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Some Integral Inequalities for p-Convex Functions

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Abstract. In this paper, we consider the class of p-convex functions. We derive some new integral inequalities of Hermite-Hadamard and Simpson type for differentiable p-convex functions using two new integral identities. Some special cases are also discussed. Interested readers may find novel and innovative applications of p-convex functions in various branches of pure and applied sciences. The ideas and techniques of this paper may stimulate further research in this field.

1. Introduction

The classical concept of convex sets and convex functions has been extended in different directions using various novel approaches, see [3, 4, 7–18, 25]. Motivated by this Zhang et al. [25] introduced an interesting class of convex functions which is called p-convex functions. For some recent investigations and extensions of p-convex functions, see [8, 15, 25].

Theory of convex functions and theory of inequalities are very closely related to each other. Hence many classical inequalities are derived for convex functions. One of the most extensively studied inequalities in the literature are Hermite-Hadamard's and Simpson's inequalities. For some interesting details on these inequalities, readers are referred to [1, 2, 4–24].

In this paper, we consider the class of p-convex functions. We derive some new integral identities for differentiable functions. Using these new auxiliary results, we obtain some new Hermite-Hadamard and Simpson type inequalities respectively. Some special cases are also discussed. Results obtained in this paper continue to hold for these problems. Our results can be viewed as significant refinement and improvement of the previously known results.

2. Definitions and Basic Results

In this section, we recall some previously known concepts and results.

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Definition 2.1 ([25]). *An interval I is said to be a p-convex set, if*

$$M_p(x,y;t) = [tx^p + (1-t)y^p]^{\frac{1}{p}} \in I, \quad \forall x,y \in I, t \in [0,1] \tag{2.1}$$

where $p = \frac{n}{m}$, $n = 2r+1, m = 2s+1$ *and* $k,r,s \in \mathbb{N}$.

Here for $t = \frac{1}{2}$, $A = M_1 = \frac{x+y}{2}$ is the arithmetic mean, see [9].

Definition 2.2. *[[25]] Let I be a p-convex set. A function* $f : I \to \mathbb{R}$ *is said to be p-convex function or belongs to the class* $PC(I)$*, if*

$$f(M_p(x,y;t)) \le tf(x) + (1-t)f(y), \ \forall x,y \in I, t \in [0,1].$$

It is obvious that, if $p = 1$, then Definition 2.2 reduces to the definition for classical convex functions.
Also note that for $t = \frac{1}{2}$ in Definition 2.2, we have Jensen p-convex functions or mid p-convex functions. That is

$$f\Big(M_p\big(x,y;1/2\big)\Big) \le \frac{f(x)+f(y)}{2}, \quad \forall x,y \in I.$$

It is worth mentioning that, if $p = -1$, then Definition 2.2 collapses to the harmonically convex functions.

Definition 2.3. *A function* $f : \mathbb{R}^* = (-\infty,0)\bigcup(0,+\infty) \to \mathbb{R}$ *is said to be harmonically convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le (1-t)f(x) + tf(y), \quad \forall x,y \in I, t \in [0,1]. \tag{2.2}$$

For the Harmoniaclly convex functions and their characterization, see, Noor et al. [14] and Noor et al. [16] and the references therein.
It is clear that p-convex functions includes convex functions and Harmonically convex functions as special cases. This shows that results obtained in this paper continue to hold for these classes of convex functions and their variant forms.

Theorem 2.4 (Hermite-Hadamard's Inequality). *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a convex function, where* $a,b \in I$ *with* $a < b$*. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{2.3}$$

Theorem 2.5 (Simpson's Inequality). *Let* $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a four time continuously differentiable on* I°*, where* I° *is the interior of I and* $\|f^{(4)}\|_\infty < \infty$*. Then*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^4. \tag{2.4}$$

3. Hermite-Hadamard Like Inequalities

In this section, we derive some new Hermite-Hadamard type inequalities for differentiable p-convex functions.

Lemma 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I), where $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$, then*

$$\Xi(a,b;p;n) = \frac{b^p - a^p}{(n+1)^2 p}\Big[\int_0^1 \Big(\frac{1}{2} - t\Big) f'\Big(\Big[\frac{n+t}{n+1}a^p + \frac{1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)dt + \int_0^1 \Big(\frac{1}{2} - t\Big) f'\Big(\Big[\frac{t}{n+1}a^p + \frac{n+1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)dt\Big],$$

where

$$\begin{aligned}
&\Xi(a,b;p;n)\\
&= \frac{1}{2(n+1)}\left[\frac{f(a)}{(a^p)^{\frac{1}{p}-1}} + \frac{f\Big(\Big[\frac{n}{n+1}a^p + \frac{1}{n+1}b^p\Big]^{\frac{1}{p}}\Big)}{\Big[\frac{n}{n+1}a^p + \frac{1}{n+1}b^p\Big]^{\frac{1}{p}-1}} + \frac{f\Big(\Big[\frac{1}{n+1}a^p + \frac{n}{n+1}b^p\Big]^{\frac{1}{p}}\Big)}{\Big[\frac{1}{n+1}a^p + \frac{n}{n+1}b^p\Big]^{\frac{1}{p}-1}} + \frac{f(b)}{(b^p)^{\frac{1}{p}-1}}\right]\\
&- \frac{p}{b^p - a^p}\left[\int_a^{\left(\frac{n}{n+1}a^p + \frac{1}{n+1}b^p\right)^{\frac{1}{p}}} x^{2p-2} f(x)dx + \int_{\left(\frac{1}{n+1}a^p + \frac{n}{n+1}b^p\right)^{\frac{1}{p}}}^{b} x^{2p-2} f(x)dx\right].
\end{aligned}$$

Proof. Let

$$\begin{aligned}
I &= \int_0^1 \Big(\frac{1}{2} - t\Big) f'\Big(\Big[\frac{n+t}{n+1}a^p + \frac{1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)dt + \int_0^1 \Big(\frac{1}{2} - t\Big) f'\Big(\Big[\frac{t}{n+1}a^p + \frac{n+1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)dt\\
&= I_1 + I_2. \qquad (3.1)
\end{aligned}$$

Now

$$\begin{aligned}
I_1 &= \int_0^1 \Big(\frac{1}{2} - t\Big) f'\Big(\Big[\frac{n+t}{n+1}a^p + \frac{1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)dt\\
&= \frac{(n+1)p}{2(b^p - a^p)}\left[\frac{f(a)}{(a^p)^{\frac{1}{p}-1}} + \frac{f\Big(\Big[\frac{n}{n+1}a^p + \frac{1}{n+1}b^p\Big]^{\frac{1}{p}}\Big)}{\Big[\frac{n}{n+1}a^p + \frac{1}{n+1}b^p\Big]^{\frac{1}{p}-1}}\right] - \frac{(n+1)^2 p^2}{(b^p - a^p)^2}\int_a^{\left(\frac{n}{n+1}a^p + \frac{1}{n+1}b^p\right)^{\frac{1}{p}}} x^{2p-2} f(x)dx. \qquad (3.2)
\end{aligned}$$

Similarly

$$\begin{aligned}
I_2 &= \int_0^1 \Big(\frac{1}{2} - t\Big) f'\Big(\Big[\frac{t}{n+1}a^p + \frac{n+1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)dt\\
&= \frac{(n+1)p}{2(b^p - a^p)}\left[\frac{f\Big(\Big[\frac{1}{n+1}a^p + \frac{n}{n+1}b^p\Big]^{\frac{1}{p}}\Big)}{\Big[\frac{1}{n+1}a^p + \frac{n}{n+1}b^p\Big]^{\frac{1}{p}-1}} + \frac{f(b)}{(b^p)^{\frac{1}{p}-1}}\right] - \frac{(n+1)^2 p^2}{(b^p - a^p)^2}\int_{\left(\frac{1}{n+1}a^p + \frac{n}{n+1}b^p\right)^{\frac{1}{p}}}^{b} x^{2p-2} f(x)dx. \qquad (3.3)
\end{aligned}$$

Combining (3.1), (3.2), (3.3) and multiplying by $\frac{b^p - a^p}{(n+1)^2 p}$ completes the proof. □

Remark 3.1. *For $p = 1 = n$ in Lemma 3.1, we have Lemma 2.1 of [23].*

Theorem 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$ and $|f'|$ is p-convex function, then*

$$\left|\Xi(a,b;p;n)\right| \leq \frac{b^p - a^p}{2(n+1)^2 p}\Big[\frac{|f'(a)| + |f'(b)|}{2}\Big].$$

Proof. Using Lemma 3.1 and the fact that $|f'|$ is p-convex function, we have

$$\begin{aligned}
&\left|\Xi(a,b;p;n)\right| \\
&= \left|\frac{b^p - a^p}{(n+1)^2 p}\Big[\int_0^1 \Big(\frac{1}{2} - t\Big) f'\Big(\Big[\frac{n+t}{n+1}a^p + \frac{1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)dt \right. \\
&\qquad \left. + \int_0^1 \Big(\frac{1}{2} - t\Big) f'\Big(\Big[\frac{t}{n+1}a^p + \frac{n+1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)dt\Big]\right| \\
&\leq \frac{b^p - a^p}{(n+1)^2 p}\Big[\int_0^1 \Big|\frac{1}{2} - t\Big|\Big|f'\Big(\Big[\frac{n+t}{n+1}a^p + \frac{1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)\Big|dt \\
&\qquad + \int_0^1 \Big|\frac{1}{2} - t\Big|\Big|f'\Big(\Big[\frac{t}{n+1}a^p + \frac{n+1-t}{n+1}b^p\Big]^{\frac{1}{p}}\Big)\Big|dt\Big] \\
&\leq \frac{b^p - a^p}{(n+1)^2 p}\Big[\int_0^1 \Big|\frac{1}{2} - t\Big|\Big[\Big(\frac{n+t}{n+1}\Big)|f'(a)| + \Big(\frac{1-t}{n+1}\Big)|f'(b)|\Big]dt \\
&\qquad + \int_0^1 \Big|\frac{1}{2} - t\Big|\Big[\Big(\frac{t}{n+1}\Big)|f'(a)| + \Big(\frac{n+1-t}{n+1}\Big)|f'(b)|\Big]dt\Big] \\
&= \frac{b^p - a^p}{2(n+1)^2 p}\Big[\frac{|f'(a)| + |f'(b)|}{2}\Big],
\end{aligned}$$

where

$$\int_0^1 \Big|\frac{1}{2} - t\Big|\Big(\frac{n+t}{n+1}\Big)dt = \frac{1+2n}{8(1+n)},$$

$$\int_0^1 \Big|\frac{1}{2} - t\Big|\Big(\frac{1-t}{n+1}\Big)dt = \frac{1}{8(1+n)},$$

$$\int_0^1 \Big|\frac{1}{2} - t\Big|\Big(\frac{t}{n+1}\Big)dt = \frac{1}{8(1+n)},$$

and

$$\int_0^1 \Big|\frac{1}{2} - t\Big|\Big(\frac{n+1-t}{n+1}\Big)dt = \frac{1+2n}{8(1+n)},$$

respectively. This completes the proof. □

Theorem 3.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$ and $|f'|^q$, $\frac{1}{r} + \frac{1}{q} = 1$, $r, q \geq 1$ is p-convex function, then*

$$\begin{aligned}&\left|\Xi(a, b; p; n)\right|\\&\leq \frac{b^p - a^p}{2^{\frac{1}{q}}(n+1)^{2+\frac{1}{q}}p}\left[\left(\frac{1}{2^r(1+r)}\right)^{\frac{1}{r}}\left\{\left((1+2n)|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}} + \left(|f'(a)|^q + (1+2n)|f'(b)|^q\right)^{\frac{1}{q}}\right\}\right].\end{aligned}$$

Proof. Using Lemma 3.1, Holder's inequality and the fact that $|f'|^q$ is p-convex function, we have

$$\begin{aligned}&\left|\Xi(a, b; p; n)\right|\\&= \left|\frac{b^p - a^p}{(n+1)^2 p}\left[\int_0^1 \left(\frac{1}{2} - t\right) f'\left(\left[\frac{n+t}{n+1}a^p + \frac{1-t}{n+1}b^p\right]^{\frac{1}{p}}\right)dt\right.\right.\\&\qquad\left.\left. + \int_0^1 \left(\frac{1}{2} - t\right) f'\left(\left[\frac{t}{n+1}a^p + \frac{n+1-t}{n+1}b^p\right]^{\frac{1}{p}}\right)dt\right]\right|\\&\leq \frac{b^p - a^p}{(n+1)^2 p}\left[\left(\int_0^1 \left|\frac{1}{2} - t\right|^r dt\right)^{\frac{1}{r}}\left\{\left(\int_0^1 \left|f'\left(\left[\frac{n+t}{n+1}a^p + \frac{1-t}{n+1}b^p\right]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\right.\right.\\&\qquad\left.\left. + \left(\int_0^1 \left|f'\left(\left[\frac{t}{n+1}a^p + \frac{n+1-t}{n+1}b^p\right]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\right]\\&\leq \frac{b^p - a^p}{(n+1)^2 p}\left[\left(\frac{1}{2^r(1+r)}\right)^{\frac{1}{r}}\left\{\left(\int_0^1 \left[\left(\frac{n+t}{n+1}\right)|f'(a)|^q + \left(\frac{1-t}{n+1}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right.\right.\\&\qquad\left.\left. + \left(\int_0^1 \left[\left(\frac{t}{n+1}\right)|f'(a)|^q + \left(\frac{n+1-t}{n+1}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}\right]\\&= \frac{b^p - a^p}{2^{\frac{1}{q}}(n+1)^{2+\frac{1}{q}}p}\left[\left(\frac{1}{2^r(1+r)}\right)^{\frac{1}{r}}\left\{\left((1+2n)|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}} + \left(|f'(a)|^q + (1+2n)|f'(b)|^q\right)^{\frac{1}{q}}\right\}\right].\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$ and $|f'|^q$, $q \geq 1$ is p-convex function, then*

$$\begin{aligned}&\left|\Xi(a, b; p; n)\right|\\&\leq \frac{b^p - a^p}{4(n+1)^{2+\frac{1}{q}}p}\left[\left(\frac{1}{2}\right)^{\frac{1}{q}}\left\{\left((1+2n)|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}} + \left(|f'(a)|^q + (1+2n)|f'(b)|^q\right)^{\frac{1}{q}}\right\}\right].\end{aligned}$$

Proof. Using Lemma 3.1, power-mean inequality and the fact that $|f'|^q$ is p-convex function, we have

$$\begin{aligned}
&\left|\Xi(a,b;p;n)\right| \\
&= \left|\frac{b^p-a^p}{(n+1)^2p}\left[\int_0^1\left(\frac{1}{2}-t\right)f'\left(\left[\frac{n+t}{n+1}a^p+\frac{1-t}{n+1}b^p\right]^{\frac{1}{p}}\right)dt+\int_0^1\left(\frac{1}{2}-t\right)f'\left(\left[\frac{t}{n+1}a^p+\frac{n+1-t}{n+1}b^p\right]^{\frac{1}{p}}\right)dt\right]\right| \\
&\le \frac{b^p-a^p}{(n+1)^2p}\left[\left(\int_0^1\left|\frac{1}{2}-t\right|dt\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1\left|\frac{1}{2}-t\right|\left|f'\left(\left[\frac{n+t}{n+1}a^p+\frac{1-t}{n+1}b^p\right]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\right.\right. \\
&\qquad \left.\left.+\left(\int_0^1\left|\frac{1}{2}-t\right|\left|f'\left(\left[\frac{t}{n+1}a^p+\frac{n+1-t}{n+1}b^p\right]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\right] \\
&\le \frac{b^p-a^p}{(n+1)^2p}\left[\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1\left|\frac{1}{2}-t\right|\left[\left(\frac{n+t}{n+1}\right)|f'(a)|^q+\left(\frac{1-t}{n+1}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right.\right. \\
&\qquad \left.\left.+\left(\int_0^1\left|\frac{1}{2}-t\right|\left[\left(\frac{t}{n+1}\right)|f'(a)|^q+\left(\frac{n+1-t}{n+1}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}\right] \\
&= \frac{b^p-a^p}{4(n+1)^{2+\frac{1}{q}}p}\left[\left(\frac{1}{2}\right)^{\frac{1}{q}}\left\{\left((1+2n)|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}+\left(|f'(a)|^q+(1+2n)|f'(b)|^q\right)^{\frac{1}{q}}\right\}\right].
\end{aligned}$$

This completes the proof. □

4. Simpson Type Inequalities

In this section, we derive some Simpson type inequalities for differentiable p-convex functions.

Lemma 4.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ differentiable function I° where $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$, then*

$$\Theta(a,b;p)-\frac{p^2}{b^p-a^p}\int_a^b x^{2p-2}f(x)dx=(b^p-a^p)\int_0^1\mu(t)f'([(1-t)a^p+tb^p]^{\frac{1}{p}})dt,$$

where

$$\Theta(a,b;p)=\frac{p}{6}\left[\frac{f(a)}{(a^p)^{\frac{1}{p}-1}}+4\frac{f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)}{\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}-1}}+\frac{f(b)}{(b^p)^{\frac{1}{p}-1}}\right],$$

and

$$\mu(t)=\begin{cases} t-\frac{1}{6}, & if\, t\in\left[0,\frac{1}{2}\right), \\ t-\frac{5}{6}, & if\, t\in\left[\frac{1}{2},1\right]. \end{cases}$$

Proof. Let

$$\begin{aligned} I &= \int_0^1 \mu(t) f'([(1-t)a^p + tb^p]^{\frac{1}{p}})dt \\ &= \int_0^{\frac{1}{2}} \Big(t - \frac{1}{6}\Big) f'([(1-t)a^p + tb^p]^{\frac{1}{p}})dt + \int_{\frac{1}{2}}^1 \Big(t - \frac{5}{6}\Big) f'([(1-t)a^p + tb^p]^{\frac{1}{p}})dt \\ &= I_1 + I_2. \end{aligned} \tag{4.1}$$

Now

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \Big(t - \frac{1}{6}\Big) f'([(1-t)a^p + tb^p]^{\frac{1}{p}})dt \\ &= \frac{p}{b^p - a^p} \left[\frac{1}{3} \frac{f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big)}{\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}-1}} + \frac{1}{6} \frac{f(a)}{(a^p)^{\frac{1}{p}-1}} \right] - \frac{p}{b^p - a^p} \int_0^{\frac{1}{2}} \frac{f([(1-t)a^p + tb^p]^{\frac{1}{p}})}{[(1-t)a^p + tb^p]^{\frac{1}{p}-1}} dt. \end{aligned} \tag{4.2}$$

Similarly

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^1 \Big(t - \frac{5}{6}\Big) f'([(1-t)a^p + tb^p]^{\frac{1}{p}})dt \\ &= \frac{p}{b^p - a^p} \left[\frac{1}{3} \frac{f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big)}{\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}-1}} + \frac{1}{6} \frac{f(b)}{(b^p)^{\frac{1}{p}-1}} \right] - \frac{p}{b^p - a^p} \int_{\frac{1}{2}}^1 \frac{f([(1-t)a^p + tb^p]^{\frac{1}{p}})}{[(1-t)a^p + tb^p]^{\frac{1}{p}-1}} dt. \end{aligned} \tag{4.3}$$

On summation of (4.1), (4.2), (4.3) and multiplying by $(b^p - a^p)$ completes the proof. □

Remark 4.1. *For $p = 1$ in Lemma 4.1 we have Lemma 1 of [2]*

Theorem 4.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ differentiable function I° where $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$ and $|f'|$ is p-convex function, then*

$$\left| \Theta(a, b; p) - \frac{p^2}{b^p - a^p} \int_a^b x^{2p-2} f(x)dx \right| \le \frac{5(b^p - a^p)}{72} \Big[|f'(a)| + |f'(b)| \Big].$$

Proof. Using Lemma 4.1 and the fact that $|f'|$ is p-convex function, we have

$$\begin{aligned}
&\left|\Theta(a,b;p) - \frac{p^2}{b^p - a^p}\int_a^b x^{2p-2}f(x)dx\right| \\
&= \left|(b^p - a^p)\int_0^1 \mu(t)f'([(1-t)a^p + tb^p]^{\frac{1}{p}})dt\right| \\
&= \left|(b^p - a^p)\left[\int_0^{\frac{1}{2}}\left(t - \frac{1}{6}\right)f'([(1-t)a^p + tb^p]^{\frac{1}{p}})dt + \int_{\frac{1}{2}}^1\left(t - \frac{5}{6}\right)f'([(1-t)a^p + tb^p]^{\frac{1}{p}})dt\right]\right| \\
&\leq (b^p - a^p)\left[\int_0^{\frac{1}{2}}\left|t - \frac{1}{6}\right|[(1-t)|f'(a)| + t|f'(b)|]dt + \int_{\frac{1}{2}}^1\left|t - \frac{5}{6}\right|[(1-t)|f'(a)| + t|f'(b)|]dt\right] \\
&= \frac{5(b^p - a^p)}{72}\Big[|f'(a)| + |f'(b)|\Big].
\end{aligned}$$

This completes the proof. □

Theorem 4.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ differentiable function I° where $a, b \in I$ with $a < b$. If $f' \in L_1([a,b])$ and $|f'|^q$, $\frac{1}{r} + \frac{1}{q} = 1$, $r, q \geq 1$, is p-convex function, then*

$$\begin{aligned}
&\left|\Theta(a,b;p) - \frac{p^2}{b^p - a^p}\int_a^b x^{2p-2}f(x)dx\right| \\
&\leq (b^p - a^p)\left(\frac{6^{-1-r}(1 + 2^{1+r})}{1 + r}\right)^{\frac{1}{r}}\left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. Using Lemma 4.1, Holder's inequality and the fact that $|f'|^q$ is p-convex function, we have

$$\begin{aligned}
&\left|\Theta(a,b;p) - \frac{p^2}{b^p - a^p}\int_a^b x^{2p-2}f(x)dx\right| \\
&\leq (b^p - a^p)\left\{\left(\int_0^{\frac{1}{2}}\left|t - \frac{1}{6}\right|^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}|f'([(1-t)a^p + tb^p]^{\frac{1}{p}})|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\int_{\frac{1}{2}}^1\left|t - \frac{5}{6}\right|^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^1|f'([(1-t)a^p + tb^p]^{\frac{1}{p}})|^q dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

$$\leq (b^p - a^p)\Big(\frac{6^{-1-r}(1+2^{1+r})}{1+r}\Big)^{\frac{1}{r}}\Bigg[\Bigg(\int_0^{\frac{1}{2}}[(1-t)|f'(a)|^q + t|f'(b)|^q]dt\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\int_{\frac{1}{2}}^{1}[(1-t)|f'(a)|^q + t|f'(b)|^q]dt\Bigg)^{\frac{1}{q}}\Bigg]$$

$$= (b^p - a^p)\Big(\frac{6^{-1-r}(1+2^{1+r})}{1+r}\Big)^{\frac{1}{r}}\Bigg[\Bigg(\frac{3|f'(a)|^q + |f'(b)|^q}{8}\Bigg)^{\frac{1}{q}} + \Bigg(\frac{|f'(a)|^q + 3|f'(b)|^q}{8}\Bigg)^{\frac{1}{q}}\Bigg].$$

This completes the proof. □

Theorem 4.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ differentiable function I° where $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$ and $|f'|^q$, $q \geq 1$, is p-convex function, then*

$$\Big|\Theta(a, b; p) - \frac{p^2}{b^p - a^p}\int_a^b x^{2p-2} f(x)dx\Big|$$

$$\leq \frac{5(b^p - a^p)}{72}\Big(\frac{1}{90}\Big)^{\frac{1}{q}}\Big[\Big(61|f'(a)|^q + 29|f'(b)|^q\Big)^{\frac{1}{q}} + \Big(29|f'(a)|^q + 61|f'(b)|^q\Big)^{\frac{1}{q}}\Big]$$

Proof. Using Lemma 4.1, power-mean inequality and the fact that $|f'|^q$ is p-convex function, we have

$$\Big|\Theta(a, b; p) - \frac{p^2}{b^p - a^p}\int_a^b x^{2p-2} f(x)dx\Big|$$

$$\leq (b^p - a^p)\Bigg\{\Bigg(\int_0^{\frac{1}{2}}\Big|t - \frac{1}{6}\Big|dt\Bigg)^{1-\frac{1}{q}}\Bigg(\int_0^{\frac{1}{2}}\Big|t - \frac{1}{6}\Big||f'([(1-t)a^p + tb^p]^{\frac{1}{p}})|^q dt\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\int_{\frac{1}{2}}^{1}\Big|t - \frac{5}{6}\Big|dt\Bigg)^{1-\frac{1}{q}}\Bigg(\int_{\frac{1}{2}}^{1}\Big|t - \frac{5}{6}\Big||f'([(1-t)a^p + tb^p]^{\frac{1}{p}})|^q dt\Bigg)^{\frac{1}{q}}\Bigg\}$$

$$\leq (b^p - a^p)\Big(\frac{5}{72}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\int_0^{\frac{1}{2}}\Big|t - \frac{1}{6}\Big|[(1-t)|f'(a)|^q + t|f'(b)|^q]dt\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\int_{\frac{1}{2}}^{1}\Big|t - \frac{5}{6}\Big|[(1-t)|f'(a)|^q + t|f'(b)|^q]dt\Bigg)^{\frac{1}{q}}\Bigg]$$

$$= \frac{5(b^p - a^p)}{72}\Big(\frac{1}{90}\Big)^{\frac{1}{q}}\Big[\Big(61|f'(a)|^q + 29|f'(b)|^q\Big)^{\frac{1}{q}} + \Big(29|f'(a)|^q + 61|f'(b)|^q\Big)^{\frac{1}{q}}\Big]$$

This completes the proof. □

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