(x)$ and $g_i(x), i = 1, 2, ..., m$ are η-invex at x^*;

(b) $f(x)$ is η-pseudo-invex at x^* and $g_i(x), i = 1, 2, ..., m$ are η-quasi-invex at x^*;

(c) $f(x)$ is η-pseudo-invex at x^* and $(\mu^*)^T g(x)$ is η-quasi-invex at x^*;

(d) The Lagrangian function $f(x) + (\mu^*)^T g_i(x)$ is η-pseudo-invex at x^* with respect to an arbitrary η (i.e., the Lagrangian function $f(x) + (\mu^*)^T g_i(x)$ is η-invex at x^*).

If there exists $0 \leq \mu^* \in R^m$, such that (x^*, μ^*) satisfies the following conditions.

(i) $\nabla f(x^*) + \sum_{i=1}^{m} \mu_i^* \nabla g_i(x^*) = 0;$

(ii) $\mu_i^* \nabla g_i(x^*) = 0, i = 1, 2, ..., m.$

Then x^* solves problem (P).

Next in this section, we present some KKT conditions for the problem (IVP1), which are obtained by using gH-differentiability of interval valued functions. For this we consider (IVP1) with the feasible set $X = \{x \in R^n : g_i(x) \leq 0, i = 1, 2, ..., m\}$. That is, we consider the following problem,

(IVP2)

$\min f(x) = [f^L(x), f^U(x)],$

subject to $g_i(x) \leq 0, i = 1, 2, ..., m.$

Remark 5.2. *For the problem (IVP2), the KKT conditions are obtained in*

(i) *[7], if objective function f is LU-convex and continuously weakly differentiable. Also the real valued constraint functions g_i are assumed to be convex and continuously differentiable for $i = 1, 2, ..., m$.*

(ii) *[11], if objective function f is $LU - \eta$-preinvex and weakly continuously differentiable and the constraint functions $g_i, i = 1, 2, ..., m$ are η-invex.*

(iii) *[34], if objective function f is LU-convex and continuously gH-differentiable and the constraint functions $g_i, i = 1, 2, ..., m$ are convex and continuously differentiable.*

Now we shall present KKT conditions for the case of generalized convexity and generalized Hukuhara differentiability.

Theorem 5.3. *Let $f \in \mathcal{I}_k$ be continuously gH-differentiable, $LU - \eta$-preinvex and each $g_i, i = 1, 2, ..., m$ is continuously differentiable, η-invex functions at $x^* \in K$. If there exist (Lagrangian) multipliers $0 < \lambda \in R$ and $\mu^* \in R^m$ with $0 \leq \mu_i^* \in R$ for $i = 1, 2, ..., m$, such that (x^*, μ^*) satisfy the following KKT conditions;*

(i) $\lambda \nabla (f^L + f^U)(x^*) + \sum_{i=1}^{m} \mu^* \nabla g_i(x^*) = 0;$

(ii) $\mu_i^* g_i(x^*) = 0, i = 1, 2, ..., m.$

Then x^* is an optimal LU-solution and an optimal LS-solution of problem (IVP2).

Proof. We define a real valued function for $x \in K$

$$F(x) = \lambda(f^L + f^U)(x).$$

Since f is continuously gH-differentiable and $LU-\eta$-preinvex at x^*, then by Propositions 2.9 and 4.9 (iii) we see that $f^L + f^U$ is continuously differentiable and η-invex at x^*. Therefore we have

$$\nabla F(x^*) = \lambda\nabla(f^L + f^U)(x^*).$$

From above we have new conditions as follows:

(iii) $\nabla F(x^*) + \sum_{i=1}^{m} \mu^* \nabla g_i(x^*) = 0$;

(iv) $\mu_i^* g_i(x^*) = 0, i = 1, 2, ..., m.$

According to the Theorem 5.1 (a), x^* is an optimal solution of the function F subject to the same constraints of problem (IVP2). That is

$$F(x^*) \leq F(\hat{x}), \hat{x} \in X. \tag{5.1}$$

If possible suppose x^* is not optimal LU-solution of (IVP2). Then from Definition 3.1, there exists $\hat{x}(\neq x^*) \in X$ such that $f(\hat{x}) \prec_{LU} f(x^*)$. That is,

(a1) $f^L(\hat{x}) < f^L(x^*)$ and $f^U(\hat{x}) \leq f^U(x^*)$ or

(a2) $f^L(\hat{x}) \leq f^L(x^*)$ and $f^U(\hat{x}) < f^U(x^*)$ or

(a3) $f^L(\hat{x}) < f^L(x^*)$ and $f^U(\hat{x}) < f^U(x^*)$

is satisfied. Since $\lambda > 0$, we have from above three conditions $F(\hat{x}) \leq F(x^*)$, which contradicts (5.1). It shows that x^* is an optimal LU-solution of (IVP2). From Theorem 3.4, it can be shown that x^* is also an optimal LS-solution of (IVP2). □

The following example shows the advantages of Theorem 5.3.

Example 5.4. *Consider the following problem:*

$$\begin{aligned} &\min f(x),\\ &subject\ to\\ &x-1 \leq 0,\\ &-x-1 \leq 0. \end{aligned} \tag{5.2}$$

Consider $f(x) = [-1, 1]|x|$, then f is not LU-convex function (see fig. 4), therefore Theorem 4.2 of [7] cannot be employed (see [34]).

Now let us consider a function defined as

$$\eta(x, y) = x - y \quad if\ \ x, y \geq 0\ \ or\ \ x, y \leq 0. \tag{5.3}$$

Then f is $LU-\eta$-preinvex. However f is not weakly continuously differentiable at 0. Therefore Theorem (4.4) of [11] cannot be employed. But $f(x)$ is continuously gH-differentiable at 0 and the conditions of Theorem 5.3 are verified. Therefore we can say that solves problem (5.2).

Again consider $f(x) = [|x|, |x|+1]$, then clearly f is not LU-convex (see fig. 5) and hence $f^L + f^U$ is not convex.

Therefore Theorem 6 of [34] cannot be employed. But f is $LU-\eta$-preinvex, where η is given by (5.2). Also f is continuously gH-differentiable at 0 and the conditions of Theorem 5.3 are satisfied. Therefore 0 is the required solution.

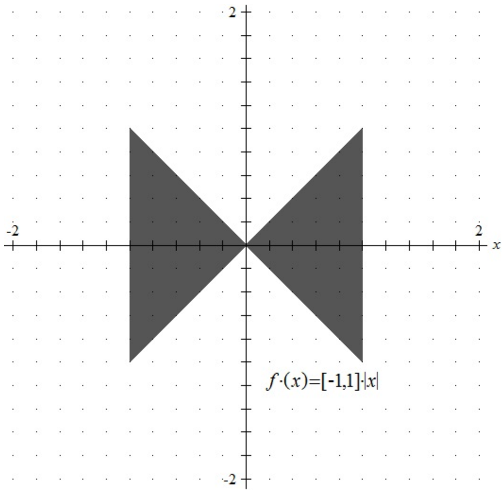

Figure 4: $f(x)$

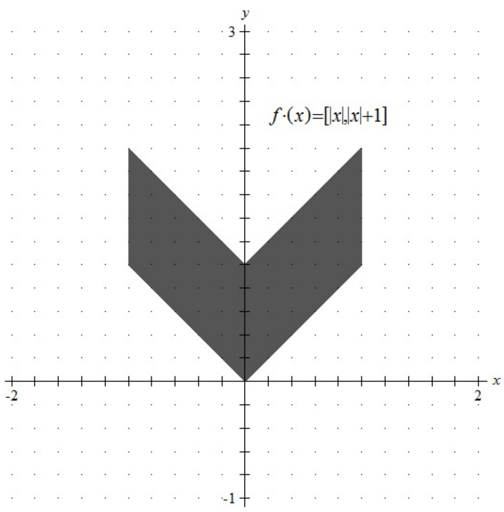

Figure 5: $f(x)$

Theorem 5.5. *Let $f \in \mathcal{I}_k$ be weakly continuously differentiable, $LS - \eta$-preinvex and each $g_i, i = 1, 2, ..., m$ is continuously differentiable, η-invex functions at $x^* \in K$. If there exist (Lagrangian) multipliers $0 < \lambda^L, \lambda^S \in R$ and $\mu^* = (\mu_1^*, ..., \mu_m^*)^T; 0 \leq \mu_i^* \in R_i, i = 1, 2, ..., m$, such that (x^*, μ^*) satisfy the following KKT conditions;*

(i) $\lambda^L \nabla f^L(x^*) + \lambda^S \nabla f^S(x^*) + \sum_{i=1}^m \mu_i \nabla g_i(x^*) = 0;$
(ii) $\mu_i^* \nabla g_i(x^*) = 0, i = 1, 2, ..., m.$

Then x^* is an optimal LS-solution of problem (IVP2).

Proof. We define a real valued function for $x \in R^n$

$$F(x) = \lambda^L f^L(x) + \lambda^S f^S(x).$$

Since f is weakly continuously differentiable at x^*, by Definition 2.1 , f^L and f^S are continuously differentiable at x^*. Also since f is LS-preinvex at x^*, then by Proposition 4.4 (ii) f^L and f^S are η-preinvex at x^*. Therefore f^L and f^U are η-preinvex and continuously differentiable at x^*. Therefore f^L and f^U are η-invex at x^*. We have

$$\nabla F(x^*) = \lambda^L \nabla f^L(x^*) + \lambda^S \nabla f^S(x^*).$$

From above we have,

(i) $\nabla F(x^*) + \sum_{i=1}^m \mu_i \nabla g_i(x^*) = 0$;
(ii) $\mu_i g_i(x^*) = 0, i = 1, 2, ..., m$.

By Theorem 5.1 (a) we conclude that x^* is an optimal solution of real valued objective function $F(x)$ subject to the same constraints of (IVP2). Now by using similar arguments of the Theorem 5.3 we can show that x^* is an optimal LS-solution of problem (IVP2). □

Theorem 5.6. *Let $f \in \mathcal{I}_k$, $x^* \in K$, $\eta : K \times K \to R^n$ and one of the following conditions is satisfied:*

(a) f is $LU - \eta$-invex at x^* and $g_i(x)$, $i = 1, 2, ..., m$ are η-invex at x^*;

(b) f is $LU - \eta$-pseudo-invex at x^* and $g_i(x)$, $i = 1, 2, ..., m$ are η-quasi-invex at x^*;

(c) f is $LU - \eta$-pseudo-invex at x^* and $(\mu^*)^T g(x)$ is η-quasi-invex at x^*;

(d) The Lagrangian function $f(x) + (\mu^*)^T g(x)$ is $LU - \eta$-pseudo-invex at x^* (that is the interval valued function $f(x) + (\mu^*)^T g_i(x)$ is $LU - \eta$-invex at x^*).

If there exist (Lagrangian) multipliers $0 < \lambda \in R$, $\mu^* = (\mu_1^*, ..., \mu_m^*)^T; 0 \leq \mu_i^* \in R, i = 1, 2, ..., m$ for continuously gH-differentiable function f and continuously differentiable functions g_i, $i = 1, 2, ..., m$ such that the following conditions hold true;

(i) $\lambda \nabla (f^L + f^U)(x^*) + \sum_{i=1}^m \mu_i^* \nabla g_i(x^*) = 0$;
(ii) $\mu_i^* g_i(x^*) = 0, i = 1, 2, ..., m$.

Then x^* is an optimal LU-solution and an LS-solution of problem (IVP2).

Proof. We define a real valued function

$$F(x) = \lambda (f^L + f^U)(x).$$

Since f is continuously gH-differentiable at x^*, then by Propositions 2.9 $f^L + f^U$ is continuously differentiable at x^*. Therefore we have,

$$\nabla F(x^*) = \lambda \nabla (f^L + f^U)(x^*). \tag{5.4}$$

(a) Since f is $LU - \eta$-invex at x^* then from Proposition 4.9 (iii) $f^L + f^U$ is η-invex at x^*. Therefore for $0 < \lambda \in R$, we see that the real valued function F is η-invex. Also g_i, $i = 1, 2, ..., m$ are η-invex. From (i), (ii) and (5.4), we have

(i) $\nabla F(x^*) + \sum_{i=1}^m \mu_i^* \nabla g_i(x^*) = 0$;
(ii) $\mu_i^* g_i(x^*) = 0, i = 1, 2, ..., m$.

By using Theorem 5.1 (a), we can say that x^* is optimal solution of F subject to the same constraints of (IVP2). Therefore proceeding similar to Theorem 5.3 we can say that x^* is an optimal LU-solution and an optimal LS-solution of problem (IVP2).

(b) Since f is $LU - \eta$-Pseudo-invex , then from Definition 4.2, F is η-Pseudo-invex for $\lambda^L = \lambda = \lambda^U > 0$. From ($i$), ($ii$), (5.4) and Theorem 5.1 (b), the result follows on similar lines as that of *(a)*.

(c) and *(d)* follows similar to that of *(a)* and *(b)*. □

Example 5.7. *Consider the following problem*

$$\begin{aligned}
&\min f(x_1, x_2) = [x_1^2 + x_2^2 - 6x_1 - 4x_2 + 12, x_1^2 + x_2^2 - 6x_1 - 4x_2 + 14],\\
&\textit{subject to}\\
&g_1(x_1, x_2) = x_1^2 + x_2^2 - 5 \le 0;\\
&g_2(x_1, x_2) = x_1 + 2x_2 - 4 \le 0;\\
&g_3(x_1, x_2) = -x_1 \le 0;\\
&g_4(x_1, x_2) = -x_2 \le 0.
\end{aligned} \tag{5.5}$$

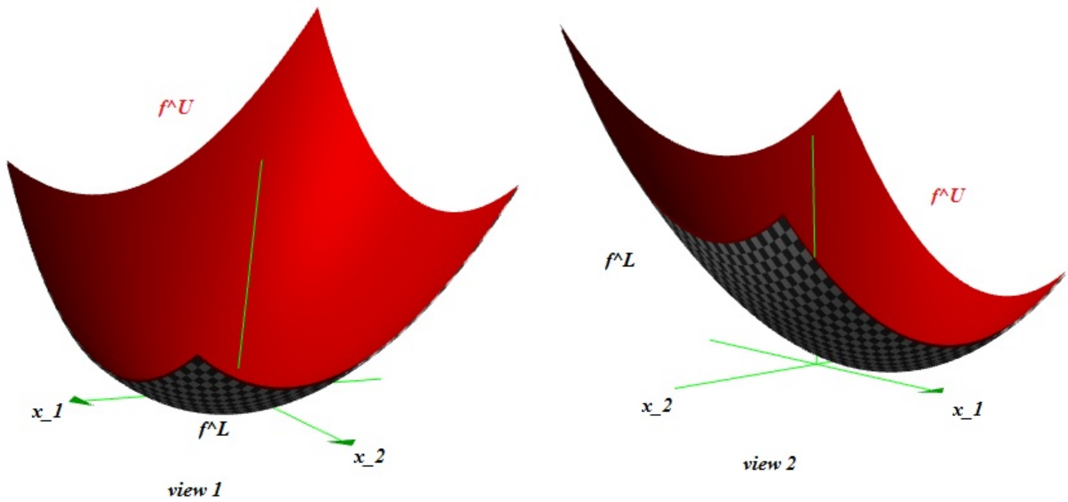

Figure 6: $f(x_1, x_2)$

It is easy to see that the interval valued objective function f is $LU - \eta$-invex and constraint functions g_i, $i = 1, 2, 3, 4$are η-invex, where for $x^T = (x_1, x_2)$ and $y^T = (y_1, y_2)$, η is given by

$$\eta^T(x, y) = (x_1 - y_1, x_2 - y_2).$$

It is easy to see that the problem (5.5) satisfies the conditions of Theorem 5.6 (a). Then we have

$$\lambda(4x_1 - 12, 4x_2 - 8)^T + \mu_1^*(2x_1, 2x_2)^T + \mu_2^*(1, 2)^T + \mu_3^*(-1, 0)^T + \mu_4^*(0, -1)^T = (0, 0)^T.$$

After some algebraic calculations, we obtain

$$(x^*)^T = (2, 1), \textit{for } (\mu^*)^T = \left(\tfrac{2}{3}, \tfrac{4}{3}, 0, 0\right) \textit{ and } \lambda = 1.$$

Since $g_1(x^*) = 0$ and $g_2(x^*) = 0$, the conditions of Theorem 5.6 are satisfied. Therefore $(x^*)^T$ is an optimal LU-solution and an optimal LS-solution of problem (5.5).

Remark 5.8. *Note that η also satisfies condition C, then by using Proposition 4.10 we can say that Problem 5.5 is also solved by Theorem 5.3.*

Theorem 5.9. *Let $f \in \mathcal{I}_k$, $x^* \in K$, $\eta : K \times K \to R^n$ and one of the following conditions is satisfied:*

(a) f is $LS - \eta$-invex at x^* and $g_i(x)$, $i = 1, 2, ..., m$ are η-invex at x^*;

(b) f is $LS-\eta$-pseudo-invex at x^* and $g_i(x)$, $i = 1, 2, ..., m$ are η-quasi-invex at x^*;

(c) f is $LS-\eta$-pseudo-invex at x^* and $(\mu^*)^T g(x)$ is η-quasi-invex at x^*;

(d) The Lagrangian function $f(x) + (\mu^*)^T g(x)$ is $LS-\eta$-pseudo-invex at x^* (that is the interval valued function $f(x) + (\mu^*)^T g_i(x)$ is $LS-\eta$-invex at x^*).

If there exist (Lagrangian) multipliers $0 < \lambda^L, \lambda^S \in R$, $\mu^* = (\mu_1^*, ..., \mu_m^*)^T; 0 \leq \mu_i^* \in R, i = 1, 2, ..., m$ for weakly continuously differentiable function f and continuously differentiable functions g_i, $i = 1, 2, ..., m$ such that the following conditions hold true;

(i) $(\lambda^L \nabla f^L + \lambda^S \nabla f^S)(x^*) + \sum_{i=1}^m \mu_i^* \nabla g_i(x^*) = 0$;

(ii) $\mu_i^* g_i(x^*) = 0$, $i = 1, 2, ..., m$.

Then x^* is an optimal LS-solution (IVP2).

Proof. The proof is same as that of Theorem 5.6. □

Next we present KKT conditions for (IVP2) by using gradient of interval valued objective function f via gH-derivative. For this, let $f \in \mathcal{I}_k$. Then the gradient of f at x^* is defined as follows:

$$\nabla_g f(x^*) = \left(\left(\frac{\partial f}{\partial x_1}\right)_g (x^*), ..., \left(\frac{\partial f}{\partial x_n}\right)_g (x^*) \right),$$

where $\left(\frac{\partial f}{\partial x_j}\right)_g (x^*)$ is the j^{th} partial gH-derivative of f at x^* (see, Definition 2.8). We see from Theorem 2.4 that, if f^L and f^U are differentiable functions then f is gH-differentiable and we have

$$\left(\frac{\partial f}{\partial x_j}\right)_g (x^*) = \left[\min\left\{\frac{\partial f^L}{\partial x_j}(x^*), \frac{\partial f^U}{\partial x_j}(x^*)\right\}, \max\left\{\frac{\partial f^L}{\partial x_j}(x^*), \frac{\partial f^U}{\partial x_j}(x^*)\right\}\right],$$

which is a closed interval.

Now consider the following equation

$$\nabla_g f(x) + \sum_{i=1}^m \mu_i \nabla g_i(x) = 0, \tag{5.6}$$

where $0 \leq \mu_i \in R$, $i = 1, 2, ..., m$ are real valued functions given in (IVP2) and f is interval valued gH-differentiable at x. Since $\sum_{i=1}^m \mu_i \left(\frac{\partial g_i}{\partial x_j}\right)(x) \in R$, then $\left(\frac{\partial F}{\partial x_j}\right)_g (x) \in R$. Consequently, from Theorem 2.5, f^L and f^U are continuously differentiable at x . Therefore 5.6 is equivalent to

$$\frac{\partial f^L}{\partial x_j}(x) + \sum_{i=1}^m \mu_i \frac{\partial g_i}{\partial x_j}(x) = 0 = \frac{\partial f^U}{\partial x_j}(x) + \sum_{i=1}^m \mu_i \frac{\partial g_i}{\partial x_j}(x), \quad for \quad j = 1, 2, ..., n,$$

which can be written as

$$\nabla f^L(x) + \sum_{i=1}^m \mu_i \nabla g_i(x) = 0 = \nabla f^U(x) + \sum_{i=1}^m \mu_i \nabla g_i(x).$$

Which is equivalent to

$$\nabla f^L(x) + \nabla f^U(x) + \sum_{i=1}^m \bar{\mu}_i \nabla g_i(x) = 0, \tag{5.7}$$

where $\bar{\mu}_i = 2\mu_i$, $i = 1, 2, ..., m$.

Theorem 5.10. *Let $f \in \mathcal{I}_k$ be continuously gH-differentiable, $LU - \eta$-preinvex and each g_i, $i = 1, 2, ..., m$ is continuously differentiable, η-invex functions at $x^* \in K$. If there exist (Lagrangian) multipliers $\mu^* = (\mu_1^*, ..., \mu_m^*)^T; 0 \leq \mu_i^* \in R, i = 1, 2, ..., m$, such that (x^*, μ^*) satisfy the following KKT conditions;*

(i) $\nabla_g f(x^*) + \sum_{i=1}^m \mu_i^* \nabla g_i(x^*) = 0;$
(ii) $\mu_i^* g_i(x^*) = 0, i = 1, 2, ..., m.$

Then x^* is an optimal LU-solution and an optimal LS-solution of (IPV2).

Proof. Since condition (i) of the Theorem is equation (5.6) for $x = x^*$. Which is equivalent to (5.7). Therefore we obtain,

$$\nabla f^L(x^*) + \nabla f^U(x^*) + \sum_{i=1}^m \bar{\mu}_i^* \nabla g_i(x^*) = 0,$$

where $\bar{\mu}_i^* = 2\mu_i^*, i = 1, 2, ..., m$. Then from Theorem 5.3, the result follows. □

Theorem 5.11. *Let $f \in \mathcal{I}_k$ be continuously gH-differentiable, $LS - \eta$-preinvex and each g_i, $i = 1, 2, ...m$ is continuously differentiable, η-invex functions at $x^* \in K$. If there exist (Lagrangian) multipliers $\mu^* = (\mu_1^*, ..., \mu_m^*)^T; 0 \leq \mu_i^* \in R$, $i = 1, 2, ..., m$ such that (x^*, μ^*) satisfy the following KKT conditions;*

(i) $\nabla_g f(x^*) + \sum_{i=1}^m \mu_i^* \nabla g_i(x^*) = 0;$
(ii) $\mu_i^* g_i(x^*) = 0, i = 1, 2, ..., m.$

Then x^* is an optimal LS-solution of (IPV2).

Proof. The condition (i) of the Theorem is equation (5.6) for $x = x^*$. Therefore we obtain from (5.7)

$$\nabla f^L(x^*) + \nabla f^S(x^*) + \sum_{i=1}^m \bar{\mu}_i^* \nabla g_i(x^*) = 0.$$

Then from Theorem 5.5, we see that x^* is optimal LS-solution of (IVP2). □

Remark 5.12. *We remark that for weakly continuously differentiable interval valued function f, we can not define gradient as we can not define partial derivatives of f. Moreover gradient of $f(x, y) = [2x^2 + 3y^2, x^2 + y^2 + 1]$ using H-derivative (Wu [7]) does not exist as the partial derivative $\left(\frac{\partial f}{\partial x}\right)_H(0, 1)$ does not exist. The behaviour of $f(x, y)$ is shown in fig.* **??**.

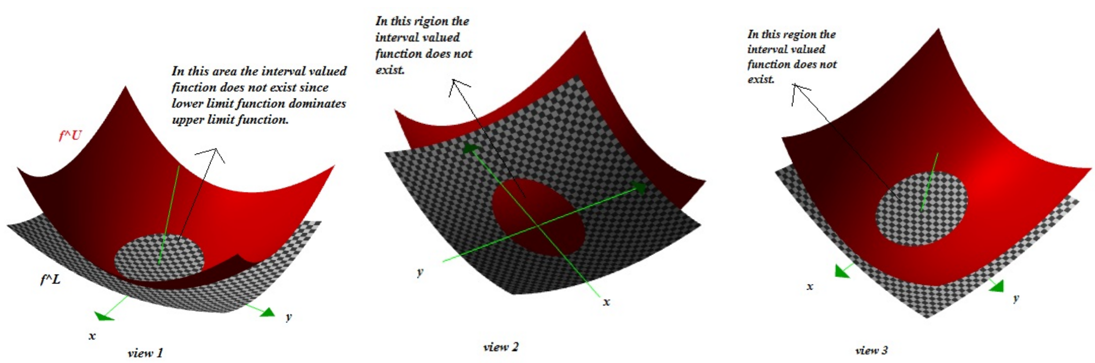

Figure 7: $f(x, y)$

However by applying Theorem 2.4, we obtain

$$\nabla_g f(x, y) = \Big([\min(4x, 2x), \max(4x, 2x)], \min(6y, 2y), \max(6y, 2y)]\Big).$$

Therefore we have

$$\nabla_g f(x,y) = \begin{cases} \big([2x,4x],[2y,4y]\big) \ : \ x,y \geq 0, \\ \big([4x,2x],[6y,2y]\big) \ : \ x,y < 0. \end{cases}$$

Therefore the gradient of f using gH-derivative is more general and more robust for optimization.

Example 5.13. *Consider the problem*

$$\begin{aligned} &\min f(x,y) = [x^2, x^2+y^2], \\ &\textit{subject to} \\ &g_1(x,y) = x+y-1 \leq 0, \\ &g_2(x,y) = -x \leq 0. \end{aligned} \tag{5.8}$$

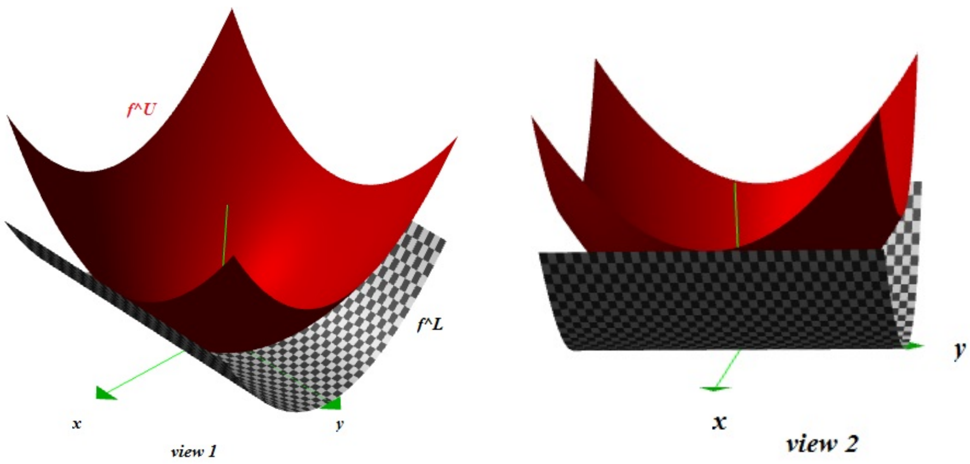

Figure 8: $f(x,y)$

Let $\eta(x,y) = x-y$. Then f is $LU-\eta$-preinvex and $g_i, i = 1,2$ are η-invex at $(0,0)$. The interval valued function f is continuously gH-differentiable on R^2. Also the conditions of Theorem 5.10 are satisfied at $(0,0)$. Therefore $(0,0)$ is optimal LU-solution and optimal LS-solution of problem (5.8).

6. Conclusions

The KKT optimality conditions for interval valued nonlinear programming problems under the condition of invexity, preinvexity, pseudo-invexity, quasi-invexity and generalized Hukuhara differentiability are represented in this paper. Our results generalize the results of Wu [7], Zhang et al. [11], Chalco-Cano et al. [28]. In fact Theorem 5.6 generalizes the similar result of Zhang et al. [11] and Hanson [19]. Also the results for the case of LS order relation are novel.

Although the equality constraints are not considered in this paper we can use similar methodology proposed in this paper to handle equality constraints. The constraint functions in this paper are still real valued, in future research, one may extend to consider the constraint functions as the interval valued functions.

References

[1] A. Bhurjee, G. Panda, Efficient solution of interval optimization problem, Math. Meth. Oper. Res. 76 (2012) 273–288.
[2] A. G. M. Ibrahim, On the differentiability of set-valued functions defined on a Banach space and mean value theorem, Appl. Math. Comput. 74 (1996) 76–94.
[3] B. Bede, S. G. Gal, Generalization of the differentiability of fuzzy number valued functions with applications to fuzzy differential equation. Fuzzy sets syst. 151 (2005) 581–599.
[4] F. S. De Blasi, On the differentiability of multifunctions, Pacific J. of Math. 66 (1976) 67–91.
[5] G. J. Zalmai, Generalized sufficiency criteria in continuous time programming with application to a class of variational type inequalities, J. Math. Anal. Appl. 153 (1990) 331–355.
[6] H. Ishibuchi, H. Tanaka, Multiobjective programming in optimization of interval valued objective functions, European J. Oper. Res. 48 (1990) 219–225.
[7] H. C. Wu, The Karush Kuhn Tuker optimality conditions in an optimization problem with interval valued objective functions, European J. Oper. Res. 176 (2007) 46–59.
[8] H. T. Banks, M. Q. Jacobs, A differential calculus for multifunctions, J. Math. Anal. Appl. 29 (1970) 246–272.
[9] I. Ahmad, Efficiency and duality in nondifferentiable multiobjective programs involving directional derivative, Appl. Math. 2(2011) 452–460.
[10] I. Ahmad, A. Jayswal, J. Banerjee, On interval-valued optimization problems with generalized invex functions, J. Inequal. Appl. (2013)1–14.
[11] J. Zhang, S. Liu, L. Li, Q. Feng, The KKT optimality conditions in a class of generalized convex optimization problems with an interval-valued objective function, Optim. Lett. 8 (2014) 607–631.
[12] J. P. Aubin, A. Cellina, Differential inclusions, NY, Springer, 1984.
[13] J. P. Aubin, H. Franskowska, Set-valued analysis, Boston: Birkhauser, 1990.
[14] J. P. Aubin, H. Franskowska, Introduction: set-valued analysis in control theory, Set-valued Anal. 8 (2000), 1–9.
[15] J. P. Vial, Strong and weak convexity set and functions, Math. Oper. Res. 8 (1983) 231–259.
[16] L. Li, S. Liu, J. Zhang, Univex interval-valued mapping with differentiability and its application in nonlinear programming, J. Appl. Math, Art. ID 383692, (2013). http://dx.doi.org/10.1155/2013/383692.
[17] L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval valued functions and interval differential equations, Nonlinear Anal, 71 (2009) 1311–1328.
[18] M. Hukuhara, Integration des applications measurable dont la valeuerestun compact convexe, Funkcialaj Ekvacioj 10 (1967) 205–223.
[19] M. A. Hanson, On sufficiency of the KKT conditions, J. Math. Anal. Appl. 80 (1981) 545–550.
[20] M. A. Hanson, B. Mond, Necessary and sufficient conditions in constrained optimization, Math. Program. 37 (1987) 51–58.
[21] M. A. Hanson, R. Pini, C. Singh, Multiobjective programming under generalization type I invexity, J. Math. Anal. Appl. 262 (2001) 562–577.
[22] M. S. Bazaraa, H. D. Sherali, C. M. Shetty, Nonlinear programming, Wiley, YN, 1993.
[23] P. Mandal, C. Nahak, Symmetric duality with $(p,r)-\rho-(\eta,\theta)$-invexity. Appl. Math. Comput. 217 (2011) 8141–8148.
[24] R. E. Moore, Interval Analysis, Prentice Hall, Englewood Cliffs, NJ, (1966).
[25] R, Pini, Invex and generalized convexity, Optim. 22 (1991) 513–525.
[26] S. M. Assev, Quasilinear operators and their applications in the theory of multivalued mappings, Proceedings of the Steklov Inst. of Math. 2 (1986) 23–52.
[27] S. R. Mohan, S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995) 901–908.
[28] T. Antczak, (ρ,r)-invex sets and functions, J. Math. Anal. Appl. 263 (2001) 355–379.
[29] T. R. Gulati, I. Ahmad, D. Agarwal, Sufficiency and duality in multiobjective programming under generalized type I functions, J. Optim. Theory Appl. 135 (2007) 411–427.
[30] T. Weir, B. Mond, Preinvex functions in multiple-objective optimization, J. Math. Anal. Appl. 136 (1988) 29–38.
[31] T. Weir, V. Jeyakumar, A class of non-convex functions and mathematical programming, Bull. Austral. Math. Soc. 38 (1988) 177–189.
[32] V. Jeyakumar, B. Mond, On generalized convex mathematical programming, J. austral. Math. Soc. Ser. 34 (1992) 43–53.
[33] Y. Chalco-Cano, H. Roman-Flores, M. D. Jimenez-Gamero, Generalized derivative and π-derivative for set valued functions. Inf. Sci. 181 (2011) 2177–2188.
[34] Y. Chalco-Cano, W. A. Lodwick, A. Rufian-Lizana, Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative, Fuzzy Optim. Decis. Making. 12 (2013) 305–322.
[35] Z. A. Liang, H. X. Huang, P. M. Pardalos, Optimality conditions and duality for a class of nonlinear fractional programming problems, J. Optim. Theory Appl. 110 (2001) 611–619.