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Expressions and Perturbations for the Moore–Penrose Inverse of Bounded Linear Operators in Hilbert Spaces

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Abstract. Let A, X, Y be bounded linear operators. In this paper, we present the explicit expression for the Moore–Penrose inverse of $A - XY$. In virtue of the expression of $(A + X)^{\dagger}$, we get the upper bounds of $\|(A + X)^{\dagger}\|$ and $\|(A + X)^{\dagger} - A^{\dagger}\|$.

1. Introduction

Let H_1, H_2, K, K_1, K_2 be Hilbert spaces. $B(H_1, H_2)$ denote the set of the bounded linear operators from H_1 to H_2. $B(K, K)$ is abbreviated to $B(K)$. Let $A \in B(H_1, H_2)$. We denote the rang and the kernel of A by $R(A)$ and $\ker(A)$, respectively.

The operator $B \in B(H_2, H_1)$ which satisfied $ABA = A$ is called the inner inverse of A, denoted by A^-. If B is an inner inverse and satisfied $BAB = B$, then B is called a generalized inverse of A, denoted by A^+. The Moore–Penrose inverse of A, denoted by $A^{\dagger}$, is the unique solution to the following equations:

$$AA^{\dagger}A = A, \quad A^{\dagger}AA^{\dagger} = A^{\dagger}, \quad (AA^{\dagger})^* = AA^{\dagger}, \quad (A^{\dagger}A)^* = A^{\dagger}A,$$

in which A^* denote the adjoint operator of A. It is well known that A has an Moore–Penrose inverse iff $R(A)$ is closed. From [17, Proposition 3.5.3], we know $A^{\dagger} = A^*(AA^*)^{\dagger} = (A^*A)^{\dagger}A^*$ if $A^{\dagger}$ exists.

The perturbation for the Moore–Penrose inverse of bounded linear operators has been studying by many authors. G.Chen and Y.Xue introduce the notation so–called the stable perturbation in [2, 4]. This notation is an extension of the rank–preserving perturbation of matrices. Using this notation, they give the estimation of upper bounds about the perturbation of Moore–Penrose and Drazin inverse in the work of Chen and Xue et al (cf.[2–4, 17–20]). A classical results for the perturbation analysis of the Moore–Penrose inverse is

$$\|\bar{T}^{\dagger}\| \leq \frac{\|T^{\dagger}\|}{1 - \|T^{\dagger}\|\|\delta T\|}, \qquad \frac{\|\bar{T}^{\dagger} - T^{\dagger}\|}{\|T^{\dagger}\|} \leq \frac{1 + \sqrt{5}}{2} \frac{\|T^{\dagger}\|}{1 - \|T^{\dagger}\|\|\delta T\|}$$

when $\|T^{\dagger}\|\|\delta T\| < 1$ and $\bar{T}$ is a stable perturbation of T, i.e., $R(\bar{T}) \cap R(T)^{\perp} = \{0\}$ ($T \in B(H, K), \bar{T} = T + \delta T \in B(H, K)$).

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Later this notation is generalized to the set of Banach algebras by Y.Xue in [18] and to the set of Hilbert C^*–module by Xu et al. in [16]. Motivated by the stable perturbation, we want to investigate the general perturbation analysis for the Moore–Penrose inverse of bounded linear operators. In order to do this, we must study the expression for the Moore–Penrose inverse of $A - XY$.

The Moore–Penrose inverse of $A - XY$ has many applications in statistics, networks, optimizations etc. (see [10, 11, 14]). For a long time, the expression of $(A - XY)^\dagger$ has been studied by many authors and been obtained lots of results under certain conditions(see [1, 5–7, 9, 13, 15]).

In this paper, we first investigate the Moore–Penrose inverse of $A - XY$ and give the explicit expression of the Moore–Penrose inverse $(A - XY)^\dagger$ under the weaker conditions. Using the expression of $(A - XY)^\dagger$, we estimate the upper bounds for the perturbation of the Moore–Penrose inverse $(A + X)^\dagger$. Our results are new and generalize the stable perturbation to the case $I + A^\dagger X$ is not invertible.

2. Preliminaries

In this section, we give some lemmas which will be used in the context. The first two lemmas which come from [3, 9, 17] play an important role in this paper.

Lemma 2.1. [3, 9, 17] *Let $S \in B(K)$ be an idempotent, then $I - S - S^*$ is invertible and $O(S) = S(S + S^* - I)^{-1}$ is a projection (i.e. $(O(S))^2 = O(S) = (O(S))^*$) and $O(S) = SS^\dagger, O(I - S) = I - S^\dagger S$.*

Lemma 2.2. [3, 9, 17] *Let $A \in B(H_1, H_2)$ with $R(A)$ closed. Then*

$$A^\dagger = [I - O(I - A^+A)]A^+O(AA^+) = (I - P - P^*)^{-1}A^+(I - Q - Q^*)^{-1}.$$

Here, $P = A^+A, Q = AA^+$.

From Lemma 2.2, we get $A^\dagger$ exists iff A^+ exists and have the following equations:

$$AA^\dagger = O(AA^+) = AA^+(AA^+ + (AA^+)^* - I)^{-1},$$

$$A^\dagger A = I - O(I - A^+A) = (A^+A + (A^+A)^* - I)^{-1}A^+A.$$

Remark 2.3. *Assume that A^- is an inner inverse of A, then A^-AA^- is a generalized inverse of A. Thus, we have*

$$A^\dagger = [I - O(I - A^-A)]A^-O(AA^-)$$

when there is an inner inverse A^- of A. (Please see [3, 9, 17] for details).

Lemma 2.4. [12, 17] *Let $A \in B(H_1, H_2), B \in B(H_2, H_1)$. Then $(AB)^\dagger = B^\dagger A^\dagger$ iff $R(A^*AB) \subseteq R(B)$ and $R(BB^*A^*) \subseteq R(A^*)$.*

3. The Expression for the Moore–Penrose Inverse

Let $A \in B(H_1, H_2)$ with $R(A)$ closed and $X \in B(K, H_2), Y \in B(H_1, K)$. Let $E_A = I - AA^\dagger, F_A = I - A^\dagger A, U = E_A X, V = YF_A$ with $R(U), R(V)$ closed throughout this paper.

Theorem 3.1. *Let $A \in B(H_1, H_2)$ with $R(A)$ closed and $X \in B(K, H_2), Y \in B(H_1, K), Z = I - YA^\dagger X, S = E_V Z F_U$. If $R(S)$ is closed, then*

$$\Lambda = A^\dagger - V^\dagger YA^\dagger + (V^\dagger Z + A^\dagger X)[S^\dagger YA^\dagger - (I - S^\dagger Z)U^\dagger] \tag{1}$$

is an inner inverse of $A - XY$ and

$$\begin{aligned}
(A - XY)^\dagger &= \{I - (A^\dagger X + V^\dagger Z)F_U F_S Y - ((A^\dagger X + V^\dagger Z)F_U F_S Y)^*\}^{-1} \\
&\quad \times \{A^\dagger - V^\dagger YA^\dagger + (V^\dagger Z + A^\dagger X)[S^\dagger YA^\dagger - (I - S^\dagger Z)U^\dagger]\} \\
&\quad \times \{I - XE_S E_V(YA^\dagger + ZU^\dagger) - (XE_S E_V(YA^\dagger + ZU^\dagger))^*\}^{-1} \\
&= \{I - ((A^\dagger X + V^\dagger Z)F_U F_S Y)((A^\dagger X + V^\dagger Z)F_U F_S Y)^\dagger\} \\
&\quad \times \{A^\dagger - V^\dagger YA^\dagger + (V^\dagger Z + A^\dagger X)[S^\dagger YA^\dagger - (I - S^\dagger Z)U^\dagger]\} \\
&\quad \times \{I - (XE_S E_V(YA^\dagger + ZU^\dagger))^\dagger (XE_S E_V(YA^\dagger + ZU^\dagger))\}.
\end{aligned}$$

Proof. Let $\Lambda = A^\dagger - V^\dagger YA^\dagger + (V^\dagger Z + A^\dagger X)[S^\dagger YA^\dagger - (I - S^\dagger Z)U^\dagger]$. Noting that

$$U^\dagger = U^\dagger E_A, \quad V^\dagger = F_A V^\dagger, \quad S^\dagger = F_U S^\dagger = S^\dagger E_V = F_U S^\dagger E_V,$$

we have

$$\begin{aligned} &YV^\dagger = VV^\dagger, \quad AV^\dagger = 0, \quad US^\dagger = 0, \quad E_V ZS^\dagger = SS^\dagger, \\ &U^\dagger X = U^\dagger U, \quad U^\dagger A = 0, \quad S^\dagger V = 0, \quad S^\dagger ZF_U = S^\dagger S. \end{aligned}$$

Hence,

$$\begin{aligned} &(A - XY)\Lambda = AA^\dagger + UU^\dagger - XE_S E_V(YA^\dagger + ZU^\dagger), \\ &\Lambda(A - XY) = A^\dagger A + V^\dagger V - (A^\dagger X + V^\dagger Z)F_U F_S Y \end{aligned}$$

and $(A - XY)\Lambda(A - XY) = (A - XY)$. This indicate Λ is an inner inverse of $A - XY$.

Since

$$\begin{aligned} &(I - 2A^\dagger A - 2V^\dagger V)^{-1} = (I - 2A^\dagger A - 2V^\dagger V), \\ &(I - 2AA^\dagger - 2UU^\dagger)^{-1} = (I - 2AA^\dagger - 2UU^\dagger), \end{aligned}$$

and

$$(I - 2A^\dagger A - 2V^\dagger V)\Lambda(I - 2AA^\dagger - 2UU^\dagger) = \Lambda,$$

we have, by Lemma 2.2,

$$\begin{aligned} (A - XY)^\dagger = &\{I - (A^\dagger X + V^\dagger Z)F_U F_S Y - ((A^\dagger X + V^\dagger Z)F_U F_S Y)^*\}^{-1} \\ &\times \{A^\dagger - V^\dagger YA^\dagger + (V^\dagger Z + A^\dagger X)[S^\dagger YA^\dagger - (I - S^\dagger Z)U^\dagger]\} \\ &\times \{I - XE_S E_V(YA^\dagger + ZU^\dagger) - (XE_S E_V(YA^\dagger + ZU^\dagger))^*\}^{-1}. \end{aligned}$$

Since

$$\begin{aligned} Y(A^\dagger X + V^\dagger Z)F_U F_S &= (YA^\dagger X + VV^\dagger Z)F_U F_S \\ &= (I - Z + VV^\dagger Z)F_U F_S \\ &= (I - E_V Z)F_U F_S \\ &= F_U F_S \\ E_S E_V(YA^\dagger + ZU^\dagger)X &= E_S E_V(YA^\dagger X + ZU^\dagger X) \\ &= E_S E_V(I - ZF_U) \\ &= E_S E_V, \end{aligned}$$

we have $(A^\dagger X + V^\dagger Z)F_U F_S Y$ and $XE_S E_V(YA^\dagger + ZU^\dagger)$ are idempotent, Thus, $R((A^\dagger X + V^\dagger Z)F_U F_S Y)$ and $R(XE_S E_V(YA^\dagger + ZU^\dagger))$ are closed. So, $[(A^\dagger X + V^\dagger Z)F_U F_S Y]^\dagger$ and $[XE_S E_V(YA^\dagger + ZU^\dagger)]^\dagger$ exist.

Noting that $I - (A^\dagger X + V^\dagger Z)F_U F_S Y$ and $I - XE_S E_V(YA^\dagger + ZU^\dagger)$ are idempotent too and

$$\begin{aligned} &\{I - (A^\dagger X + V^\dagger Z)F_U F_S Y\}\Lambda\{I - XE_S E_V(YA^\dagger + ZU^\dagger)\} \\ &= \{I - A^\dagger A - V^\dagger V + \Lambda(A - XY)\}\Lambda\{I - AA^\dagger - UU^\dagger + (A - XY)\Lambda\} \\ &= \Lambda(A - XY)\Lambda \\ &= \Lambda(I - XE_S E_V(YA^\dagger + ZU^\dagger)). \end{aligned}$$

We get $\bar{\Lambda} = \Lambda(I - XE_S E_V(YA^\dagger + ZU^\dagger))$ is a generalized inverse of $A - XY$ and

$$(A - XY)\Lambda = (A - XY)\bar{\Lambda}, \quad \bar{\Lambda}(A - XY) = \Lambda(A - XY).$$

Since $(AA^{\dagger} + UU^{\dagger})X = AA^{\dagger}X + UU^{\dagger}X = X - U + U = X$, we have

$$(AA^{\dagger} + UU^{\dagger})XE_SE_V(YA^{\dagger} + ZU^{\dagger}) = XE_SE_V(YA^{\dagger} + ZU^{\dagger}) = XE_SE_V(YA^{\dagger} + ZU^{\dagger})(AA^{\dagger} + UU^{\dagger}).$$

This indicate $XE_SE_V(YA^{\dagger} + ZU^{\dagger})$ commute with $I - 2AA^{\dagger} - 2UU^{\dagger}$.

Noting that the result of the Lemma 2.2 is independent of the choice of A^{+}. Hence, by Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
(A - XY)^{\dagger} &= [I - O(\bar{\Lambda}(A - XY))]\bar{\Lambda}O((A - XY)\bar{\Lambda})\\
&= \{I - (A^{\dagger}X + V^{\dagger}Z)F_UF_SY - ((A^{\dagger}X + V^{\dagger}Z)F_UF_SY)^*\}^{-1}\\
&\times \Lambda(I - XE_SE_V(YA^{\dagger} + ZU^{\dagger}))\\
&\times \{I - XE_SE_V(YA^{\dagger} + ZU^{\dagger}) - (XE_SE_V(YA^{\dagger} + ZU^{\dagger}))^*\}^{-1}\\
&= \{I - (A^{\dagger}X + V^{\dagger}Z)F_UF_SY - ((A^{\dagger}X + V^{\dagger}Z)F_UF_SY)^*\}^{-1}\\
&\times \{I - (A^{\dagger}X + V^{\dagger}Z)F_UF_SY\}\Lambda\{I - XE_SE_V(YA^{\dagger} + ZU^{\dagger})\}\\
&\times \{I - XE_SE_V(YA^{\dagger} + ZU^{\dagger}) - (XE_SE_V(YA^{\dagger} + ZU^{\dagger}))^*\}^{-1}\\
&= \{I - O[(A^{\dagger}X + V^{\dagger}Z)F_UF_SY]\}\Lambda O[I - (XE_SE_V(YA^{\dagger} + ZU^{\dagger})]\\
&= \{I - ((A^{\dagger}X + V^{\dagger}Z)F_UF_SY)((A^{\dagger}X + V^{\dagger}Z)F_UF_SY)^{\dagger}\}\\
&\times \{A^{\dagger} - V^{\dagger}YA^{\dagger} + (V^{\dagger}Z + A^{\dagger}X)[S^{\dagger}YA^{\dagger} - (I - S^{\dagger}Z)U^{\dagger}]\}\\
&\times \{I - (XE_SE_V(YA^{\dagger} + ZU^{\dagger}))^{\dagger}(XE_SE_V(YA^{\dagger} + ZU^{\dagger}))\}.
\end{aligned}$$

□

Corollary 3.2. *Let $A \in B(H_1, H_2)$ with $R(A)$ closed and $X \in B(K, H_2), Y \in B(H_1, K), Z = I - YA^{\dagger}X$. If $R(X) \subseteq R(A), \ker(A) \subseteq \ker(Y)$ and $R(Z)$ is closed, then $A^{\dagger} + A^{\dagger}XZ^{\dagger}YA^{\dagger}$ is an inner inverse of $A - XY$ and*

$$\begin{aligned}
(A - XY)^{\dagger} &= \{I - (A^{\dagger}XF_ZY) - (A^{\dagger}XF_ZY)^*\}^{-1}\{A^{\dagger} + A^{\dagger}XZ^{\dagger}YA^{\dagger}\}\\
&\times \{I - (XE_ZYA^{\dagger}) - (XE_ZYA^{\dagger})^*\}^{-1}\\
&= \{I - (A^{\dagger}XF_ZY)(A^{\dagger}XF_ZY)^{\dagger}\}\{A^{\dagger} + A^{\dagger}XZ^{\dagger}YA^{\dagger}\}\\
&\times \{I - (XE_ZYA^{\dagger})^{\dagger}(XE_ZYA^{\dagger})\}.
\end{aligned}$$

Proof. Since $R(X) \subseteq R(A), \ker(A) \subseteq \ker(Y)$, we have $U = 0 = V$. Hence, the results follow by Theorem 3.1. □

Corollary 3.3. *Let $A, X \in B(H_1, H_2)$ with $R(A)$ closed and $Z = I + A^{\dagger}X, S = A^{\dagger}AZF_U$. If $R(S)$ is closed, then*

$$\Lambda = A^{\dagger} + (F_A - A^{\dagger}X)[S^{\dagger}A^{\dagger} + (I - S^{\dagger}Z)U^{\dagger}] \tag{2}$$

is an inner inverse of $A + X$ and

$$\begin{aligned}
(A + X)^{\dagger} = \{I - (F_UF_S) - (F_UF_S)^*\}^{-1}\{A^{\dagger} + (F_A - A^{\dagger}X)[S^{\dagger}A^{\dagger} + (I - S^{\dagger}Z)U^{\dagger}]\}\\
\times\{I + (XE_S(A^{\dagger} - A^{\dagger}AZU^{\dagger})) + (XE_S(A^{\dagger} - A^{\dagger}AZU^{\dagger}))^*\}^{-1}.
\end{aligned}$$

Especially, if $U = 0$, then $S = A^{\dagger}AZ$ and $\Lambda = (I + S^{\dagger} - SS^{\dagger})A^{\dagger}$ is an inner inverse of $A + X$ and

$$(A + X)^{\dagger} = (2S^{\dagger}S - I)(I + S^{\dagger} - SS^{\dagger})A^{\dagger}\{I + XE_SA^{\dagger} + (XE_SA^{\dagger})^*\}^{-1}.$$

Proof. Replacing Y by $-I$ in Eq.(1), we get Eq.(2). If $U = 0$, then $S = A^{\dagger}AZ$. By Eq.(2),

$$\begin{aligned}
\Lambda &= A^{\dagger} + (F_A - A^{\dagger}X)S^{\dagger}A^{\dagger}\\
&= A^{\dagger} = (I - S)S^{\dagger}A^{\dagger}\\
&= (I + S^{\dagger} - SS^{\dagger})A^{\dagger}.
\end{aligned}$$

Thus, the results are obtained by Theorem 3.1. □

Corollary 3.4. *Let $A, Y \in B(H_1, H_2)$ with $R(A)$ closed and $Z = I + YA^\dagger, T = E_V ZAA^\dagger$. If $R(T)$ is closed, then*

$$\Lambda = V^\dagger + (A^\dagger - V^\dagger Z)[T^\dagger + (I - T^\dagger Z)AA^\dagger] \tag{3}$$

is an inner inverse of $A + Y$ and

$$\begin{aligned}(A+Y)^\dagger = &\{I - (V^\dagger Z - A^\dagger)AA^\dagger F_T Y - ((V^\dagger Z - A^\dagger)AA^\dagger F_T Y)^*\}^{-1}\\ &\times \{V^\dagger + (A^\dagger - V^\dagger Z)[T^\dagger + (I - T^\dagger Z)AA^\dagger]\}\{I - E_T E_V - (E_T E_V)^*\}^{-1}.\end{aligned}$$

Especially, if $V = 0$, then $T = ZAA^\dagger$ and $A^\dagger(I + T^\dagger - T^\dagger T)$ is an inner inverse of $A + Y$ and

$$(A+Y)^\dagger = \{I + A^\dagger F_T Y + (A^\dagger F_T Y)^*\}^{-1} A^\dagger (I + T^\dagger - T^\dagger T)(2TT^\dagger - I).$$

Proof. Replacing X by $-I$ in Eq.(1), we have

$$\begin{aligned}\Lambda &= A^\dagger - V^\dagger YA^\dagger + (V^\dagger Z - A^\dagger)[T^\dagger YA^\dagger + (I - T^\dagger Z)E_A]\\ &= A^\dagger - V^\dagger(Z - I) + (V^\dagger Z - A^\dagger)[T^\dagger YA^\dagger - T^\dagger ZE_A + E_A]\\ &= V^\dagger + (A^\dagger - V^\dagger Z)[I - T^\dagger YA^\dagger + T^\dagger ZE_A - E_A]\\ &= V^\dagger + (A^\dagger - V^\dagger Z)[T^\dagger + (I - T^\dagger Z)AA^\dagger].\end{aligned}$$

Noting that $YA^\dagger - ZE_A = ZAA^\dagger - I$, by Theorem 3.1, the results are obtained. □

Corollary 3.5. *Let $X \in B(H_2, H_1), Y \in B(H_1, H_2)$. Then $R(I - XY)$ is closed iff $R(I - YX)$ is closed and $I + X(I - YX)^\dagger Y$ is an inner inverse of $I - XY$. Put $Z = I - YX$, then*

$$(I - XY)^\dagger = \{I - XF_Z Y - (XF_Z Y)^*\}^{-1}(I + XZ^\dagger Y)\{I - XE_Z Y - (XE_Z Y)^*\}^{-1}.$$

Proof. The assertion follows by Corollary 3.2. □

In [6, 13], the expression for the Moore–Penrose inverse of $A - XGY$ was obtained under some complex conditions, respectively. Now, we get this expression under simpler conditions.

Proposition 3.6. *Let $A \in B(H_1, H_2), X \in B(K_2, H_2), Y \in B(H_1, K_1), G \in B(K_1, K_2)$ with $R(A), R(G)$ closed. If $R(X) \subseteq R(A), \ker(A) \subseteq \ker(Y)$ and $\ker(X)^\perp \subseteq R(G), R(Y) \subseteq \ker(G)^\perp$ and $R(G^\dagger - YA^\dagger X)$ closed, then*

$$\Lambda = A^\dagger + A^\dagger X(G^\dagger - YA^\dagger X)^\dagger YA^\dagger$$

is an inner inverse of $A - XGY$ and

$$\begin{aligned}(A - XGY)^\dagger &= \{I - (A^\dagger XF_W GY) - (A^\dagger XF_W GY)^*\}^{-1}\{A^\dagger + A^\dagger XW^\dagger YA^\dagger\}\\ &\quad\times \{I - (XGE_W YA^\dagger) - (XGE_W YA^\dagger)^*\}^{-1}\\ &= \{I - (A^\dagger XF_W GY)(A^\dagger XF_W GY)^\dagger\}\{A^\dagger + A^\dagger XW^\dagger YA^\dagger\}\\ &\quad\times \{I - (XGE_W YA^\dagger)^\dagger(XGE_W YA^\dagger)\}.\end{aligned}$$

Here, $W = G^\dagger - YA^\dagger X$.

Proof. Let $W = G^\dagger - YA^\dagger X$ and $\Lambda = A^\dagger + A^\dagger X(G^\dagger - YA^\dagger X)^\dagger YA^\dagger$.

Since $R(X) \subseteq R(A), \ker(A) \subseteq \ker(Y)$ and $\ker(X)^\perp \subseteq R(G), R(Y) \subseteq \ker(G)^\perp$, we have

$$AA^\dagger X = X, \quad YA^\dagger A = Y, \quad XGG^\dagger = X, \quad G^\dagger GY = Y.$$

Thus,

$$(A - XGY)\Lambda = AA^\dagger - XGE_W YA^\dagger, \quad \Lambda(A - XGY) = A^\dagger A - A^\dagger XF_W GY,$$

and $(A - XGY)\Lambda(A - XGY) = (A - XGY)$. This shows Λ is an inner inverse of $A - XGY$. By Lemma 2.2,

$$(A - XGY)^\dagger = \{I - (A^\dagger XF_W GY) - (A^\dagger XF_W GY)^*\}^{-1}\{A^\dagger + A^\dagger XW^\dagger YA^\dagger\} \\ \times \{I - (XGE_W YA^\dagger) - (XGE_W YA^\dagger)^*\}^{-1}.$$

Noting that $W = G^\dagger - YA^\dagger X = G^\dagger - G^\dagger GYA^\dagger XGG^\dagger$, we have $R(W) \subseteq R(G^\dagger)$ and $\ker(G^\dagger) \subseteq \ker(W)$. Thus,

$$G^\dagger GW = W, \qquad WGG^\dagger = W.$$

Again, $\ker(G) \subseteq \ker(W^\dagger)$ and $R(W^\dagger) \subseteq R(G)$ by $R(W) = \ker(W^\dagger)^\perp$, $R(G^\dagger) = \ker(G)^\perp$, $\ker(G^\dagger) = R(G)^\perp$, $\ker(W) = R(G^\dagger)^\perp$. We have

$$GG^\dagger W^\dagger = W^\dagger, \qquad W^\dagger G^\dagger G = W^\dagger.$$

Thus,

$$\begin{aligned}(A^\dagger XF_W GY)(A^\dagger XF_W GY) &= A^\dagger XF_W G(YA^\dagger X)F_W GY \\ &= A^\dagger XF_W G(G^\dagger - W)F_W GY \\ &= A^\dagger XF_W GY.\end{aligned}$$

Similarly, $XGE_W YA^\dagger$ is an idempotent too. Thus, $R(A^\dagger XF_W GY)$ and $R(XGE_W YA^\dagger)$ are closed.
Since $(I - 2A^\dagger A)\Lambda(I - 2AA^\dagger) = \Lambda$ and

$$\begin{aligned}(I - A^\dagger XF_W GY)\Lambda(I - XGE_W YA^\dagger) &= (I - A^\dagger XF_W GY)(I - 2A^\dagger A)\Lambda(I - 2AA^\dagger)(I - XGE_W YA^\dagger) \\ &= (I - 2A^\dagger A + A^\dagger XF_W GY)\Lambda(I - 2AA^\dagger + XGE_W YA^\dagger) \\ &= (I - A^\dagger A - \Lambda(A - XGY))\Lambda(I - AA^\dagger - (A - XGY)) \\ &= \Lambda(A - XGY)\Lambda \\ &= \Lambda(I - XGE_W YA^\dagger),\end{aligned}$$

we have $\bar{\Lambda} = \Lambda(I - XGE_W YA^\dagger)$ is a generalized inverse of $A - XGY$ and

$$(A - XGY)\bar{\Lambda} = (A - XGY)\Lambda, \quad \bar{\Lambda}(A - XGY) = \Lambda(A - XGY).$$

Noting that $I - XGE_W YA^\dagger$ commute with $I - 2AA^\dagger$ and consequently, by Lemma 2.1 and Lemma 2.2,

$$(A - XGY)^\dagger = \{I - (A^\dagger XF_W GY)(A^\dagger XF_W GY)^\dagger\}\{A^\dagger + A^\dagger XW^\dagger YA^\dagger\}\{I - (XGE_W YA^\dagger)^\dagger(XGE_W YA^\dagger)\}.$$

□

4. The Perturbation Analysis of the Moore–Penrose Inverse

In this section, we study the perturbation of the Moore–Penrose inverse and give the upper bound of $\|(A + X)^\dagger\|$ and $\|(A + X)^\dagger - A^\dagger\|$ on general case.

Theorem 4.1. *Let $A, X \in B(H_1, H_2)$ with $R(A)$ closed and $Z = I + A^\dagger X, U = E_A X, F_U = I - U^\dagger U, S = A^\dagger AZF_U$. If $R(S)$ is closed, then*

$$\|(A + X)^\dagger\| \leq (1 + \|S^\dagger\|)(\|A^\dagger\| + \|Z\|\|U^\dagger\|) + \|U^\dagger\|.$$

$$\|(A + X)^\dagger - A^\dagger\| \leq \left\{\|(I + S^\dagger - SS^\dagger)(A^\dagger - A^\dagger AZU^\dagger) + U^\dagger\|^2 + \|A^\dagger\|^2\right\}\|X\|.$$

Proof. Since $R(S)$ is closed, we have $(A+X)^\dagger$ exists by Corollary 3.3. Noting that $A^\dagger AZS^\dagger = SS^\dagger, S^\dagger A^\dagger A = S^\dagger$, by Corollary 3.3, we have

$$\begin{aligned}
\Lambda &= A^\dagger + (F_A - A^\dagger X)[S^\dagger(A^\dagger - ZU^\dagger) + U^\dagger] \\
&= A^\dagger + (I - A^\dagger AZ)[S^\dagger(A^\dagger - ZU^\dagger) + U^\dagger] \\
&= A^\dagger + S^\dagger(A^\dagger - ZU^\dagger) + U^\dagger - A^\dagger AZ[S^\dagger(A^\dagger - ZU^\dagger) + U^\dagger] \\
&= A^\dagger + S^\dagger(A^\dagger - ZU^\dagger) + U^\dagger - A^\dagger AZS^\dagger(A^\dagger - ZU^\dagger) - A^\dagger AZU^\dagger \\
&= A^\dagger + S^\dagger A^\dagger - S^\dagger ZU^\dagger + U^\dagger - SS^\dagger(A^\dagger - ZU^\dagger) - A^\dagger AZU^\dagger \\
&= A^\dagger + S^\dagger A^\dagger - S^\dagger ZU^\dagger + U^\dagger - SS^\dagger A^\dagger + SS^\dagger ZU^\dagger - A^\dagger AZU^\dagger \\
&= (I + S^\dagger - SS^\dagger)A^\dagger + (I - S^\dagger Z + SS^\dagger Z - A^\dagger AZ)U^\dagger \\
&= (I + S^\dagger - SS^\dagger)A^\dagger + [I - (I + S^\dagger - SS^\dagger)A^\dagger AZ]U^\dagger \\
&= (I + S^\dagger - SS^\dagger)A^\dagger - (I + S^\dagger - SS^\dagger)A^\dagger AZU^\dagger + U^\dagger \\
&= (I + S^\dagger - SS^\dagger)(A^\dagger - A^\dagger AZU^\dagger) + U^\dagger.
\end{aligned}$$

Since Λ is an inner inverse of $A+X$, we have

$$\begin{aligned}
\|(A+X)^\dagger\| &\le \|(A+X)^-\| \\
&= \|(I + S^\dagger - SS^\dagger)(A^\dagger - A^\dagger AZU^\dagger) + U^\dagger\| \\
&\le (1 + \|S^\dagger\|)(\|A^\dagger\| + \|Z\|\|U^\dagger\|) + \|U^\dagger\|
\end{aligned}$$

From the identity(cf. [4, Eq.(21)]),

$$(A+X)^\dagger - A^\dagger = -(A+X)^\dagger XA^\dagger + (A+X)^\dagger((A+X)^\dagger)^* X^*(I - AA^\dagger) + [I - (A+X)^\dagger(A+X)]X^*(A^\dagger)^* A^\dagger,$$

by applying the orthogonality of the operators on the right side, we get

$$\begin{aligned}
\|(A+X)^\dagger - A^\dagger\|^2 &= \|-(A+X)^\dagger XA^\dagger + (A+X)^\dagger((A+X)^\dagger)^* X^*(I - AA^\dagger)\|^2 + \|[I - (A+X)^\dagger(A+X)]X^*(A^\dagger)^* A^\dagger\|^2 \\
&\le \|-(A+X)^\dagger XA^\dagger\|^2 + \|(A+X)^\dagger((A+X)^\dagger)^* X^*(I - AA^\dagger)\|^2 + \|X\|^2\|A^\dagger\|^4 \\
&\le \|(A+X)^\dagger\|^2\|X\|^2\|A^\dagger\|^2 + \|(A+X)^\dagger\|^4\|X\|^2 + \|X\|^2\|A^\dagger\|^4 \\
&\le \left\{\|(A+X)^\dagger\|^2\|A^\dagger\|^2 + \|(A+X)^\dagger\|^4 + \|A^\dagger\|^4\right\}\|X\|^2 \\
&\le \left\{\|(A+X)^\dagger\|^2 + \|A^\dagger\|^2\right\}^2\|X\|^2 \\
&\le \left\{\|(I + S^\dagger - SS^\dagger)(A^\dagger - A^\dagger AZU^\dagger) + U^\dagger\|^2 + \|A^\dagger\|^2\right\}^2\|X\|^2.
\end{aligned}$$

□

Corollary 4.2. *Let $A, X \in B(H_1, H_2)$ with $R(A)$ closed and $Z = I + A^\dagger X, S = A^\dagger AZ$. If $R(X) \subseteq R(A)$ and $R(S)$ are closed, then*

$$\|(A+X)^\dagger\| \le (1 + \|S^\dagger\|)\|A^\dagger\|.$$

$$\frac{\|(A+X)^\dagger - A^\dagger\|}{\|A^\dagger\|} \le (\sqrt{2} + \|S^\dagger\|)\|X\|\|A^\dagger\|.$$

Proof. Since $R(X) \subset R(A)$, we have $U = 0$. By Theorem4.1, we have

$$\|(A+X)^\dagger\| \le (1 + \|S^\dagger\|)\|A^\dagger\|$$

and

$$\begin{aligned}\|(A+X)^{\dagger}-A^{\dagger}\| &\le \left\{\|(I+S^{\dagger}-SS^{\dagger})A^{\dagger}\|^2+\|A^{\dagger}\|^2\right\}\|X\| \\ &\le \{1+(1+\|S^{\dagger}\|)^2\}\|X\|\|A^{\dagger}\|^2 \\ &\le (\sqrt{2}+\|S^{\dagger}\|)\|X\|\|A^{\dagger}\|^2.\end{aligned}$$

Thus,

$$\frac{\|(A+X)^{\dagger}-A^{\dagger}\|}{\|A^{\dagger}\|} \le (\sqrt{2}+\|S^{\dagger}\|)\|X\|\|A^{\dagger}\|.$$

□

Remark 4.3. *The condition $R(X) \subseteq R(A)$ in Corollary 4.2 shows it is a perturbation of range–preserving . In additon, if $Z = I + A^{\dagger}X$ is invertible, then $(A+X)^{+} = Z^{-1}A^{\dagger}$. This is a special case of stable perturbation (Please see [17] for details).*

Corollary 4.4. *Let $A, X \in B(H_1, H_2)$ with $R(A)$ closed and $Z = I + XA^{\dagger}, T = ZAA^{\dagger}$. If $\ker(A) \subseteq \ker(X)$ and $R(T)$ are closed, then*

$$\|(A+X)^{\dagger}\| \le (1+\|T^{\dagger}\|)\|A^{\dagger}\|.$$

$$\frac{\|(A+X)^{\dagger}-A^{\dagger}\|}{\|A^{\dagger}\|} \le (\sqrt{2}+\|T^{\dagger}\|)\|X\|\|A^{\dagger}\|.$$

Proof. Note that $\|T^{\dagger}\| = \|(T^*)^{\dagger}\|$ and $A^{\dagger}(I + T^{+} - T^{+}T)$ is an inner inverse of $A + X$ by Corollary 3.4. Similar to the proof in Corollary 4.2, the results can be obtained easily. □

Remark 4.5. *The condition $\ker(A) \subseteq \ker(X)$ in Corollary 4.4 shows it is a perturbation of kernel–preserving. In additon, if $Z = I + XA^{\dagger}$ is invertible, then $(A+X)^{+} = A^{\dagger}Z^{-1}$. This is also a special cases of stable perturbation (Please see [17] for details).*

Corollary 4.6. *Let $A, X \in B(H_1, H_2)$ with $R(A)$ closed. If $R(X) \subseteq R(A), \ker(A) \subseteq \ker(X)$ and $R(I + A^{\dagger}X)$ is closed, then*

$$\|(A+X)^{\dagger}\| \le \|(I+A^{\dagger}X)^{\dagger}\|\|A^{\dagger}\|$$

and

$$\frac{\|(A+X)^{\dagger}-A^{\dagger}\|}{\|A^{\dagger}\|} \le \frac{1+\sqrt{5}}{2}\|(I+A^{\dagger}X)^{\dagger}\|\|A^{\dagger}\|\|X\|.$$

Proof. Since $R(I + A^{\dagger}X)$ is closed, we have $R(I + XA^{\dagger})$ is closed by Corollary 3.5. Since $R(X) \subseteq R(A), \ker(A) \subseteq \ker(X)$, we have $AA^{\dagger}X = X, XA^{\dagger}A = X$. Thus,

$$(I+A^{\dagger}X)A^{\dagger}A = A^{\dagger}A(I+A^{\dagger}X).$$

Combining with $(A^{\dagger}A)^* = A^{\dagger}A$, we get

$$R(A^{\dagger}A(I+A^{\dagger}X)) \subseteq R(I+A^{\dagger}X),$$

$$R((I+A^{\dagger}X)(I+A^{\dagger}X)^*(A^{\dagger}A)) \subseteq R(A^{\dagger}A).$$

Hence, by Lemma 2.4

$$(A^{\dagger}A+A^{\dagger}X)^{\dagger} = (I+A^{\dagger}X)^{\dagger}A^{\dagger}A.$$

Therefore,

$$\begin{aligned}&(I+A^{\dagger}X)^{\dagger}A^{\dagger}(A+X)(I+A^{\dagger}X)^{\dagger}A^{\dagger} \\ &= (I+A^{\dagger}X)^{\dagger}A^{\dagger}A(A^{\dagger}A+A^{\dagger}X)(I+A^{\dagger}X)^{\dagger}A^{\dagger}AA^{\dagger} \\ &= (I+A^{\dagger}X)^{\dagger}A^{\dagger}.\end{aligned}$$

It is easy to obtain $A + X = (A + X)(I + A^{\dagger}X)^{\dagger}A^{\dagger}(A + X)$. The above indicate $(I + A^{\dagger}X)^{\dagger}A^{\dagger}$ is a generalized inverse of $A + X$. Similarly, we get $A^{\dagger}(I + XA^{\dagger})^{\dagger}$ is a generalized inverse of $A + X$ too. Hence,

$$\|(A + X)^{\dagger}\| \leq \|(A + X)^{+}\| \leq \|(I + A^{\dagger}X)^{\dagger}\|\|A^{\dagger}\|.$$

Noting that $R(X) \subseteq R(A)$ and $\ker(A) \subseteq \ker(X)$ mean $R(A + X) \cap R(A)^{\perp} = \{0\}$ and $(\ker(A + X))^{\perp} \cap \ker A = \{0\}$, respectively. Thus, from [20], we have

$$\begin{aligned}\frac{\|(A + X)^{\dagger} - A^{\dagger}\|}{\|A^{\dagger}\|} &\leq \frac{1 + \sqrt{5}}{2}\|(A + X)^{\dagger}\|\|X\| \\ &\leq \frac{1 + \sqrt{5}}{2}\|(I + A^{\dagger}X)^{\dagger}\|\|A^{\dagger}\|\|X\|.\end{aligned}$$

□

Proposition 4.7. *Let $A, X \in B(H_1, H_2)$ with $R(A)$ closed. Let $Z = I + A^{\dagger}X, G = AZ$. If $R(G)$are closed and $R(A + X) \cap R(G)^{\perp} = \{0\}$, then*

$$\|(A + X)^{\dagger}\| \leq \|G^{\dagger}\|$$

and

$$\|(A + X)^{\dagger} - A^{\dagger}\| \leq \frac{1 + \sqrt{5}}{2}\|G^{\dagger}\|^2\|X\| + (\sqrt{2} + \|(A^{\dagger}AZ)^{\dagger}\|)\|X\|\|A^{\dagger}\|^2.$$

Proof. Put $G = AZ = A(I + A^{\dagger}X), U = E_A X$. Then we have $A + X = G + U$. For $\forall x \in \ker G$, we have $G^{\dagger}(G+U)x = 0$ for $G^{\dagger}U = 0$. This means $(G+U)x \in R(A+X)\cap R(G)^{\perp} = \{0\}$. Hence, we have $\ker G = \ker(G+U)$. It is easy to verify $G^{\dagger}$ is a generalized inverse of $G + U$, i.e.,

$$(A + X)^{+} = (G + U)^{+} = G^{\dagger}.$$

Thus, $\|(A + X)^{\dagger}\| \leq \|(A + X)^{+}\| = \|G^{\dagger}\|$.

Since $R(A + X) \cap R(G)^{\perp} = \{0\}$ and $\ker(G + U)^{\perp} \cap \ker G = \{0\}$, from [20], we have

$$\begin{aligned}\|(G + U)^{\dagger} - G^{\dagger}\| &\leq \frac{1 + \sqrt{5}}{2}\|(G + U)^{\dagger}\|\|G^{\dagger}\|\|U\| \\ &\leq \frac{1 + \sqrt{5}}{2}\|G^{\dagger}\|^2\|X\|.\end{aligned}$$

Since $G = AZ = A + AA^{\dagger}X$ and $R(AA^{\dagger}X) \subseteq R(A)$, by Corollary 4.2, we have

$$\|G^{\dagger} - A^{\dagger}\| \leq (\sqrt{2} + \|(A^{\dagger}AZ)^{\dagger}\|)\|X\|\|A^{\dagger}\|^2.$$

Thus,

$$\begin{aligned}\|(A + X)^{\dagger} - A^{\dagger}\| &= \|(A + X)^{\dagger} - G^{\dagger} + G^{\dagger} - A^{\dagger}\| \\ &\leq \|(A + X)^{\dagger} - G^{\dagger}\| + \|G^{\dagger} - A^{\dagger}\| \\ &\leq \frac{1 + \sqrt{5}}{2}\|G^{\dagger}\|^2\|X\| + (\sqrt{2} + \|(A^{\dagger}AZ)^{\dagger}\|)\|X\|\|A^{\dagger}\|^2.\end{aligned}$$

□

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